Approximate Relativistic Coulomb Scattering Wave Function*

W. R. JOHNSON AND C. J. MULLIN
University of Notre Dame, Notre Dame, Indiana

(Received April 4, 1960)

A relativistic scattering wave function, which is valid to second order in αZ for all electron velocities, is developed. The wave function has as its dominant term the Sommerfeld-Maue function. The computational simplicity of the closed form Sommerfeld-Maue wave function is retained in this modified Sommerfeld-Maue function.

Applications of the wave function to the calculation of cross sections for the scattering of polarized electrons and bremsstrahlung production at the short wave limit of the spectrum are given. A large first-order correction to the Sauter-Fano formula for the bremsstrahlung cross section is obtained.

I. INTRODUCTION

CALCULATIONS of various electrodynamic processes, such as electron scattering and bremsstrahlung production, require the use of a continuum Coulomb wave function. Because of the complexity of the exact series solution of the Dirac equation, an approximate solution is desirable. One such approximate wave function is provided by the truncated Born series. As is well known, the Born series solution for a pure Coulomb field diverges. This difficulty is overcome by replacing the Coulomb potential by a screened Coulomb potential and allowing the screening to vanish at the end of the computation.[1] The criterion for the convergence of the resulting series is $\alpha Z/\beta<1$. For problems such as the computation of the bremsstrahlung intensity at the short wave limit of the spectrum, the velocity of the outgoing electron is zero, and the Born series cannot be used. On the other hand, the Sommerfeld-Maue wave function provides a closed form solution to the Dirac equation which is valid to order αZ for all values of β.

We shall give a modification of the Sommerfeld-Maue wave function accurate to order α^2Z^2. For $\beta\cong1$, this function is equivalent to three terms in the Born series; it has the obvious advantage, however, that it remains valid for low velocities. In practice, this wave function retains the computational advantages of the closed form Sommerfeld-Maue wave function.

Some examples illustrating the relative simplicity of computations utilizing this wave function are given in Secs. III and IV.

II. MODIFIED SOMMERFELD-MAUE WAVE FUNCTION

An approximate Coulomb wave function which behaves asymptotically like a plane wave plus an outgoing spherical wave is given by[2–4]

$$\psi_{\mathrm{SM}}=Ne^{i\mathbf{p}\cdot\mathbf{r}}[1-(i/2W)\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}]\times{}_1F_1(i\alpha Z/\beta;1;ipr-i\mathbf{p}\cdot\mathbf{r})u(\mathbf{p}),\quad(1)$$

where

$$N=\Gamma(1-i\alpha Z/\beta)\exp(\pi\alpha Z/2\beta),$$

where p and W denote the electron momentum and energy, respectively, and ${}_1F_1(a;b;x)$ denotes the confluent hypergeometric function. This wave function differs from the exact Coulomb wave function[3] by terms of order α^2Z^2/r. It is the purpose of this paper to modify the Sommerfeld-Maue solution to include terms of order α^2Z^2.

Setting $\psi=\psi_{\mathrm{SM}}+\psi_C$, the Dirac equation gives

$$(H-W)\psi_C=-R,\quad(2)$$

where

$$R=(H-W)\psi_{\mathrm{SM}}=Ne^{i\mathbf{p}\cdot\mathbf{r}}(i\alpha Z/2Wr)\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}Fu(\mathbf{p}),$$

with $F={}_1F_1(i\alpha Z/\beta;1;ipr-i\mathbf{p}\cdot\mathbf{r})$. Since the asymptotic form of ψ_{SM} gives an exact description of the distorted plane incident wave, the α^2Z^2 corrections are required only in the particular solution of the inhomogeneous equation (2). The solution to Eq. (2), subject to the boundary condition that ψ_C represent a spherical outgoing wave, is given by

$$\psi_C(\mathbf{r})=-\int G(\mathbf{r},\mathbf{r}')R(\mathbf{r}')d\mathbf{r}',\quad(3)$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function for the operator $(H-W)$. Since we are interested in the α^2Z^2 contribution to ψ_C, we replace $G(\mathbf{r},\mathbf{r}')$ by $G_0(\mathbf{r},\mathbf{r}')$, the Green's function for the force-free Dirac equation. Thus

$$\psi(\mathbf{r})=\psi_{\mathrm{SM}}(\mathbf{r})-\int G_0(\mathbf{r},\mathbf{r}')R(\mathbf{r}')d\mathbf{r}'.\quad(4)$$

To evaluate $\psi(\mathbf{r})$ explicitly, it is convenient to introduce the Fourier transforms:

$$\phi(\mathbf{k})=\frac{1}{(2\pi)^3}\int\psi(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r};$$
$$G_0(\mathbf{k},\mathbf{r}')=\frac{1}{(2\pi)^3}\int G_0(\mathbf{r},\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}.\quad(5)$$

---

* This work was supported in part by the U. S. Atomic Energy Commission.

