zero-prong events ar rest are about $(2.5\pm1.2)\%$ and at 80 Mev about 6% of the total number of events. With improved statistics and a better resolution of the π^0 events, we believe the above theoretical estimates can be checked more correctly.

$N\bar{p}$ ANNIHILATION

For $N\bar{p}$ annihilation, the values of S_n/S_2 for different values of λ are given in Table VII. The values of $\bar{n}$ thus determined are also given. As in the $p\bar{p}$ annihilation, the selection rules change significantly the number distribution of the outgoing pions without changing the average multiplicity. If, as remarked earlier, we ignore partial transmission in ${}^3D_3{}^3$ and ${}^3F_4{}^1$ states, then the results at 140 and 260 Mev would be identical.

In the collaboration emulsion experiment,[14] the value of $\bar{n}$ was observed to be 5.3 ± 0.4. Here 35 events were recorded out of which 21 annihilations occurred in flight at an average laboratory energy of 140 Mev. In another recent emulsion experiment,[15] $\bar{n}$ was observed to be 5.36 ± 0.3. There were 221 events recorded out of which 95 events occurred in flight at an average laboratory energy of 140 Mev. In the propane bubble-chamber experiment, the $\bar{n}$ value was observed to be 4.7 ± 0.5.[12] Here there were 337 $\bar{p}$C events recorded out of which 166 occurred in flight at an average laboratory energy of 80 Mev.

We see that for $\lambda\sim10$ a good agreement with experiment is obtained. It is interesting to note that $\lambda=n$ also gives the multiplicity close to the experimental values. This might suggest that there is a strong pion-pion interaction in the final state.[16]

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Threshold Effects in Three-Body Channels

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The possibility of obtaining threshold anomalies in reactions leading to three-particle channels is studied in detail. It is found that a threshold cusp or rounded step exists in reactions whose final three-body channels have at least one particle in common. The effect appears as a function of the momentum of the common particle while the total energy is fixed.

I. INTRODUCTION

THE anomalous energy dependence of a scattering or reaction cross section at the threshold of a new inelastic process (the so-called "Wigner cusp"[1]) has been investigated in a number of recent theoretical papers.[2–9] The analysis of this effect, apart from the information one can obtain about scattering phase shifts, proves to be particularly useful for the determination of parities and spins of the reaction products.[2–13]

It is now well understood that the physical reason for the infinite energy derivative of old cross sections at the threshold of a new channel is the sudden removal of flux from the incident beam due to the opening of a new cross section which starts with an infinite slope. There is consequently no such cusp (or rounded step)

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if both particles in the new channel are charged; for then the new inelastic cross section starts with zero slope (in the repulsive case) or with finite slope (in the attractive case) but with nonzero value. In the latter case another type of anomaly is to be observed.[6,8] It is clear that no observable anomaly exists at the onset of a continuum channel such as a three or more particle production process,[6] except in the second derivative where it is not expected to be experimentally detectable.[9]

In the present paper we want to draw attention to another type of cross-section measurement in which an energy anomaly is to be expected, and derive the quantitative details on quite general grounds. It is perhaps simplest to describe the type of experiment we have in mind by a special example which has been recently discussed.[14]

Suppose one were to measure the process

(a) $$K^- + d \rightarrow \Lambda + p + \pi^-$$

at a fixed total energy. One may then observe the production process for various energies of the emerging pion. Concomitantly to process (a), the process

(b) $$K^- + d \rightarrow \Sigma^0 + p + \pi^-$$

is also possible, but the energy of the pion has a smaller maximum than for process (a). Therefore the pions seen near the maximum energy for (a) must all come from (a), but if we look at pions below the maximum energy for (b) then they may come from either (a) or (b). Thus there is a *threshold* in the cross section as a function of the pion energy, with *fixed total energy*, below which channel (b) is open, and above which it is closed. One may expect a corresponding anomaly in the dependence of the cross section for (a) as a function of the pion energy, the energy of the K^- beam being fixed. Moreover, the total counting rate of pions from both processes (a) and (b) will exhibit the characteristic cusp (or rounded step) too.

