Calculation of α-Transition Probabilities*

HANS J. MANG†
Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received February 15, 1960)

The decay rates for the ground-state transitions of all polonium isotopes and the odd-even astatine isotopes are discussed on the basis of the nuclear shell model. Good agreement with experimental data is obtained. In particular the behavior of the reduced width as a function of the neutron number around the magic number $N=126$ is well reproduced.

INTRODUCTION

IN an earlier paper[1] a theory of alpha decay was developed whose aim was to take into account the influence of nuclear structure on alpha decay. The theory was found successful in explaining the fine structure of the Po^{211} alpha decay. The relative intensities of transitions leading to various states in the daughter nucleus and the coefficients of alpha-gamma angular correlations were obtained in good agreement with experimental values.

In this paper we shall first outline again briefly the derivation of some formulas which are then applied to discuss the ground-state transitions of even-even, even-odd, and odd-even alpha emitters in the region of Pb^{208}. The nuclear wave functions are approximated by shell-model wave functions and good agreement is found with experimental data.[2]

I. DERIVATION OF AN EXPRESSION FOR THE DECAY CONSTANT

We consider a system of A nucleons (Z protons and N neutrons) and describe it by means of the time-dependent Schrödinger equation

$$H\Phi(1\cdots A;t)=i\hbar\partial\Phi(1\cdots A;t)/\partial t. \tag{I.1}$$

We assume for the Hamiltonian the following form:

$$H=-\sum_{i=1}^{A}\left(\frac{\hbar^2}{2m}\right)\Delta_i+V(1,2\cdots A). \tag{I.2}$$

V describes the interaction of all the particles. It is further assumed that the wave function $\Phi_0(1\cdots A)=\Phi(1\cdots A;t=0)$ is known and that this wave function is the wave function of the parent nucleus.

Next we separate off the trivial center-of-mass motion of the system and decompose the remaining Hamiltonian into several terms

$$H=H_\alpha(1234)+H_K(5\cdots A)-(\hbar^2/2M)\Delta_{\rm rel}+W(\alpha,K), \tag{I.3}$$

where 1, 2, are protons, 3, 4, are neutrons.

H_α describes the internal motion of a system consisting of two protons and two neutrons. H_K describes the internal motion of the remaining $A-4$ nucleons. $\mathbf{R}$ is the relative coordinate between the center of mass of particles 1; 2; 3; 4 and the center of mass of the remaining particles 5; 6; $\cdots A$; and $-(\hbar^2/2M)\Delta_{\rm rel}$ is the kinetic energy operator associated with the relative motion of the two groups of nucleons. $W(\alpha K)$ accounts for the interaction between the two groups of particles. We should mention that the Hamiltonian H has not lost its symmetry properties.

For obvious reasons we shall refer from now on to these two groups of nucleons as α particle and daughter nucleus. Consequently, we call the solutions of the equations

$$\begin{aligned} H_\alpha\chi_\alpha{}^\tau&=\epsilon_\alpha\chi_\alpha{}^\tau,\\ H_K\Psi_K{}^\sigma&=E_K\Psi_K{}^\sigma, \end{aligned} \tag{I.4}$$

the internal wave functions of the α particle and the daughter nucleus. In these equations τ and σ are short-hand notations for those sets of quantum numbers that are necessary to determine the solutions $\chi_\alpha{}^\tau$ and $\Psi_K{}^\sigma$ completely.

Furthermore, if $\chi_\alpha{}^\tau$, $\chi_\alpha{}^{\tau'}$ and $\Psi_K{}^\sigma$, $\Psi_K{}^{\sigma'}$ are bound-state solutions of (I.4) the following relation holds:

$$\langle\chi_\alpha{}^{\tau'}\Psi_K{}^{\sigma'}|W(\alpha K)|\chi_\alpha{}^\tau\Psi_K{}^\sigma\rangle=\delta_{\tau\tau'}\delta_{\sigma\sigma'}V(\mathbf{R}), \tag{I.5}$$

for $R=|\mathbf{R}|\geq R_0$. R_0 turns out to be about 9×10^{-13} cm for natural α emitters. The meaning of the relation (I.5) is that when the α particle and the daughter nucleus are well separated in space the interaction between them is always described by a simple potential. Furthermore $V(\mathbf{R})$ approaches the Coulomb potential $2(Z-2)e^2/R$ rapidly. For most purposes therefore it will be sufficient to use $2(Z-2)e^2/R$ instead of $V(\mathbf{R})$.

These considerations indicate that it might be reasonable to write for the time-dependent wave function $\Phi(1\cdots A;t)$:

$$\Phi(1\cdots A;t)=a(t)\Phi_0(1\cdots A)+\sum_{\tau\sigma Lm}\int d\epsilon\,\mathfrak{A}\times\{\chi_\alpha{}^\tau\Psi_K{}^\sigma\varphi_L(R,\epsilon)Y_L{}^m(\mathbf{R}/R)\}b_{\tau\sigma Lm}(\epsilon,t). \tag{I.6}$$

---

* This work was performed under the auspices of the U. S. Atomic Energy Commission while the author was on leave of absence from the University of Heidelberg, Heidelberg, Germany.
† Present address: University of Heidelberg, Heidelberg, Germany.
[1] H. J. Mang, Z. Physik **148**, 572 (1957), for details also see H. J. Mang, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, p. 299 (1959) and H. J. Mang, University of California Radiation Laboratory Report UCRL-8931, October, 1959 (unpublished).
[2] J. O. Rasmussen, Phys. Rev. **113**, 1593 (1959); and **115**, 1675 (1959).