[1] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[2] A. Sommerfeld and A. W. Maue, Ann. Physik **22**, 629 (1935).
[3] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).
[4] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg und Sohn, Braunschweig, 1951), 2nd ed., Vol. 2, p. 408.

Then $\phi(\mathbf{k})=\phi_{SM}(\mathbf{k})+\phi_C(\mathbf{k})$, where

$$\phi_C(\mathbf{k})=-\int G_0(\mathbf{k},\mathbf{r})R(\mathbf{r})d\mathbf{r}. \tag{6}$$

Using the fact that

$$G_0(\mathbf{k},\mathbf{r})=\frac{1}{(2\pi)^3}\frac{H_0(\mathbf{k})+W}{k^2-p^2-i\delta}e^{-i\mathbf{k}\cdot\mathbf{r}},$$

where $H_0=\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m$, and noting that

$$\nabla f(pr-\mathbf{p}\cdot\mathbf{r})=-(p/r)\nabla_p f(pr-\mathbf{p}\cdot\mathbf{r}),$$

we find to lowest order in αZ:

$$R(\mathbf{r})=N\frac{\alpha^2Z^2}{2r^2}e^{i\mathbf{p}\cdot\mathbf{r}}\boldsymbol{\alpha}\cdot\nabla_p\lim_{a\to0}\frac{d}{da}\,{}_1F_1(a;1;ipr-i\mathbf{p}\cdot\mathbf{r})u(\mathbf{p}). \tag{7}$$

In this and subsequent expressions it is to be understood that ∇_p does not operate on the spin functions $u(\mathbf{p})$. Thus we obtain:

$$\phi_C(\mathbf{k})=-N\frac{\alpha^2Z^2}{16\pi^3}\frac{H_0(\mathbf{k})+W}{k^2-p^2-i\delta}\lim_{\mu\to0}\int_\mu^\infty\boldsymbol{\alpha}\cdot\nabla_p\lim_{a\to0}\frac{d}{da}\times I(\mu,a)d\mu u(\mathbf{p}), \tag{8}$$

where

$$I(\mu,a)=\int\frac{e^{i\mathbf{q}\cdot\mathbf{k}-\mu r}}{r}\,{}_1F_1(a;1;ipr-i\mathbf{p}\cdot\mathbf{r})d\mathbf{r}=\frac{4\pi}{q^2+\mu^2}\left[\frac{q^2+\mu^2}{q^2-2\mathbf{p}\cdot\mathbf{q}-2ip\mu+\mu^2}\right]^a, \tag{9}$$

with $\mathbf{q}=\mathbf{p}-\mathbf{k}$. Using Eq. (9) we may write,

$$\phi_C(\mathbf{k})=-N\frac{\alpha^2Z^2}{2\pi^2}\frac{H_0(\mathbf{k})+W}{k^2-p^2-i\delta}\lim_{\mu\to0}J(\mu)u(\mathbf{p}), \tag{10}$$

where

$$J(\mu)=\int_\mu^\infty\frac{(i\boldsymbol{\alpha}\cdot\mathbf{p}\mu+p\boldsymbol{\alpha}\cdot\mathbf{q})d\mu}{p(q^2+\mu^2)(q^2-2\mathbf{p}\cdot\mathbf{q}-2ip\mu+\mu^2)}. \tag{11}$$

We compute this elementary integral and use the relations

$$(H_0(\mathbf{k})+W)\boldsymbol{\alpha}\cdot\mathbf{p}u(\mathbf{p})=(\boldsymbol{\alpha}\cdot\mathbf{k}\boldsymbol{\alpha}\cdot\mathbf{p}+p^2)u(\mathbf{p}),$$
$$(H_0(\mathbf{k})+W)\boldsymbol{\alpha}\cdot\mathbf{q}u(\mathbf{p})=(p^2-k^2)u(\mathbf{p}),$$