The reason why an infinite derivative appears in the type of cross section described above while it fails to appear at the threshold of a three-particle process as a function of the *total* energy[6] is that in the process with fixed incident energy the threshold acts essentially as a two-particle threshold, as though the pion acted only to test the energy of the two-particle process

(c) $$\Lambda + p \rightarrow \Sigma^0 + p.$$

It is therefore not surprising to find, as we do, that the cusp size depends on the size of the cross section for (c). In other words, in the physically implausible event that (c) (in which the pion does not partake at all) does not occur, no threshold anomaly would exist in (a) as a function of the pion energy.

Apart from phase factors which may be observed by detailed cusp measurements, and to which discussions as in references 2 and 5 are applicable, the cusp observation may in principle be a tool for the observation of relative parities. Since it is the S or P wave in the $\Lambda - p$ system which leads to the S wave in the $\Sigma - p$ system at threshold, depending on the $\Lambda - \Sigma$ parity, observation of the angular distribution of (a) near the threshold for (b) may thus determine that parity. This is admittedly a difficult experiment. We suggest that it may be easier to look at (a) near the threshold for (b) as a function of the *proton* energy. In that case the cross section for the fundamental process

(d) $$\Lambda + \pi^- \rightarrow \Sigma^0 + \pi^-$$

enters, in which the relative $\Lambda - \pi$ momentum is considerably smaller at threshold and hence the P wave may be expected to be much smaller than the S wave. It may then be possible to determine the $\Lambda - \Sigma$ parity without looking at the angular distribution of (a).

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II. FORMULATION OF THE PROBLEM

We work in the center-of-mass coordinate system and introduce the customary set of coordinates for three particles of masses m_1, m_2, m_3, coordinates $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, and momenta $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$:

$$\begin{aligned} \mathbf{r}_1 &= (m_1/\mu_1)\mathbf{R}_1, \\ \mathbf{r}_2 &= \mathbf{R}_3 - \mathbf{R}_2, \end{aligned} \tag{1}$$

and their canonical momenta:

$$\begin{aligned} \mathbf{p}_1 &= \hbar\mathbf{k}_1 = \mathbf{P}_1 = \mu_1 d\mathbf{r}_1/dt, \\ \mathbf{p}_2 &= \hbar\mathbf{k}_2 = (m_2\mathbf{P}_3 - m_3\mathbf{P}_2)/(m_2+m_3) \\ &= \mu_2\left(\frac{d\mathbf{R}_3}{dt} - \frac{d\mathbf{R}_2}{dt}\right) = \mu_2\frac{d\mathbf{r}_2}{dt}, \end{aligned} \tag{2}$$

with the reduced masses

$$\mu_2 = \frac{m_2 m_3}{m_2+m_3}, \qquad \mu_1 = \frac{m_1(m_2+m_3)}{m_1+m_2+m_3}. \tag{3}$$

Thus $\mathbf{p}_2$ is directly related to the relative velocity of particles 2 and 3, and $\mathbf{p}_1$ is equal to the momentum of the first.

We are interested in cross sections for reactions leading from two particles to three, in which the first particle obtains a momentum between $\mathbf{p}_1$ and $\mathbf{p}_1 + d\mathbf{p}_1$, and particles 2 and 3 receive a relative momentum $\mathbf{p}_2$ in the direction between Ω_2 and $\Omega_2 + d\Omega_2$:

$$\frac{d\sigma_{fi}}{d^3k_1' d\Omega_2'} = \frac{1}{2s+1} \sum_{\nu\nu_1'\nu_2'} |\Theta_{fi}(\xi_1', \xi_2'; \xi)|^2, \tag{4}$$

with

$$\Theta_{fi}(\xi_1', \xi_2'; \xi) = -(2\pi)^2 \hbar k^{-1} (\mu_1' k_1')^{-\frac{1}{2}} \times (\phi_0^{(f)}(\xi_1', \xi_2') | H_I | \psi_+^{(i)}(\xi)). \tag{5}$$