$\mathfrak{A}$ is an antisymmetrization and normalization operator, while $\varphi_L(R,\epsilon)$ is a solution of the equation

$$\left[-\frac{\hbar^2}{2M}\frac{1}{R}\frac{d^2}{dR^2}R+\frac{\hbar^2}{2M}\frac{L(L+1)}{R^2}+\frac{2(Z-2)e^2}{R}-\epsilon\right]\varphi_L(R,\epsilon)=0, \quad \text{(I.7)}$$

and is normalized as follows:

$$\int \varphi_L(R,\epsilon)\varphi_L(R,\epsilon')R^2dR=\delta(\epsilon-\epsilon'). \quad \text{(I.8)}$$

To get unique solutions[3] for the coefficients $a(t)$ and $b_{\tau\sigma Lm}(\epsilon,t)$, it is necessary to impose the condition

$$\left\langle \Phi_0(1\cdots A)\left|\sum_{\tau\sigma Lm}\int d\epsilon\,\mathfrak{A}\times\{\chi_\alpha{}^\tau\Psi_K{}^\sigma\varphi_L(R,\epsilon)Y_L{}^m\}b_{\tau\sigma Lm}(\epsilon,t)\right.\right\rangle=0. \quad \text{(I.9)}$$

Of course $\Phi(1\cdots A;t)$ may always be expanded in the above form (I.6) as long as we sum over a complete set of functions $\chi_\alpha{}^\tau$ and $\Psi_K{}^\sigma$. But we know that for energetic reasons the terms in the sum over τ and σ corresponding to unbound states cannot contribute to the alpha decay. Therefore, we split the sum into one over bound states only and one in which τ and σ or both correspond to unbound states and neglect the latter sum. With this approximation and taking into account angular momentum conservation as well as the fact that there is only a single bound state of the alpha particle, we rewrite $\Phi(1\cdots A;t)$ in the following form:

$$\Phi(1\cdots A;t)=a(t)\Phi_{0J}{}^M(1\cdots A)+\sum_{j\sigma L}\int d\epsilon b_{j\sigma L}(\epsilon,t)\mathfrak{A}\times\{\chi_\alpha\varphi_L(R,\epsilon)\sum_m C(LjJ;\,m\,M-m)\times Y_L{}^m(\mathbf{R}/R)\Psi_{j\sigma}{}^{M-m}\}, \quad \text{(I.10)}$$

where J is the angular momentum of the parent nucleus, j the angular momentum of the daughter nucleus, and L the angular momentum of the α particle. The Clebsch-Gordan coefficient $C(LjJ;\,m\,M-m)$ couples L and j to give the resulting angular momentum J. The functions χ_α and $\psi_{j\sigma}{}^{M-m}$ are now, as already mentioned, only the bound state solutions of (I.4).

Introducing this expression for Φ into the Schrödinger equation (I.1), we get a system of coupled integro-differential equations for $a(t)$ and $b_{j\sigma L}(\epsilon,t)$. But this system of equations decouples and becomes rather simple if one more approximation is introduced. We neglect nondiagonal matrix elements of the type

$$\langle\Phi_{Jj\sigma L\epsilon}{}^M|H-H_\alpha-H_K+(\hbar^2/2M)\Delta_{\rm rel}-2(Z-2)e^2/R|\Phi_{Jj'\sigma'L'\epsilon'}{}^M\rangle,$$

where

$$\Phi_{Jj\sigma L\epsilon}{}^M=\mathfrak{A}\{\chi_\alpha\varphi_L(R,\epsilon)\sum_m C(LjJ;\,m\,M-m)\times Y_L{}^m\Psi_{j\sigma}{}^{M-m}\},$$

which means we neglect the interaction of the alpha particle with the daughter nucleus via the nuclear forces[4] and take only into account the Coulomb interaction; a reasonable approximation at the alpha energies involved in natural alpha decays.

The simplified equations read:

$$\begin{aligned} i\hbar\dot a(t)&=a(t)E_0+\sum_{j\sigma L}\int d\epsilon\, b_{j\sigma L}(\epsilon,t)\times\langle\Phi_{0J}{}^M|H-E_0|\Phi_{Jj\sigma L\epsilon}{}^M\rangle,\\ i\hbar[&\dot b_{j\sigma L}(\epsilon,t)+\dot a(t)\langle\Phi_{Jj\sigma L\epsilon}|\Phi_{0J}{}^M\rangle]\\ &=(\epsilon_\alpha+E_K+\epsilon)b_{j\sigma L}(\epsilon,t)+a(t)\times\langle\Phi_{Jj\sigma L\epsilon}{}^M|H|\Phi_{0J}{}^M\rangle, \end{aligned} \quad \text{(I.11)}$$

where $E_0=\langle\Phi_{0J}{}^M|H|\Phi_{0J}{}^M\rangle$.

Using standard techniques[5] to solve the equations approximately we get:

$$\begin{aligned} a(t)&=\exp[-(i/\hbar)(E_0+F-i\gamma)t],\\ b_{j\sigma L}(\epsilon,t)&=\frac{\exp[-(i/\hbar)(E_0+F-i\gamma)t]-\exp[-(i/\hbar)(E_K+\epsilon_\alpha+\epsilon)t]}{E_0+F-E_K-\epsilon_\alpha-\epsilon-i\gamma}\langle\Phi_{0J}{}^M|H-E_0|\Phi_{Jj\sigma L\epsilon}{}^M\rangle,\\ \gamma&=\pi\sum_{jL\sigma}|\langle\Phi_{Jj\sigma L\epsilon}{}^M|H-E_0|\Phi_{0J}{}^M\rangle|_{\epsilon=E_0-\epsilon_\alpha-E_K+F},\\ F&=\sum_{j\sigma L}P\int d\epsilon\frac{|\langle\Phi_{Jj\sigma L\epsilon}{}^M|H-E_0|\Phi_{0J}{}^M\rangle|^2}{E_0+F-E_K-\epsilon_\alpha-\epsilon}, \end{aligned} \quad \text{(I.12)}$$

where P means principal value. The conditions for the approximate solution to be a good one are:

$$\gamma\ll|E_0-E_K-\epsilon_\alpha|=\epsilon_0, \quad F\ll\epsilon_0, \quad \text{(I.13)}$$

and

$$|\langle\Phi_{Jj\sigma L\epsilon}{}^M|H-E_0|\Phi_{0J}{}^M\rangle|^2$$

has to vary slowly compared to

$$[(E_0+F-\epsilon_\alpha-E_K-\epsilon)^2+\gamma^2]^{-1}$$

if ϵ varies between $\epsilon_0-\gamma$ and $\epsilon_0+\gamma$. For all natural alpha emitters these conditions are well fulfilled.