to find

$$\phi_C(\mathbf{k})=N\frac{\alpha^2Z^2}{8\pi^2(k^2-p^2-i\delta)k(\mathbf{p}\times\mathbf{q})^2}\times\lim_{\mu\to0}\Bigg[\frac{1}{p}(\boldsymbol{\alpha}\cdot\mathbf{k}\boldsymbol{\alpha}\cdot\mathbf{p}+p^2)\times\left(\pi pqk+i\mathbf{p}\cdot\mathbf{q}k\ln\frac{q^2}{k^2-p^2-2ip\mu}+ip(\mathbf{p}\cdot\mathbf{q}-q^2)\ln\frac{p+k}{p-k+i\mu}\right)+\frac{k^2-p^2}{q}\times\left(\pi\mathbf{p}\cdot\mathbf{q}k+ikpq\ln\frac{q^2}{k^2-p^2-2ip\mu}+iq(p^2-\mathbf{p}\cdot\mathbf{q})\ln\frac{p+k}{p-k+i\mu}\right)\Bigg]u(\mathbf{p}). \tag{12}$$

We have retained μ in $\phi_C(\mathbf{k})$ in order to facilitate the choice of phase of the ln terms in the applications given below.

The wave function which behaves asymptotically like a plane wave with an incoming spherical wave can be obtained similarly. Writing $\psi=\psi_{SM}+\psi_C$, and denoting the adjoint of ψ and $\psi^\dagger$, we find:

$$\psi_{SM}{}^\dagger=N^*u^\dagger(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{r}}[1+(i/2W)\boldsymbol{\alpha}\cdot\nabla]\times{}_1F_1(i\alpha Z/\beta;1;ipr+i\mathbf{p}\cdot\mathbf{r}), \tag{13}$$

and

$$\psi_C{}^\dagger=\int d\mathbf{k}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\phi_C{}^\dagger(\mathbf{k}), \tag{14}$$

where

$$\phi_C{}^\dagger(\mathbf{k})=N^*u^\dagger(\mathbf{p})\frac{\alpha^2Z^2}{8\pi^2(k^2-p^2-i\delta)k(\mathbf{p}\times\mathbf{q})^2}\times\lim_{\mu\to0}\Bigg[\frac{1}{p}(\boldsymbol{\alpha}\cdot\mathbf{p}\boldsymbol{\alpha}\cdot\mathbf{k}+p^2)\times\left(\pi pqk+i\mathbf{p}\cdot\mathbf{q}k\ln\frac{q^2}{k^2-p^2-2ip\mu}+ip(\mathbf{p}\cdot\mathbf{q}-q^2)\ln\frac{p+k}{p-k+i\mu}\right)+\frac{k^2-p^2}{q}\times\left(\pi\mathbf{p}\cdot\mathbf{q}k+ikpq\ln\frac{q^2}{k^2-p^2-2ip\mu}+iq(p^2-\mathbf{p}\cdot\mathbf{q})\ln\frac{p+k}{p-k+i\mu}\right)\Bigg]. \tag{15}$$

Equations (12) and (15) give the Fourier transforms of the α^2Z^2 corrections to the Sommerfeld-Maue wave functions.

III. APPLICATION TO COULOMB SCATTERING OF POLARIZED ELECTRONS

Using the modified Sommerfeld-Maue wave function we can find the differential cross section for elastic scattering of polarized electrons. For large r, the Coulomb scattering wave function has the form

$$\psi(\mathbf{r})=\exp[i\mathbf{p}\cdot\mathbf{r}-i(\alpha Z/\beta)\ln 2pr\sin^2(\theta/2)]u(\mathbf{p})$$
$$+f(\theta)u(\mathbf{p})\frac{\exp[ipr-i(\alpha Z/\beta)\ln 2pr]}{r}. \quad (16)$$

The differential cross section for scattering of electrons with spin orientations specified by $\boldsymbol{\zeta}$ (in the rest system of the electron) is given by

$$d\sigma=(1/4W)\,\mathrm{Tr}[f^\dagger(\theta)f(\theta)(1-iS_\mu\gamma_\mu\gamma_5)$$
$$\times(H_0(\mathbf{p})+W)]d\Omega, \quad (17)$$

where S_μ is the 4-vector with components

$$S=\left(\boldsymbol{\zeta}+\frac{\mathbf{p}\cdot\boldsymbol{\zeta}}{m(W+m)}\mathbf{p},\, i\frac{\mathbf{p}\cdot\boldsymbol{\zeta}}{m}\right)$$

Using the modified Sommerfeld-Maue wave function, we can write $f(\theta)=f_{\mathrm{SM}}(\theta)+f_C(\theta)$ where f_{SM} is the contribution from the asymptotic form of the Sommerfeld-Maue wave function and f_C is the contribution from the correction term. We use the asymptotic form of the confluent hypergeometric function to find

$$f_{\mathrm{SM}}(\theta)=\frac{\Gamma(1-i\alpha Z/\beta)}{\Gamma(1+i\alpha Z/\beta)}\frac{\alpha Z}{4p^2\sin^2(\theta/2)}$$
$$\times[H_0(\mathbf{p}')+W]\exp\left(i\frac{\alpha Z}{\beta}\ln\sin^2(\theta/2)\right), \quad (18)$$

where $\mathbf{p}'=p\mathbf{r}/r$.