The notation is such that ξ indicates the momentum, the spin, and its z-component:

$$\xi = (\mathbf{k}, s, \nu);$$

ϕ_0 is the "unperturbed" three-particle wave function; in the coordinate representation,

$$\phi_0(\xi_1,\xi_2;\mathbf{r}_1,\mathbf{r}_2)=\phi_0(\xi_1,\mathbf{r}_1)\phi_0(\xi_2,\mathbf{r}_2),$$
$$\phi_0(\xi,\mathbf{r})=(\mu k/\hbar^2)^{\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\chi_s^\nu e^{i\mathbf{k}\cdot\mathbf{r}},$$

which is normalized so that

$$\int(d\mathbf{r})\phi_0^*(\xi,\mathbf{r})\phi_0(\xi',\mathbf{r})=\delta(E-E')\delta(\Omega_k-\Omega_{k'})\delta_{ss'}\delta_{\nu\nu'},$$

χ_s^ν being the appropriate spin function; $\psi_+^{(i)}(\xi)$ is the complete wave function obeying the outgoing wave boundary condition and with the incident plane wave consisting of two particles of relative momentum and spin ξ in channel i.

If we make an angular momentum expansion we obtain

$$\Theta_{fi}(\xi_1',\xi_2';\xi)=-2\pi i\hbar k^{-1}(\mu_1'k_1')^{-\frac{1}{2}}\sum_{JMl_1l_2j_1j_2}Y_J^M(\lambda',\xi_1',\xi_2')\times S_{f\lambda',i\lambda}^J(E_1',E_2';E)Y_J^{M*}(\lambda,\xi),\quad(6)$$

with the following abbreviations:

$$\lambda\equiv l,\ s\quad\text{for a two-body channel}$$
$$\equiv l_1,\ s_1,\ j_1;\ l_2,\ s_2,\ j_2\quad\text{for a three-body channel,}$$
$$Y_J^M(\lambda,\xi)\equiv\sum_m C_{ls}(J,M;m,\nu)Y_l^m(\mathbf{k}),$$
$$Y_J^M(\lambda,\xi_1,\xi_2)\equiv\sum_{M_1M_2}C_{j_1j_2}(J,M;M_1,M_2)\times Y_{j_1}^{M_1}(\lambda_1,\xi_1)Y_{j_2}^{M_2}(\lambda_2,\xi_2),$$

$C_{ls}(J,M;m,\nu)$ and Y_l^m being the Clebsch-Gordan coefficients and the spherical harmonics in the notation and with the phase convention of Blatt and Weisskopf.[15]

With these definitions and the time reversal operator[16]:

$$\vartheta=\prod_j(i\sigma_y^{(j)})K,$$

where K is the complex conjugation operator and $\sigma_y^{(j)}$ is the Pauli spin matrix for the jth particle if it has spin $\frac{1}{2}$, or $i\sigma_y^{(j)}=1$ if it has spin zero, we have the simple properties

$$\vartheta Y_J^M(\lambda,\xi)=(-)^{J+M+s+\nu}Y_J^{-M}(\lambda,-\xi),$$
$$\vartheta Y_J^M(\lambda,\xi_1,\xi_2)=(-)^{J+M+s_1+\nu_1+s_2+\nu_2}Y_J^{-M}(\lambda,-\xi_1,-\xi_2),\quad(7)$$
$$\vartheta\phi_0(\xi,\mathbf{r})=(-)^{s+\nu}\phi_0(-\xi,\mathbf{r})$$

provided that we use real spin functions χ_s^ν and we mean

$$-\xi=(-\mathbf{k},s,-\nu).$$

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Equation (5) then leads to the reciprocity theorem

$$\Theta_{fi}(\xi_1',\xi_2';\xi)=(-)^{s+\nu+s_1'+\nu_1'+s_2'+\nu_2'}\Theta_{if}(-\xi;-\xi_1',-\xi_2'),\quad(8)$$

if H_I is invariant under time reversal. A similar relation, of course, holds for the two-particle to two-particle reaction and scattering amplitudes. Comparison of (8) with (6) shows that therefore the time-reversal invariance of H_I implies the *symmetry* of the matrix $S_{f\lambda',i\lambda}^J$ no matter whether the elements refer to two or three particle channels.