[3] H. Casimir, Physica **1**, 193 (1934).

[4] If we use $V(\mathbf{R})$ as defined in (I.5) instead of the Coulomb potential, we take into account a part of the nuclear interaction.

[5] W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), 3rd ed., p. 181, and the references quoted there.

The problem of determining the decay constant is now reduced to the calculation of the matrix elements $\langle\Phi_{0J}{}^{M}|H-E_0|\Phi_{Jj\sigma L\epsilon}{}^{M}\rangle$. But those matrix elements cannot be calculated unless $\Phi_{0J}{}^{M}$ is defined in a rigorous way. The somewhat vague statement "the wave function $\Phi_{0J}{}^{M}$ should describe the parent nucleus of the alpha decay" cannot be considered as a definition of $\Phi_{0J}{}^{M}$.

A natural way to define $\Phi_{0J}{}^{M}$ would be to obtain it from a consideration of the formation process of the alpha emitter. But on a nuclear time scale all alpha emitters are nearly stable. If it were not for the repulsive Coulomb interaction between the alpha particle and the daughter nucleus there would be no alpha decay at all. Therefore, another definition of $\Phi_{0J}{}^{M}$ is suggested. It has to be a bound-state solution of a Schrödinger equation

$$H_0\Phi_{0J}{}^{M}=E^{(0)}\Phi_{0J}{}^{M}. \tag{I.14}$$

H_0 is defined as follows:

$$H_0=H,$$

if all nucleons are confined in a spherical[6] volume Ω_0 with radius r_0;

$$H_0=H-V_C(r)+V_C(r_0),$$

if one or more nucleons are outside the above defined volume Ω_0. $V_C(r)$ is the Coulomb interaction between the nucleons inside and those outside Ω_0.

r_0 is so defined that the interaction between one selected proton (neutron) and all the others that are in Ω_0 is repulsive (zero) if the selected proton (neutron) is outside Ω_0 but mostly attractive if the proton (neutron) is inside Ω_0. Of course there is a maximum value for r_0 because of the condition that the eigenvalue problem with H_0 (I.14) should have bound-state solutions. This definition assures that H_0 is equal to the exact Hamiltonian H in that part of the configuration space where the nuclear forces between the nucleons play an important role and therefore $\Phi_{0J}{}^{M}$ should be a very close approximation to an exact wave function as derived for instance from considering the formation process of an alpha emitter. By means of these assumptions about $\Phi_{0J}{}^{M}$ the matrix element can be simplified considerably.

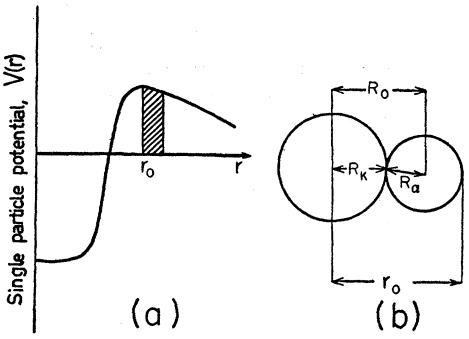

FIG. 1. (a) Relation between the single-particle potential $V(r)$ and the constant r_0. (b) Connection between r_0 and R_0. R_α and R_K are some sort of radii of the α particle and daughter nucleus.

[6] Of course one may assume Ω_0 to be nonspherical if one deals with a deformed nucleus.

$$\begin{aligned}
&\langle\Phi_{0J}{}^{M}|H-E_0|\Phi_{Jj\sigma L\epsilon}{}^{M}\rangle\\
&=\left[\binom{Z}{2}\binom{N}{2}\right]^{\frac{1}{2}}\langle\Phi_{0J}{}^{M}|H-H_0|\chi_\alpha\varphi_L(R,\epsilon)\\
&\qquad\times\sum_m C(LjJ;\,m\,M-m)Y_L{}^{m}\Psi_{j\sigma}{}^{M-m}\rangle\\
&=\int_{R_0}^{\infty}R^2dR\int d\Omega_{\rm rel}\int d\xi_\alpha d\xi_K\,\Phi_{0J}{}^{M*}(H-H_0)\\
&\qquad\times\chi_\alpha\varphi_L(R,\epsilon)\sum_m C(LjJ;\,m\,M-m)Y_L{}^{m}(\mathbf{R}/R)\\
&\qquad\times\Psi_{j\sigma}{}^{M-m}\left[\binom{Z}{2}\binom{N}{2}\right]^{\frac{1}{2}},
\end{aligned}\tag{I.15}$$

where ξ_α and ξ_K are the internal coordinates of the alpha particle and the daughter nucleus.

The first equality holds because of $H-E_0$ being a symmetric operator and $\Phi_{0J}{}^{M}$ an antisymmetric wave function and the fact that $E_0=E^{(0)}$ is a very good approximation. The second equality holds because of the definition of $\Phi_{0J}{}^{M}(1\cdots A)$ (I.14).