In order to find $f_C(\theta)$ we replace the Green's function $G_0(\mathbf{r},\mathbf{r}')$ by its asymptotic form

$$G_0(\mathbf{r},\mathbf{r}')\underset{r\to\infty}{\longrightarrow}-\frac{1}{4\pi}[H_0(\mathbf{p})+W]e^{i\mathbf{p}'\cdot\mathbf{r}'}\frac{e^{ipr}}{r}. \quad (19)$$

Thus we obtain:

$$\psi_C\underset{r\to\infty}{\longrightarrow}-\frac{1}{4\pi}[H_0(\mathbf{p}')+W]\frac{e^{ipr}}{r}\int e^{-i\mathbf{p}'\cdot\mathbf{r}'}R(\mathbf{r}')d\mathbf{r}'. \quad (20)$$

From Eqs. (20) and (6) we find

$$f_C(\theta)=2\pi^2\lim_{\mathbf{k}\to\mathbf{p}'}(k^2-p^2)\phi_C(\mathbf{k}). \quad (21)$$

Consequently from Eq. (12)

$$f_C(\theta)=\frac{\alpha^2Z^2}{8p^3\cos^2(\theta/2)}(\boldsymbol{\alpha}\cdot\mathbf{p}'\boldsymbol{\alpha}\cdot\mathbf{p}+p^2)$$
$$\times\{\pi[\csc(\theta/2)-1]+i\ln\sin^2(\theta/2)\}. \quad (22)$$

We write $d\sigma=d\sigma_{\mathrm{SM}}+d\sigma_C$, where $d\sigma_{\mathrm{SM}}$ is the contribution from f_{SM} and $d\sigma_C$ is the αZ correction and find:

$$d\sigma_{\mathrm{SM}}=\frac{\alpha^2Z^2}{4p^2\beta^2\sin^4(\theta/2)}[1-\beta^2\sin^2(\theta/2)]d\Omega,$$

$$d\sigma_C=\frac{\alpha^3Z^3}{4p^2\beta^2\sin^4(\theta/2)}\{\pi\beta\sin(\theta/2)[1-\sin(\theta/2)] \quad (23)$$
$$+\beta(1-\beta^2)^{\frac{1}{2}}\tan(\theta/2)\sin^2(\theta/2)$$
$$\times\ln\sin^2(\theta/2)\mathbf{n}\cdot\boldsymbol{\zeta}\}d\Omega,$$

where $\mathbf{n}=(\mathbf{p}\times\mathbf{p}')/|\mathbf{p}\times\mathbf{p}'|$ is the unit normal to the plane of scattering. This result is in agreement with that obtained by use of an expansion of the exact Coulomb wave function.[5] The well-known McKinley-Feshbach[6] cross section for scattering of unpolarized electrons follows immediately from Eq. (23) upon averaging over spin orientations.

IV. APPLICATION TO THE SHORT WAVELENGTH LIMIT OF THE BREMSSTRAHLUNG SPECTRUM

Recent calculations of the tip of the bremsstrahlung spectrum using an exact wave function for the outgoing electron and the Born approximation for the incident electron show that to lowest order in αZ the cross section does not vanish as might be expected from the Bethe-Heitler formula but remains finite.[7] Using the corrected Sommerfeld-Maue wave function we can easily obtain the αZ correction to the formula given by Fano for the intensity at the tip of the spectrum. In order to obtain results which are mathematically simple, we restrict ourselves to the special case $W_1\gg m$.

The bremsstrahlung cross section is given by

$$d\sigma=\frac{\alpha}{(2\pi)^4}\frac{W_1}{p_1}p_2W_2kdkd\Omega_kd\Omega_p|N_2|^2|N_1|^2$$
$$\times\tfrac{1}{2}\sum|M_{21}|^2, \quad (24)$$

where the subscripts 1 and 2 refer to the initial and final states, respectively. The sum is over electron spins and photon polarizations.