For a given total energy E the S-matrix element $S_{\beta\lambda',i\lambda}^J(E_1',E_2';E)$ leads to a continuous range of energy distributions E_1' and E_2' of the three-particle channel β, with $E_1'+E_2'=E$. Suppose then that there are two different three-particle channels open with the same particle 1, as, for example, in the case discussed in the Introduction. One may then fix the incident energy E and observe the number of particles 1 as a function of their energy E_1'. When E_1' is near its maximum then all observed particles 1 must come from the lighter channel; but when E_1' is small enough, some of the observed particles 1 may come from the heavier three particle channel. Thus there is a threshold, and we may expect to observe a corresponding anomaly in the energy dependence of the cross section. It is this anomaly which we want to study in detail.

The approach to the proof is analogous to that of reference 5. We first eliminate from each matrix element of S its usual threshold energy dependence. It is well known that there is a simple factor of $k^{l+\frac{1}{2}}$ if the channel contains two particles. We show in the Appendix that for three-particle channels the corresponding factor is $k_1^{l_1+\frac{1}{2}}k_2^{l_2+\frac{1}{2}}$. Therefore we write in the angular momentum representation in simple matrix notation

$$S=1-K_L\mathfrak{M}K_L,\quad(9)$$

where the matrix K_L is diagonal and has a $k_\gamma^{l+\frac{1}{2}}$ if the γ channel is a two-body channel, and a $k_{1\beta}^{l_1+\frac{1}{2}}k_{2\beta}^{l_2+\frac{1}{2}}$ if the β channel is a three-body channel. If the interaction Hamiltonian H_I is sufficiently well behaved then $\mathfrak{M}$ is finite at all thresholds.

III. THRESHOLD EFFECTS

The submatrix S_+ of S referring to open channels only is unitary; we can therefore introduce a real symmetric matrix $\mathfrak{K}$ so that

$$K_L\mathfrak{K}K_L(S_++1)=-i(S_+-1),\quad(10)$$

where the matrix multiplication for three-particle channels includes an integration over one of the energies. Comparison of (9) and (10) shows that

$$\mathfrak{M}_+-i\mathfrak{K}K_L^2\mathfrak{M}_+=-2i\mathfrak{K}.\quad(11)$$

We differentiate this equation with respect to the momentum $k_{2\alpha}$ of the three-body channel α and then set $k_{2\alpha}=0$. In this process we keep the total energy fixed but set $k_{1\beta}=k_{1\alpha}$, the β channel being another, lighter, three-body channel open at that energy and with the same particle 1. This corresponds to the observation of particles 1 from both channels α and β. Thus $k_{1\alpha}$, $k_{1\beta}$, and $k_{2\beta}$ are all (even) functions of $k_{2\alpha}$. Other channel momenta being quite independent of $k_{2\alpha}$, we get an effect only if we take elements of (11) whose final states are in the α or β channels. The initial channel is a two-particle channel for obvious experimental reasons. Now it is shown in the Appendix that the elements of $\mathfrak{K}$ are even functions of all open-channel momenta and that they are in general finite at thresholds. Differentiation of (11) with respect to $k_{2\alpha}$ at $k_{2\alpha}=0$ thus would yield

$$(\partial/\partial k_{2\alpha})\mathfrak{M}_+=0,$$

except for the possibility that $\mathfrak{K}$ contains a term proportional to $\delta(E_1-E_1')$. For the differentiation then acts on the corresponding element of $\mathfrak{M}_+$ on the right-hand side of (11). Such a δ-function term is indeed contained in $\mathfrak{K}$, since the interaction Hamiltonian H_I undoubtedly includes two-body potentials for particles 2 and 3 in the α and β channels. These potentials are independent of $\mathbf{r}_1$ and one can easily see that consequently

$$\mathfrak{K}(E_1',E_2';E_1,E_2)=\delta(E_1-E_1')\delta_{\lambda\lambda'}\mathfrak{K}^{(0)}(E_2)+\cdots, \quad (12)$$