R_0 is somewhat smaller than r_0 because of the finite size of the alpha particle. The relation between R_0 and r_0 is schematically illustrated in Fig. 1. Because of the freedom in choosing r_0 and consequently R_0, the distance r_0 and hence R_0 is chosen so that for $R\geq R_0$:

$$\begin{aligned}
&H\chi_\alpha\varphi_L(R,\epsilon)\textstyle\sum_m C(LjJ;\,m\,M-m)Y_L{}^{m}(\mathbf{R}/R)\Psi_{j\sigma}{}^{M-m}\\
&\quad=(E_K+\epsilon_\alpha+\epsilon)\chi_\alpha\varphi_L(R,\epsilon)\textstyle\sum_m C(LJj;\,m\,M-m)\\
&\qquad\times Y_L{}^{m}(\mathbf{R}/R)\Psi_{j\sigma}{}^{M-m}.
\end{aligned}\tag{I.16}$$

Using (I.15) and (I.16) and integrating by parts on R, the following expression is obtained for the matrix element in question

$$\begin{aligned}
&\langle\Phi_{0J}{}^{M}|H-E_0|\Phi_{Jj\sigma L\epsilon_0}{}^{M}\rangle\\
&=\left[\binom{Z}{2}\binom{N}{2}\right]^{\frac{1}{2}}\left(\frac{\hbar^2}{2M}\right)\int d\xi_\alpha d\xi_K R_0{}^2 d\Omega_{\rm rel}\\
&\qquad\times\left\{\Phi_{0J}{}^{M*}\frac{\partial\varphi_L(R,\epsilon_0)}{\partial R}\bigg|_{R=R_0}\right.\\
&\qquad\left.-\frac{\partial\Phi_{0J}{}^{M*}}{\partial R}\bigg|_{R=R_0}\varphi_L(R,\epsilon_0)\right\}\\
&\qquad\times\chi_\alpha\sum_m C(LjJ;\,m\,M-m)Y_L{}^{m}\Psi_{j\sigma}{}^{M-m}.
\end{aligned}\tag{I.17}$$

Introducing for $\varphi_L(R,\epsilon_0)$ the WKB approximation

$$\varphi_L(R,\epsilon_0)=\left(\frac{2M}{\pi\hbar^2}\right)^{\frac{1}{2}}\left(\frac{1}{R_0 q_L{}^{\frac{1}{2}}}\right)\times\tfrac{1}{2}\exp\left(-\int_R^{R_u} q_L dR\right),$$

$$q_L=\left[\frac{2M}{\hbar^2}\left(\frac{2(Z-2)e^2}{R}+\frac{\hbar^2}{2M}\frac{L(L+1)}{R^2}-\epsilon_0\right)\right]^{\frac{1}{2}}, \tag{I.18}$$

where the outer turning point R_u is defined by

$$q_L(R_u)=0,$$

and defining a function $G_{Jj\sigma L}(R)$,

$$G_{Jj\sigma L}(R)=R_0^{\frac{3}{2}}\int d\xi_K d\xi_\alpha d\Omega_{\rm rel}\times(\Phi_{0J}{}^{M*}\chi_\alpha\sum_m C(LjJ;\,m\,M-m)Y_L{}^m(\mathbf{R}/R)\times\Psi_{j\sigma}{}^{M-m})\left[\binom{Z}{2}\binom{N}{2}\right]^{\frac{1}{2}}, \tag{I.19}$$

the expression for the decay constant λ is brought into the following form:

$$\lambda=\left(\frac{1}{\hbar}\right)\sum_{jL\sigma}\exp\left(-2\int_{R_0}^{R_u} q_L dR\right)\left(\frac{\hbar^2}{2MR_0{}^2}\right)\times\frac{(R_0q_L-1)^2}{2R_0q_L}\left|G_{Jj\sigma L}(R_0)+\frac{R_0}{1-R_0q_L}\frac{\partial G_{Jj\sigma L}}{\partial R}\right|_{R=R_0}{}^2=\left(\frac{1}{\hbar}\right)\sum_{jL\sigma}P_L(\epsilon_0)\delta_{Jj\sigma L}{}^2, \tag{I.20}$$

where

$$P_L(\epsilon_0)=\exp\left(-2\int_{R_0}^{R_u} q_L dR\right),$$

$$\delta_{Jj\sigma L}{}^2=\left(\frac{\hbar^2}{2MR_0{}^2}\right)\frac{(R_0q_L-1)^2}{2R_0q_L}\left|G_{Jj\sigma L}+\frac{R_0}{1-R_0q_L}\frac{\partial G_{Jj\sigma L}}{\partial R}\right|_{R=R_0}{}^2.$$

P_L is the well-known barrier penetrability as already derived by Gamow in 1928 and $\delta_{Jj\sigma L}{}^2$ is the reduced width that accounts for the influence of the nuclear structure on alpha decay.

As far as the connection of our treatment with the treatment of other authors is concerned we refer to reference 1.

We should also mention that there is no difficulty in using more sophisticated potentials $V(\mathbf{R})$ than the simple Coulomb potential. One has to replace the Coulomb potential $2(Z-2)e^2/R$ in (I.17) by $V(\mathbf{R})$ to calculate now the somewhat different function $\varphi_L(R,\epsilon_0)$. If necessary this replacement has been made in the application of the theory, although all formulas are given for the pure Coulomb potential.

II. GROUND-STATE TRANSITIONS IN THE REGION OF Pb^{208}

To apply the theory developed in the preceding chapter we shall approximate the nuclear wave functions by some sort of shell-model wave functions.

The following assumptions determine completely all the wave functions.