The matrix element occurring in Eq. (24) is given by

$$M_{21}=\int\psi_2^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_1d\mathbf{r}, \quad (25)$$

and the common normalization factors have been removed from the matrix element. The incident

[5] F. Gürsey, Phys. Rev. **107**, 1734 (1957); L. J. Tassie, Phys. Rev. **107**, 1452 (1957).

[6] W. A. McKinley and H. Feshbach, Phys. Rev. **74**, 1759 (1948).

[7] U. Fano, Phys. Rev. **116**, 1156 (1959). The bremsstrahlung cross section at the tip of the spectrum was previously obtained by Nagasaka [F. Nagasaka, thesis, University of Notre Dame, 1955 (unpublished)], using the Sommerfeld-Maue wave function for both incident and final electron states. A similar calculation has been made by Mihailovic [M. V. Mihailovic, Nuovo cimento **9**, 331 (1958)].

electron state is represented by a plane wave plus an outgoing spherical wave while the outgoing electron state is represented by a plane wave plus an incoming spherical wave. Writing $\psi_{\rm SM}=\psi_a+\psi_b$ the corrected Sommerfeld-Maue wave function becomes $\psi=\psi_a+\psi_b+\psi_C$, where:

$$\psi_{1a}=e^{i\mathbf{p}_1\cdot\mathbf{r}}\,{}_1F_1(i\alpha Z/\beta_1;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})u(\mathbf{p}_1),$$
$$\psi_{1b}=-(i/2W_1)e^{i\mathbf{p}_1\cdot\mathbf{r}}\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\,{}_1F_1(i\alpha Z/\beta_1;1;ip_1r-i\mathbf{p}_1\cdot\mathbf{r})u(\mathbf{p}_1), \quad (26)$$
$$\psi_{1C}=\int e^{i\mathbf{k}\cdot\mathbf{r}}\phi_{1C}(\mathbf{k})d\mathbf{k},$$

$$\psi_{2a}^\dagger=u^\dagger(\mathbf{p}_2)e^{-i\mathbf{p}_2\cdot\mathbf{r}}\,{}_1F_1(i\alpha Z/\beta_2;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r}),$$
$$\psi_{2b}^\dagger=(i/2W_2)u^\dagger(\mathbf{p}_2)e^{-\mathbf{p}_2\cdot\mathbf{r}}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\,{}_1F_1(i\alpha Z/\beta_2;1;ip_2r+i\mathbf{p}_2\cdot\mathbf{r}), \quad (27)$$
$$\psi_{2C}^\dagger=\int e^{-i\mathbf{k}\cdot\mathbf{r}}\phi_{2C}^\dagger(\mathbf{k})d\mathbf{k}.$$

Equations (12) and (15) are to be used for evaluating $\phi_{1C}(\mathbf{k})$ and $\phi_{2C}^\dagger(\mathbf{k})$, respectively.

We write $M_{21}=M_{21}^{\rm BM}+M_{21}^{(1)}$, where $M_{21}^{\rm BM}$ is given by

$$M_{21}^{\rm BM}=\int\psi_{2a}^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1a}d\mathbf{r}+\int\psi_{2b}^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1a}d\mathbf{r}+\int\psi_{1a}^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1b}d\mathbf{r}. \quad (28)$$

This is the matrix element used in the theory of bremmstrahlung by Bethe and Maximon.[3] The correction term is given by

$$M_{21}^{(1)}=\int\psi_{2b}^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1b}d\mathbf{r}+\int\psi_{2C}^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1a}d\mathbf{r}+\int\psi_{2a}^\dagger\boldsymbol{\alpha}\cdot\mathbf{e}e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1C}d\mathbf{r}. \quad (29)$$

$M_{21}^{\rm BM}$ contains both first- and second-order αZ contributions while $M_{21}^{(1)}$ contains second and higher order terms. We may consider, however, only terms through second order in $M_{21}^{\rm BM}$ and only the second-order terms in $M_{21}^{(1)}$.