where $\mathfrak{K}^{(0)}$ is the matrix $\mathfrak{K}$ referring to the two-particle scatterings and reactions of particles 2 and 3 of channels α and β, in which particle 1 is entirely ignored. The remainder in (12) is free of such δ functions. Thus we obtain at $k_{2\alpha}=0$ from (11)

$$(1-i\mathfrak{K}^{(0)}K_L{}^2)(\partial/\partial k_{2\alpha})\mathfrak{M}_+=i\mathfrak{K}^{(0)}P^{(\alpha,0)}\mathfrak{M}_+, \quad (13)$$

where $P^{(\alpha,0)}$ is the projection on the $l_2=0$ part of the channel α. At the same time $\mathfrak{K}^{(0)}$ satisfies the two-particle equivalent of (11):

$$(1-i\mathfrak{K}^{(0)}K_L{}^2)\mathfrak{M}_+{}^{(0)}=-2i\mathfrak{K}^{(0)}. \quad (11')$$

We conclude therefore that at $k_{2\alpha}=0$

$$(\partial/\partial k_{2\alpha})\mathfrak{M}_+=-\tfrac{1}{2}\mathfrak{M}_+{}^{(0)}P^{(\alpha,0)}\mathfrak{M}_+ \quad (14)$$

for elements leading to the α or β channel. The meaning of this is that the part of the matrix element that refers to particle 1 is the same for $\mathfrak{M}_+$ on both sides of the equation, while the parts that refer to particles 2 and 3 in channel α or β are the matrix multiplied by $\mathfrak{M}_+{}^{(0)}$.

The argument proceeds as in reference 5. We infer from (14) that in the vicinity of the threshold $k_{2\alpha}=0$ the linear term in the Taylor expansion of an element of $\mathfrak{M}_+$ leading to the α or β channel is[17]

$$\Delta_\alpha\mathfrak{M}\equiv\mathfrak{M}-\mathfrak{M}|_{k_{2\alpha}=0}$$
$$=-\frac{1}{2}\binom{1}{i}|k_{2\alpha}|\mathfrak{M}^{(0)}P^{(\alpha,0)}\mathfrak{M}|_{k_{2\alpha}=0}, \quad (15)$$

where the "1" is used above the threshold (as a function of E_2) and the "i" below.

The threshold anomaly arises from the linear term in $k_{2\alpha}$, as it does in the two-particle case. We are interested in the behavior of the elements of S leading to the α and β channels as functions of $k_{1\alpha}=k_{1\beta}$, with the total energy E fixed. The momenta $k_{2\alpha}$ and $k_{2\beta}$ are then functions of $k_{1\alpha}$:

$$\frac{\hbar^2k_{2\alpha}{}^2}{2\mu_{2\alpha}}=E-\frac{\hbar^2k_{1\alpha}{}^2}{2\mu_{1\alpha}},$$

and

$$\lim_{k_{2\alpha}\to0}\frac{\partial|k_{2\alpha}|}{\partial k_{1\alpha}}=-\frac{k_{1\alpha}}{|k_{2\alpha}|}\frac{\mu_{2\alpha}}{\mu_{1\alpha}},$$

which is infinite. Therefore the leading term in the derivative of $\mathfrak{M}$ is

$$\frac{\partial}{\partial k_{1\alpha}}\mathfrak{M}\bigg|_{k_{2\alpha}\sim0}=\frac{\partial}{\partial k_{1\beta}}\mathfrak{M}\bigg|_{k_{2\alpha}\sim0}$$
$$=\frac{1}{2}\binom{i}{1}k_{1\alpha}|k_{2\alpha}|^{-1}\frac{\mu_{2\alpha}}{\mu_{1\alpha}}$$
$$\times\mathfrak{M}^{(0)}P^{(\alpha,0)}\mathfrak{M}|_{k_{2\alpha}=0}, \quad (16)$$

depending on whether the derivative is evaluated from above or from below the threshold as a function of E_1. (Recall that as a function of E_1 the α channel is open below the threshold and closed above.)