(1) The alpha-particle wave function is a Gaussian type wave function.

$$\chi_\alpha(1,2,3,4)=\chi_0{}^0(1,2)\chi_0{}^0(3,4)\times\exp[-(\beta/2)(\xi_1{}^2+\xi_2{}^2+\xi_3{}^2)](2\beta^{\frac{3}{2}}/\tfrac{1}{2}!4\pi)^{\frac{3}{2}},\quad \xi_1=(1/\sqrt{2})(\mathbf{r}_1-\mathbf{r}_2),\quad \xi_2=(1/\sqrt{2})(\mathbf{r}_3-\mathbf{r}_4),\quad \xi_3=(\tfrac{1}{2})(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4), \tag{II.1}$$

where here and in the following $a!$ means $\Gamma(a+1)$ and $\chi_0{}^0(i,k)$ is a singlet spin function of particles i and k. For the constant β we are going to use the value

$$\beta=0.625\times10^{+26}\ \text{cm}^{-2}.$$

This value of β was chosen so that the rms radius of the charge density is equal to the measured value.[7]

(2) The wave functions of the parent and daughter nucleus are shell-model wave functions with seniority 0 (even-even nuclei) or 1 (even-odd and odd-even nuclei). That means an even number of particles is coupled pairwise to angular momentum zero. In an odd-mass nucleus the last odd particle is therefore responsible for the total angular momentum and the parity.

Furthermore, we assume no configurational mixing and the single-particle states involved are taken from the spectra of nuclei which have one particle more or less than Pb^{208} [8,9] (Tl^{207}, Pb^{207}, Bi^{209}, Pb^{209}).

For the radial part of the single-particle wave functions, harmonic oscillator functions are used.

radial part of the single-particle wave functions, harmonic oscillator functions are used.

$$\Phi_{jln}{}^m=\alpha^{\frac{3}{4}}\left[\frac{2\times n!}{(n+l+\frac{1}{2})!}\right]^{\frac{1}{2}}(\alpha^{\frac{1}{2}}r)^l L_n{}^{l+\frac{1}{2}}(\alpha r^2)\times\exp(-\tfrac{1}{2}\alpha r^2)\sum_\mu C(l\tfrac{1}{2}j;\,\mu\,m-\mu)\times Y_l{}^\mu(\mathbf{r}/r)\chi_{\frac{1}{2}}{}^{m-\mu}, \tag{II.2}$$

[7] R. Hofstadter, Revs. Modern Phys. 28, 3, 214 (1956).

[8] D. Strominger and J. M. Hollander, University of California Radiation Laboratory Report UCRL-8289, June, 1958 (unpublished).

[9] D. Strominger, J. M. Hollander, and G. T. Seaborg, Revs. Modern Phys. 30, 2, 585 (1958).

TABLE I. Shell-model configurations for nuclei in the region of Pb^{208}.

Element	Proton config.	Neutron config.	Element	Proton config.	Neutron config.
Po^{202}	$(1h_{9/2}^2)_0$	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0(p_{3/2}^{-2})_0$	Pb^{198}	Closed shell	$(p_{1/2}^{-2})_0(f_{5/2}^{-6})_0(p_{3/2}^{-2})_0$
Po^{202}	$(1h_{9/2}^2)_0$	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0$	Pb^{200}	Closed shell	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0(p_{3/2}^{-2})_0$
Po^{206}	$(1h_{9/2}^2)_0$	$(p_{1/2}^{-2})_0(f_{5/2}^{-2})_0$	Pb^{202}	Closed shell	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0$
Po^{208}	$(1h_{9/2}^2)_0$	$(p_{1/2}^{-2})_0$	Pb^{204}	Closed shell	$(p_{1/2}^{-2})_0(f_{5/2}^{-2})_0$
Po^{210}	$(1h_{9/2}^2)_0$	Closed shell	Pb^{206}	Closed shell	$(p_{1/2}^{-2})_0$
Po^{212}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^2)_0$	Pb^{208}	Closed shell	Closed shell
Po^{104}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^4)_0$	Pb^{210}	Closed shell	$(2g_{9/2}^2)_0$
Po^{216}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^6)_0$	Pb^{212}	Closed shell	$(2g_{9/2}^4)_0$
Po^{218}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^8)_0$	Pb^{214}	Closed shell	$(2g_{9/2}^6)_0$
At^{203}	$(h_{9/2}^3)_{9/2}$	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0(p_{3/2}^{-2})_0$	Bi^{199}	$1h_{9/2}$	$(p_{1/2}^{-2})(f_{5/2}^{-6})_0(p_{3/2}^{-2})_0$
At^{205}	$(h_{9/2}^3)_{9/2}$	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0$	Bi^{201}	$1h_{9/2}$	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0(p_{3/2}^{-2})_0$
At^{207}	$(h_{9/2}^3)_{9/2}$	$(p_{1/2}^{-2})_0(f_{5/2}^{-2})_0$	Bi^{203}	$1h_{9/2}$	$(p_{1/2}^{-2})_0(f_{5/2}^{-4})_0$
At^{209}	$(h_{9/2}^3)_{9/2}$	$(p_{1/2}^{-2})_0$	Bi^{205}	$1h_{9/2}$	$(p_{1/2}^{-2})_0(f_{5/2}^{-2})_0$
At^{211}	$(h_{9/2}^3)_{9/2}$	Closed shell	Bi^{207}	$1h_{9/2}$	$(p_{1/2}^{-2})_0$
At^{213}	$(h_{9/2}^3)_{9/2}$	$(2g_{9/2}^2)_0$	Bi^{209}	$1h_{9/2}$	Closed shell
At^{215}	$(h_{9/2}^3)_{9/2}$	$(2g_{9/2}^4)_0$	Bi^{211}	$1h_{9/2}$	$(2g_{9/2}^2)_0$
At^{217}	$(1h_{9/2}^3)_{9/2}$	$(2g_{9/2}^6)_0$	Bi^{213}	$1h_{9/2}$	$(2g_{9/2}^4)_0$
At^{219}	$(1h_{9/2}^3)_{9/2}$	$(2g_{9/2}^8)_0$	Bi^{215}	$1h_{9/2}$	$(2g_{9/2}^6)_0$
Po^{203}	$(1h_{9/2}^2)_0$	$(3p_{1/2}^{-2})_0(f_{5/2}^{-3})_{5/2}(p_{3/2}^{-2})_0$	Pb^{199}	Closed shell	$(3p_{1/2}^{-2})_0(2f_{5/2}^{-5})_{5/2}(p_{3/2}^{-2})_0$
Po^{205}	$(1h_{9/2}^2)_0$	$(3p_{1/2}^{-2})_0(f_{5/2}^{-3})_{5/2}$	Pb^{201}	Closed shell	$(3p_{1/2}^{-2})_0(2f_{5/2}^{-3})_{5/2}(p_{3/2}^{-2})_0$
Po^{207}	$(1h_{9/2}^2)_0$	$(3p_{1/2}^{-2})_0(f_{5/2}^{-1})_{5/2}$	Pb^{203}	Closed shell	$(3p_{1/2}^{-2})_0(2f_{5/2}^{-3})_{5/2}$
Po^{209}	$(1h_{9/2}^2)_0$	$(3p_{1/2}^{-1})_{1/2}$	Pb^{205}	Closed shell	$(3p_{1/2}^{-2})_0(2f_{5/2}^{-1})_{5/2}$
Po^{211}	$(1h_{9/2}^2)_0$	$2g_{9/2}$	Pb^{207}	Closed shell	$(3p_{1/2}^{-1})_{1/2}$
Po^{213}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^3)_{9/2}$	Pb^{209}	Closed shell	$2g_{9/2}$
Po^{215}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^5)_{9/2}$	Pb^{211}	Closed shell	$(2g_{9/2}^3)_{9/2}$
Po^{217}	$(1h_{9/2}^2)_0$	$(2g_{9/2}^7)_{9/2}$	Pb^{213}	Closed shell	$(2g_{9/2}^5)_{9/2}$