Writing the differential cross section as the sum of two terms $d\sigma=d\sigma_{\rm BM}+d\sigma_C$, we find

$$d\sigma_{\rm BM}=[\alpha/(2\pi)^4](W_1/p_1)p_2W_2kdkd\Omega_kd\Omega_p\times|N_1|^2|N_2|^2\tfrac{1}{2}\sum|M_{21}^{\rm BM}|^2,$$

and

$$d\sigma_C=[\alpha/(2\pi)^4](W_1/p_1)p_2W_2kdkd\Omega_kd\Omega_p\times|N_1|^2|N_2|^2\sum{\rm Re}M_{21}^{(0)*}M_{21}^{(1)}, \quad (30)$$

where $M_{21}^{(0)}$ is the leading term in the expansion of $M_{21}^{\rm BM}$ in a series in αZ. From the work of Bethe and Maximon[3] we infer that

$$\tfrac{1}{2}\sum|M_{21}^{\rm BM}|^2\xrightarrow[p_2\to 0]{}\frac{16\pi^2\alpha^2Z^2}{mW_1q^4}p_1^2\times\sin^2\theta\left[\frac{4}{q^4}\left(m^2-\frac{q^2}{4}\right)+\frac{k}{mq^2}\right](1-2\pi\alpha Z/\beta_1), \quad (31)$$

where $\mathbf{q}=\mathbf{p}_1-\mathbf{k}$ and $\cos\theta=\mathbf{p}_1\cdot\mathbf{k}/p_1k$. Here we have retained terms to order α^3Z^3 only.

In order to compute the sum occurring in $d\sigma_C$, we use

$$M_{21}^{(0)}=u^\dagger(\mathbf{p}_2)\frac{4\pi\alpha Z}{q^2}\left[\boldsymbol{\alpha}\cdot\mathbf{e}\left(\frac{2m}{q^2}-\frac{1}{m}\right)+\frac{\boldsymbol{\alpha}\cdot\mathbf{e}\boldsymbol{\alpha}\cdot\mathbf{q}}{2mk}+\frac{\boldsymbol{\alpha}\cdot\mathbf{q}\boldsymbol{\alpha}\cdot\mathbf{e}}{q^2}\right]u(\mathbf{p}_1), \quad (32)$$

which is formed by expanding $M_{21}^{\rm BM}$ for $p_2=0$ in a series in αZ. It follows from Eq. (29) that

$$M_{21}^{(1)}=u^\dagger(\mathbf{p}_2)\frac{\pi^2\alpha^2Z^2}{2}\left\{\boldsymbol{\alpha}\cdot\mathbf{e}\frac{1}{mk}\left[-\frac{2}{q}+\frac{1}{p_1}\frac{\boldsymbol{\alpha}\cdot\mathbf{q}\boldsymbol{\alpha}\cdot\mathbf{p}_1+p_1q}{\mathbf{p}_1\cdot\mathbf{q}+p_1q}\right]+\boldsymbol{\alpha}\cdot\mathbf{e}\frac{2}{q^3}+\frac{\boldsymbol{\alpha}\cdot(q\mathbf{p}_1+\mathbf{q}p_1)\boldsymbol{\alpha}\cdot\mathbf{e}\boldsymbol{\alpha}\cdot(q\mathbf{p}_1+\mathbf{q}p_1)}{p_1q^2(\mathbf{p}_1\cdot\mathbf{q}+p_1q)^2}+\frac{\boldsymbol{\alpha}\cdot\mathbf{e}q^2+\boldsymbol{\alpha}\cdot\mathbf{q}\boldsymbol{\alpha}\cdot\mathbf{e}\boldsymbol{\alpha}\cdot\mathbf{q}}{q^3(\mathbf{p}_1\cdot\mathbf{q}+p_1q)}\right\}u(\mathbf{p}_1). \quad (33)$$

We expand the normalization factors in a series in αZ for $p_2\to 0$ to find

$$|N_1|^2|N_2|^2=2\pi\alpha Z(m/p_2)(1+\pi\alpha Z/\beta_1). \quad (34)$$

Using the results of Eqs. (31) and (34) and integrating over electron and photon angles, we find for $W_1\gg m$

$$d\sigma_{\rm BM}=4\pi\alpha^2Z^3r_0^2(1-\pi\alpha Z)dk/k, \quad (35)$$

with $r_0=e^2/m$. The results of Eqs. (32), (33), and (34) are similarly combined to give for $W_1\gg m$

$$d\sigma_C=-(16/15)\pi\alpha^3Z^4r_0^2dk/k. \quad (36)$$

From Eqs. (35) and (36) we find that the bremsstrahlung cross section at the tip of the spectrum is

$$d\sigma=4\pi\alpha^2Z^2r_0^2\left(1-\frac{19}{15}\pi\alpha Z\right)\frac{dk}{k}. \quad (37)$$

The leading term in Eq. (37) agrees with the result of previous calculations.[7] Because of the large coefficient of αZ in the correction, this term will be important even for light elements.