If we choose the direction of the incident beam as the z axis (i.e., axis of quantization) then we obtain from (6) and (15)

$$\Delta_\alpha\Theta_{fi}(\xi_1',\xi_2';\xi)$$
$$=\binom{-1}{i}|k_{2\alpha}|(\mu_{1\alpha}/\mu_1')^{\frac{1}{2}}\sum_{j_{2\alpha}}\Theta_{f\alpha}{}^{(2)}(\xi_2',\xi_{2\alpha})$$
$$\times\Theta_{\alpha i}(\xi_1',\xi_{2\alpha};\xi), \quad (17)$$

where the amplitudes on the right-hand side are evaluated at the threshold, $\Theta_{f\alpha}{}^{(2)}$ is the two-particle amplitude from the (2,3) part of the α channel to the (2,3) part of the final channel; the j sum runs over the total angular momenta of particles 2 and 3 in the α channel, i.e., their total spin; and the upper value, -1, is used where the α channel is open, the lower value, i, where it is closed.

Equation (17) immediately yields the linear term in the three-particle cross sections:

$$\Delta_\alpha\frac{d\sigma_{fi}(\xi_1',\xi_2';\xi)}{d^3k_1'd\Omega_2}$$
$$=-2|k_\alpha|(\mu_{1\alpha}/\mu_1')^{\frac{1}{2}}\binom{\mathrm{Re}}{\mathrm{Im}}\sum_{j_{2\alpha}}\Theta_{fi}{}^*(\xi_1',\xi_2';\xi)$$
$$\times\Theta_{f\alpha}{}^{(2)}(\xi_2',\xi_{2\alpha})\Theta_{\alpha i}(\xi_1',\xi_{2\alpha};\xi) \quad (18)$$

[17] We will drop hereafter the subscript "+" since we shall be concerned with the open-channel part of $\mathfrak{M}$ only.

provided that the reference axis of the spin projections is the incident beam direction.

We may integrate the cross sections over angles to obtain the *spectrum*:

$$d\sigma_{fi}=\frac{k_1'^2dk_1'}{2s+1}\sum_{\nu\nu_1'\nu_2'}\int|\Theta_{fi}(\xi_1',\xi_2';\xi)|^2d\Omega_1d\Omega_2, \quad (19)$$

or

$$\frac{d\sigma_{fi}(s_1',s_2';s)}{dE_1'}=\frac{\pi m_1'}{\mu_1'}\sum_{Jl_1l_2l}\frac{2J+1}{2s+1}k_1'^{2l_1+1}k_2'^{2l_2+1}k^{2l-1}\times|\mathfrak{M}_{f\lambda',i\lambda}{}^J|^2. \quad (19')$$

Equation (15) then leads to the following linear term in the spectrum:

$$\Delta_\alpha\frac{d\sigma_{fi}}{dE_1'}=\frac{\pi m_1'}{2\mu_1'}\sum_{Jl_1l_2l}\frac{2J+1}{2s+1}k_1'^{2l_1+1}k_2'^{2l_2+1}k^{2l-1}|k_{2\alpha}|\times\begin{pmatrix}\mathrm{Im}\\-\mathrm{Re}\end{pmatrix}[\mathfrak{M}_{f\lambda',i\lambda}{}^{J*}\mathfrak{M}_{f2\lambda_2',\alpha20s_\alpha}{}^{(2)s_\alpha}\mathfrak{M}_{\alpha\lambda_1'0s_\alpha,i\lambda}{}^J] \quad (20)$$

E_1' here is the energy of particle 1:

$$E_1'=\hbar^2k_1'^2/2m_1'.$$

The essential feature of the result (18) and (20) is that the cusp size depends on the three-particle cross section to the threshold channel as well as on the two-particle cross section from the threshold to the final channel in which particle 1 is entirely ignored. Ordinary selection rules thus tell us immediately what kinds of reactions can or cannot exhibit the infinite derivative. For example, in the case of reaction (a) at the threshold for reaction (b), as given in the Introduction, in which the incident energy is kept fixed and the energy of the pion observed is varied, the two-particle matrix element involved is that of Λ-proton collisions leading to Σ production at threshold. If the relative $\Lambda-\Sigma$ parity is even, this matrix element is confined to the S wave; if it is odd, to the P wave. Hence the threshold anomaly as a function of the pion energy comes from the S wave of the Λ-proton system if the $\Lambda-\Sigma$ parity is even, from the P wave if it is odd.