where

$$L_n^{l+\frac{1}{2}}=\sum_{k=0}^{n}\binom{n+l+\frac{1}{2}}{n-k}\frac{(-1)^k}{k!}(\alpha r^2)^k.$$

For the constant α we shall use $\alpha=0.175\times10^{26}$ cm^{-2}, a value which gives rms radii of the nuclei that are consistent with the measured values.[7]

For R_0 we shall use $R_0=9.0\times10^{-13}$ cm. But only the product αR_0^2 enters into the reduced width and furthermore we shall only calculate relative transition probabilities. Therefore, the choice of α or R_0 alone is not too critical.

With the above assumptions (1) and (2) we get the reduced widths $\delta_{Jj}{}^2$, where J is the angular momentum of the parent nucleus, j the angular momentum of the daughter nucleus, and L the angular momentum of the outgoing alpha particle.

Even-even nuclei:

$$\delta_{000}{}^2=(1/16)N_1(2j_1+3-N_1)N_3(2j_3+3-N_3)\times[\Re(n_1l_1n_2l_2n_3l_3n_4l_4;R_0]^2. \quad \text{(II.3)}$$

N_1 is the number of protons in the unfilled subshell with quantum numbers $n_1l_1j_1$ in the parent nucleus. N_3 is the number of neutrons in the unfilled subshell with quantum numbers $n_2l_3j_3$ in the parent nucleus. $\Re^2$ involves the radial parts of the wave functions and is given below.

Even-odd nuclei:

$$\delta_{j_1j_10}{}^2=\tfrac{1}{16}(N_1-1)(2j_1+2-N_1)N_3(2j_3+3-N_3)[\Re(n_1l_1n_1l_1n_3l_3n_3l_3;R_0)]^2,$$

$$\delta_{j_1j_1L}{}^2=\tfrac{1}{4}(N_1-1)(2j_1+2-N_1)\times\left[\frac{2j_1+1}{2j_1-1}C(j_1j_1L;\tfrac{1}{2}-\tfrac{1}{2})\right]^2N_3(2j_3+3-N_3)\times[\Re(n_1l_1n_1l_1n_3l_3n_3l_3;R_0)]^2 \quad L\neq0, \quad \text{(II.4)}$$

$$\delta_{j_2j_1L}{}^2=\tfrac{1}{4}N_1C^2(j_1j_2L;\tfrac{1}{2}-\tfrac{1}{2})N_3(2j_3+3-N_3)\times[\Re(n_1l_1n_2l_2n_3l_3n_3l_3;R_0)]^2.$$

N_1 is again the number of nucleons in the subshell n_1, l_1, j_1. In the first two cases ($\delta_{j_1j_10}{}^2$,$\delta_{j_1j_1L}{}^2$)N_1 is odd. In the third case ($\delta_{j_2j_1L}{}^2$)N_1 is even and there is one more particle in the state $n_2l_2j_2$ in the parent nucleus.

$\Re(n_1l_1n_2l_2n_3l_3n_4l_4;R_0)$ has the following form:

$$\begin{aligned}\Re(n_1l_1n_2l_2n_3l_3n_4l_4;R_0)&=[n_1!(n_1+l_1+\tfrac{1}{2})!n_2!(n_2+l_2+\tfrac{1}{2})!n_3!(n_3+l_3+\tfrac{1}{2})!\\&\times n_4!(n_4+l_4+\tfrac{1}{2})!]^{-\frac{1}{2}}\left(\frac{2\alpha}{\alpha+\beta}\right)^{N+L/2}\\&\times\left(\frac{N!}{2^{2N+L}}\right)\left[\left(\frac{\hbar^2}{MR_0^2}\right)\frac{(R_0q_L-1)^2}{R_0q_L}(\alpha^{\frac{1}{2}}R_0)^3\right.\\&\left.\times[2(\alpha\beta)^{\frac{1}{2}}/(\alpha+\beta)]^9(\tfrac{1}{2}!)^3\right]^{\frac{1}{2}}\exp(-2\alpha R_0^2)\\&\times\sum_\rho B_\rho(n_1l_1n_2l_2n_3l_3n_4l_4)\left(\frac{\beta-\alpha}{2\alpha}\right)^{N-\rho}\frac{(-1)^\rho\rho!}{N!}\\&\times4^{N-\rho}[2(\alpha+\beta)R_0^2]^{L/2}K_\rho{}^{L+\frac{1}{2}}(2(\alpha+\beta)R_0^2),\end{aligned} \quad \text{(II.5)}$$