From the point of view of experimental application the above-described case is of limited significance. Since the Σ-proton threshold in the center-of-mass system lies at about 80 Mev for the Λ-proton system, the relative momentum is high enough so that the P wave cannot be expected to be small compared to the S wave. Hence the experimental determination of whether the cusp (or rounded step) occurs in the S or P wave would have to rely on the angular distribution of the Λ and proton, which is very difficult. On the other hand, the same system could be analyzed using the proton as particle 1 instead of the pion. One would then observe the cusp as a function of the proton energy and it would come from the $\Lambda-\pi$ S wave or P wave depending on the relative $\Lambda-\Sigma$ parity. Since the reduced mass of the $\Lambda-\pi$ system is much smaller than that of the $\Lambda-p$ system, the relative momentum at the Σ threshold is much smaller and one may expect the P wave to be small compared to the S wave. In that case the observation could rely on the cusp in the spectrum. If details could be measured, one could also obtain interesting information on the Σ production cross section by $\Lambda-\pi$ collisions.

APPENDIX

We first want to derive the threshold k-dependence of the S-matrix as stated in (9). For the case of two-body–two-body transitions this is well known and need not be rederived here. For the case in which either or both sides refer to three-body channels the dependence on the over-all energy and that on the individual momenta k_1 and k_2 has been given by Delves.[9] Our derivation is somewhat different. The proof we give is heuristic and holds provided the interaction has the requisite properties, e.g., if it vanishes beyond finite values of r_1 and r_2.

Comparison of (5) and (6) shows that

$$\sum Y_J{}^{M*}(\lambda,\xi)Y_J{}^M(\lambda',\xi_1,\xi_2)S_{f\lambda',i\lambda}{}^J(E_1',E_2';E)=-2\pi i(\phi_0{}^{(f)}|H_I|\psi_+{}^{(i)}) \quad (A.1)$$

for the case of two- to three-particle transitions. We expand the free three-body wave function $\phi_0{}^{(f)}$:

$$\phi_0{}^{(f)}(\xi_1,\xi_2;\mathbf{r}_1,\mathbf{r}_2)=(2/\pi\hbar^2)(\mu_1k_1\mu_2k_2)^{\frac{1}{2}}\sum j_{l_1}(k_1r_1)j_{l_2}(k_2r_2)\times Y_J{}^{M*}(\lambda,\xi_1,\xi_2)\mathcal{Y}_{J\lambda}{}^M(\mathbf{r}_1,\mathbf{r}_2), \quad (A.2)$$

where

$$\mathcal{Y}_{J\lambda}{}^M(\mathbf{r}_1,\mathbf{r}_2)=\sum C_{j_1j_2}(J,M;M_1,M_2)\times\mathcal{Y}_{j_1\lambda_1}{}^{M_1}(\mathbf{r}_1)\mathcal{Y}_{j_2\lambda_2}{}^{M_2}(\mathbf{r}_2),$$

$$\mathcal{Y}_{j\lambda}{}^M(\mathbf{r})=\sum_{m\nu}C_{ls}(j,M;m,\nu)Y_l{}^m(\mathbf{r})\chi_s{}^\nu i^l.$$