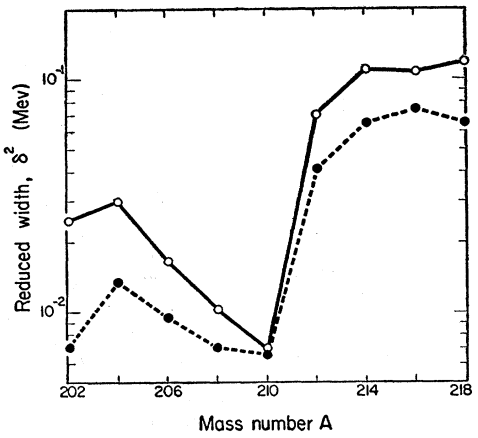

FIG. 2. Reduced widths of the even-even polonium isotopes as a function of the mass number. The open circles are the experimental values, while the closed circles are the calculated values.

where

$$2N=2(n_1+n_2+n_3+n_4)+l_1+l_2+l_3+l_4-L$$

$$K_\rho{}^{L+\frac{1}{2}}=\left(1+\frac{4\alpha R_0^2-L}{R_0q_L-1}\right)L_\rho{}^{L+\frac{1}{2}}(2(\alpha+\beta)R_0^2)$$

$$-\frac{4(\alpha+\beta)R_0^2}{R_0q_L-1}L_\rho{}^{L+\frac{1}{2}\prime}(2(\alpha+\beta)R_0^2),$$

The prime means differentiation with respect to the argument $2(\alpha+\beta)R_0^2$.

The coefficients

$$B_\rho(n_1l_1n_2l_2n_3l_3n_4l_4)$$

are defined by the following equation:

$$n_1!n_2!n_3!n_4!L_{n_1}{}^{l_1+\frac{1}{2}}(x)L_{n_2}{}^{l_2+\frac{1}{2}}(x)L_{n_3}{}^{l_3+\frac{1}{2}}(x)L_{n_4}{}^{l_4+\frac{1}{2}}(x)=\sum_\rho B_\rho(n_1l_1n_2l_2n_3l_3n_4l_4)\times x^{\rho+(L-l_1-l_2-l_3-l_4)/2}, \quad \text{(II.6)}$$

or

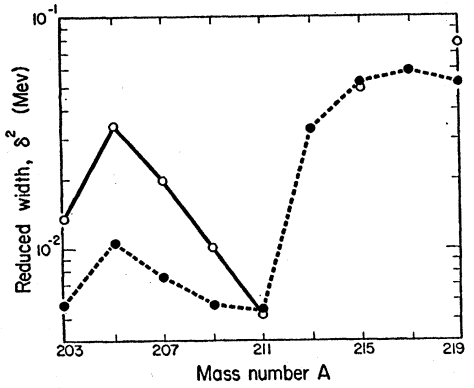

FIG. 3. Reduced widths of the odd-even astatine isotopes as a function of the mass number. The open circles are the experimental values, while the closed circles are the calculated values.

$$B_\rho(n_1l_1n_2l_2n_3l_3n_4l_4)=n_1!n_2!n_3!n_4!\times\sum_{\nu_1\nu_2\nu_3\nu_4}\binom{n_1+l_1+\frac{1}{2}}{n_1-\nu_1}\binom{n_2+l_2+\frac{1}{2}}{n_2-\nu_2}\binom{n_3+l_3+\frac{1}{2}}{n_3-\nu_3}\times\binom{n_4+l_4+\frac{1}{2}}{n_4-\nu_4}\frac{(-1)^{\nu_1+\nu_2+\nu_3+\nu_4}}{\nu_1!\nu_2!\nu_3!\nu_4!},$$

where the summation is restricted by

$$2\rho+L=2(\nu_1+\nu_2+\nu_3+\nu_4)+l_1+l_2+l_3+l_4.$$

With the help of these formulas and using the configurations listed in Table I, we have calculated the reduced widths that are compared to the experimental data[2] in Figs. 2, 3, and 4. In these diagrams the experimental and calculated reduced widths are plotted versus the mass number for the even-even polonium isotopes, the odd-even astatine isotopes and the even-odd polonium isotopes separately.

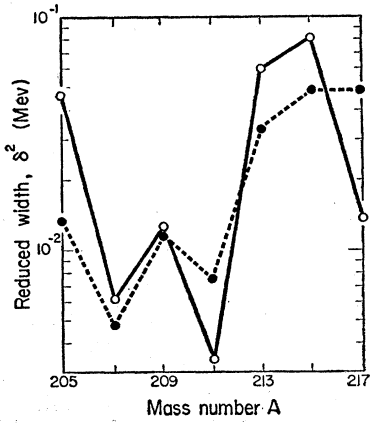

FIG. 4. Reduced widths of the even-odd polonium isotopes as a function of the mass number. The open circles are the experimental values, while the closed circles are the calculated values.

The reduced width of Po^{210} is taken as a standard and set equal to the experimental reduced width. We hope that in doing this, we minimize the ambiguities introduced by the choice of the radial wave functions and the parameters α and R_0.

It may be shown that varying the parameters within reasonable limits affects only the absolute magnitude of the radial part of the reduced width $\mathfrak{R}^2$, but leaves nearly unaffected the relative magnitudes we are mainly interested in. This result indicates also that it is sometimes allowed to use a very handy approximation for the radial part $\mathfrak{R}(n_1l_1n_2l_2n_3l_3n_4l_4;R_0)$ which arises if one sets $\beta\gg\alpha$. Then

$$\mathfrak{R}(n_1l_1n_2l_2n_3l_3n_4l_4;R_0)\sim\exp(-2\alpha R_0^2)(\alpha^{\frac{1}{2}}R_0)^{l_1+l_2+l_3+l_4}\times\left[\frac{n_1!n_2!n_3!n_4!}{(n_1+l_1+\frac{1}{2})!(n_2+l_2+\frac{1}{2})!(n_3+l_3+\frac{1}{2})!(n_4+l_4+\frac{1}{2})!}\right]^{\frac{1}{2}}\times L_{n_1}{}^{l_1+\frac{1}{2}}(\alpha R_0^2)L_{n_2}{}^{l_2+\frac{1}{2}}(\alpha R_0^2)L_{n_3}{}^{l_3+\frac{1}{2}}(\alpha R_0^2)\times L_{n_4}{}^{l_4+\frac{1}{2}}(\alpha R_0^2), \quad \text{(II.7)}$$

a very useful expression if one wants results quickly.

The expression so obtained is identical with what one would get using the formula proposed by Brussaard and Tolhoek.[10] But one has to be very careful in using this expression because it favors the formation of alpha particles from single-particle states with high angular momenta.

III. DISCUSSION

The comparison of experimental and calculated reduced widths in Figs. 2, 3, and 4 shows clearly that taking into account the nuclear structure, even in a very crude approximation, gives results that agree well with the general features of the experimental data. Especially the behavior of the reduced width when crossing the neutron number 126 is well reproduced. It seems to us that one no longer needs to introduce a sudden jump of the nuclear radius at the double magic nucleus Pb^{208} to explain the difference in the reduced widths of Po^{210} and Po^{212}. The increase of the reduced width when going from $N=126$ to $N=128$ is quite naturally brought by the change in the single-particle states involved in the alpha decay. Particles in the $2g_{9/2}$ subshell are favored over particles in the $3p_{1/2}$ and $2f_{5/2}$ subshell in forming an alpha particle, first because of the higher angular momentum (compare (II.3)] and second, because the $2g_{9/2}$ level belongs to a higher major shell and therefore the radial part of the wave function is larger too at the edge of the nucleus. This effects the quantity $\mathfrak{R}(n_1l_1n_2l_2n_3l_3n_4l_4; R_0)$ [compare (II.5)].

The remaining discrepancies may have several reasons. First, we have neglected configurational mixing. Configurational mixing under the influence of a short-range attractive force however tends to bring the particles closer together which would increase the overlap with the alpha-particle wave function. This increase would be, roughly speaking, proportional to the number of states which are available. The number of states however increases with an increasing number of particles outside or holes in closed shells. To give a more quantitative background to these considerations the reduced widths for the decays $Po^{211} \xrightarrow{\alpha} Pb^{207}$ and $Po^{210} \xrightarrow{\alpha} Pb^{206}$ have been computed with the wave functions given by True and Ford[11] and Newby.[12] The main difference between the two decays is that there is no configurational mixing in the Pb^{207} wave function (one hole in a closed shell) if the shell model has any justification at all, but there is an appreciable amount of mixing in the Pb^{206} wave function. The influence of the mixing in the Po^{210} and Po^{211} wave function (protons) drops out in first approximation if one forms the ratio $\delta^2(Po^{211})/\delta^2(Po^{210})$. The result of the calculation is $\delta^2(Po^{211})/\delta^2(Po^{210})=0.340$, whereas the experimental value is $\delta^2(Po^{211})/\delta^2(Po^{210})=0.423$ and the value without mixing 1.18.

A second reason for some discrepancies could be that even by forming ratios of transition probabilities we have not eliminated the errors introduced by the harmonic oscillator wave functions which might differ considerably from more exact single-particle wave functions.

A further reason for discrepancies could be that there is a sort of "clustering" of particles in the nuclear surface which is not even taken into account by the conventional configurational mixing. But we feel that this "clustering" would mainly effect the absolute value of the reduced width, which comes out too small with the wave functions we have used, unless rather unreasonable values for the parameters α and β are chosen.[1] A reason for this "feeling" is that if the shell model gives an adequate description of the nucleus at all, the nuclear wave functions should be approximated reasonably by products of single-particle wave functions inside the nucleus. "Clustering" should be effective in the low density region of the nuclear surface. But the "cluster wave function" has to join smoothly to the shell-model wave function. Therefore, shell-model wave functions should give the relative amplitudes of different "clusters" (for instance, α particles with different angular momenta) to a good approximation. Together with the fact that using oscillator wave functions we have greatly underestimated the magnitude of the single-particle wave functions in the surface region, these considerations would explain why shell-model wave functions give good agreement between theoretical and experimental values for the relative transition probabilities but fail to do so for the absolute values.

We may conclude with a remark about the alpha-particle wave function. Concerning this wave function we feel quite sure, that (II.1) provides a good approximation to the actual wave function. There are experimental[7] as well as some theoretical[13] reasons for this feeling. High-energy electron scattering experiments[7] show clearly that a Gaussian-type charge density is a good approximation to the real charge density.

ACKNOWLEDGMENTS

I am grateful to Professor I. Perlman and Professor J. O. Rasmussen for the hospitality extended to me at the Lawrence Radiation Laboratory and for many helpful discussions.

I also wish to thank the Deutschen Forschungsgemeinschaft for a grant.

[10] P. J. Brussaard and H. A. Tolhoek, Physica **24**, 238 (1958).
[11] W. True and K. Ford, Phys. Rev. **109**, 1675 (1958).
[12] N. Newby, thesis, Indiana University, 1958 (unpublished); N. Newby and E. J. Konopinsky, Phys. Rev. **115**, 343 (1959).
[13] H. J. Mang and W. Wild, Z. Physik **154**, 182 (1959).