Consequently we obtain

$$\sum S_{f\lambda',i\lambda}{}^J(E_1',E_2';E)Y_J{}^{M*}(\lambda,\xi)=(2/\pi i\hbar^3)(\mu_1'k_1'\mu_2'k_2'k)^{\frac{1}{2}}\int(d\mathbf{r}_1)(d\mathbf{r}_2)j_{l_1}(k_1'r_1)\times j_{l_2}(k_2'r_2)\mathcal{Y}_{J\lambda'}{}^M(\mathbf{r}_1,\mathbf{r}_2)(\mathbf{r}_1,\mathbf{r}_2|H_I|\psi_+{}^{(i)}), \quad (A.3)$$

in which the dependence on k_1' and k_2' is explicitly visible. It follows immediately from the behavior of the spherical Bessel functions that, provided the r_1 and r_2 integrations converge,

$$S_{f\lambda',i\lambda}{}^J(E_1',E_2';E)=O(k_1'^{l_1+\frac{1}{2}}k_2'^{l_2+\frac{1}{2}}), \quad (A.4)$$

as either $k_1'\to 0$ or $k_2'\to 0$. Clearly nothing is changed in this argument if the initial channel is a three-body channel. All that remains is to recall the symmetry of the S-matrix and we arrive at (9).

We now write the S-matrix in terms of the $\mathfrak{K}'$ matrix defined by

$$\mathfrak{K}' \equiv -\tfrac{1}{2}(\phi_0^{(f)}|H_I|\psi_P^{(i)}), \qquad (A.5)$$

where

$$\psi_P^{(i)} = \phi_0^{(i)} + G_P H_I \psi_P^{(i)},$$

and G_P is the "principal value Green's function"

$$G_P = \mathrm{Re}G_+.$$

We then have the Heitler integral equation,

$$(i+\mathfrak{K}')S = i - \mathfrak{K}',$$

where the matrix multiplication includes summation over discrete channels as well as integration over angles and energy distribution for three (or more) particle channels. Angular momentum analysis leads to

$$\sum Y_J^{M*}(\lambda,\xi) Y_J^M(\lambda',\xi') \mathfrak{K}_{f\lambda',i\lambda}{}'^J(E) = -\tfrac{1}{2}(\phi_0^{(f)}(\xi')|H_I|\psi_P^{(i)}(\xi)) \qquad (A.6)$$

for two-particle matrix elements, and analogous equations for three-particle elements.

We want to show that once the threshold energy dependence $k^{l+\frac{1}{2}}$ is taken out of each element, the remaining matrix is an even function of all real k's. Since the proof of this fact for discrete channels was given in reference 5 using the special methods of reference 4, we give a more direct proof here for two-particle channels which is directly generalizable to three-particle channels.

We make an angular momentum expansion of ψ in terms of radial functions:

$$\psi_{\alpha\beta}(\xi,\mathbf{r}) = \left(\frac{2\mu_\beta k_\beta}{\pi\hbar^2}\right)^{\frac{1}{2}} \sum \mathcal{Y}_{J\lambda'}{}^M(\mathbf{r}) \times \psi_{\alpha\lambda',\beta\lambda}{}^J(K,r) Y_J{}^{M*}(\lambda,\xi) k_\beta{}^l \qquad (A.7)$$

for the two-particle case. The radial "principal value wave function" then satisfies the integral equation:

$$\psi_{\alpha\lambda',\beta\lambda}{}^{(P)J}(K,r) = \delta_{\alpha\beta} j_l(k_\beta r) k_\beta{}^{-l} - \sum_{\gamma\lambda''} \int_0^\infty dr'\, r'^2 k_\alpha j_l(k_\alpha r_<) n_l(k_\alpha r_>) \times H_{\alpha\lambda',\gamma\lambda''}{}^{(I)} \psi_{\gamma\lambda'',\beta\lambda}{}^{(P)J}(K,r'). \qquad (A.8)$$

Since $j_l(kr)k^{-l}$ and $kj_l(kr)n_l(kr')$ are both even functions of k, it follows that $\psi_{\alpha\lambda',\beta\lambda}{}^{(P)J}$ is an even function of all open-channel wave numbers. (The k's of closed channels must not change sign; they must remain in the *upper* half plane.) A moment's reflection shows that the same argument applies also to a three-particle wave function.

If the angular momentum expansions of ϕ_0 and $\psi^{(P)}$ are inserted in (A.6) we are able to conclude the desired result. The matrix function,

$$\mathfrak{K} \equiv K_L{}^{-1} \mathfrak{K}' K_L{}^{-1},$$

in the angular momentum representation is an even function of all open-channel momenta.

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