Single-Particle States in Deformed Nonlocal Diffuse Boundary Potentials*

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(Received February 24, 1960)

Using the spherical wave functions generated in a previous investigation by Wyatt, Wills, and Green, the influence of spheroidal deformation is examined with the aid of perturbation theory. The combined calculations yield the energies of single-particle states for a diffuse boundary, nonlocal deformed potential. Specific calculations are performed for light nuclei around $A=25$ and in the rare-earth region between $A=150$ and $A=180$. An analysis of nuclear ground-state spins and magnetic moments is presented in terms of the computed level schemes and wave functions. The results confirm the general aspects of the Nilsson, Mottelson results as obtained with adjusted harmonic oscillator potentials although some differences arise in detail. In particular, the calculated coefficients usually show less mixing of different angular momentum states in our case. The fact that the unperturbed potentials used in this calculation were obtained in the study of Wyatt, Wills, and Green from completely independent theoretical and experimental considerations is satisfying and further tends to confirm that the phenomenological model has a strong basis in reality. A discussion of the relationship of the phenomenological model to the self-consistent nuclear model of Brueckner is given.

I. INTRODUCTION

THE successes of the unified picture of the nucleus introduced by Bohr and Mottelson in describing a variety of low-energy nuclear phenomena are now well known.[1] Basically this description of the nucleus combines the collective motion of many particles with the ideas of the strict independent particle motion incorporated in the nuclear shell model. It leads to a unified description of nuclear motion in which the bulk of the nucleons are pictured as being capable of collective vibrational and rotational modes similar to those of a liquid drop in addition to the essentially independent motion of the few loosely bound outer particles. Of particular interest is the case of strongly deformed nuclei for which it can be shown[1] that the vibrational, rotational and single-particle motions take place independently of each other and may be treated separately. This situation arises for nuclei well removed from the closed shells where the neutrons and protons can cooperate to produce a large deformation of the nucleus from a spherical shape, and the total wave function becomes simply a product of three functions describing the two collective modes and the independent motion of the outer nucleons in the average deformed field which the nucleus now effectively presents to these nucleons.

While many of the predictions of the collective model depend only on the existence of the deformed shape, a more detailed appication of the model in the analysis of nuclear properties does also require an explicit calculation of the (so-called intrinsic) wave functions describing the motion of the few outer nucleons in the deformed nuclear field. This problem has attracted much attention in recent years, especially in connection with the application of the collective model to odd A nuclei.[2] In principle such a calculation should proceed via some self-consistent manner, only assuming a knowledge of the fundamental nucleon-nucleon forces, but such a procedure poses immense difficulties and will probably not be carried out for many years to come. Indeed up to now attempts[3] to compute these intrinsic wave functions have employed phenomenological potential wells which while analytically convenient are clearly unrealistic representations of the nucleon-nucleus interaction.

In the present calculation we will also start out with some phenomenological interaction for the average nucleon-nucleus potential. However, taking advantage of advanced numerical computational facilities, the restrictions on our potential form will be considerations of physical plausibility rather than of analytic convenience. Thus, we are both guided and constrained by present experimental and theoretical information from which the predominant features of the nucleon-nucleus interaction may be inferred.

From the wealth of experimental data now available on nuclear scattering and other processes, two main facts emerge: (1) that the average potential felt by a nucleon is dependent on its velocity,[4] and (2) that the

* Portions of this work were carried out at the Oak Ridge National Laboratory and at Los Alamos Scientific Laboratory with the support of the U. S. Atomic Energy Commission. Some computations were carried out at Eglin Air Force Base through arrangements by the Air Force Office of Scientific Research.

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[1] S. A. Moszkowski, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. XXXIX.

[2] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 1, No. 8 (1959).

[3] A list of authors who have studied nucleon motion in a deformed well is given in reference 15.

[4] See for instance A. E. Glassgold, Revs. Modern Phys. **30**, 419 (1958).

average potential has a diffuse boundary at the nuclear surface.[5] The velocity dependence of the nucleon-nucleus interaction also finds theoretical justification and rather definite formulation in the self-consistent field methods for the nuclear many-body problem developed by Brueckner et al.[6] The work of Green and collaborators[7,8] has indicated the necessity of a diffuse edge in connection with the calculation of single-particle binding energies and total energies within the framework of the independent-particle model of the nucleus.

To simulate the velocity dependence we will use an "effective mass approximation" for finite nuclei[9] introduced elsewhere, which does seem to provide a good qualitative indication of the results to be expected from a more elaborate treatment of this effect which essentially arises from the nonlocality of the effective nucleon-nucleus potential in coordinate space.[10] Within the limits of this approximation it turns out that the velocity-dependent potential is not very much more difficult to treat than an ordinary potential, whether it is spherical or not. The introduction of a diffuse edge, however, is much more difficult to handle even in the case of a spherical potential well, and one has to resort to numerical methods of computation. However, it turns out that once the wave functions and level schemes of a given spherical potential are known, it is a fairly straightforward matter to compute the effects of introducing a spherical deformation into this potential field. Extensive numerical calculations of single-particle states and wave functions in velocity-dependent spherical wells having a diffuse edge, using the effective mass approximation mentioned above have previously been made and thus it is now feasible to carry out the additional computations associated with deformation effects.

A summary of the method of calculation of single-particle states in a realistic deformed potential which includes the effects of velocity dependence will be found in the next section. Detailed calculations have been carried out for light nuclei around $A=25$, and in the rare earths region between $A=150$ and $A=180$, i.e., in regions where the existence of strongly deformed nuclear shapes seem to be well established experimentally. The resulting deformed level schemes will be found in Sec. III together with an analysis of the magnetic moments of selected nuclei. A brief discussion of the results in the light of other existing calculations of the same type is presented in Sec. IV.

[5] D. L. Hill, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. XXXIX.
[6] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).
[7] A. E. S. Green, Phys. Rev. **99**, 1410 (1955); A. E. S. Green, K. Lee, and R. Berkley, Phys. Rev. **104**, 1625 (1956).
[8] A. E. S. Green, Revs. Modern Phys. **30**, 569 (1958).
[9] W. E. Frahn and R. H. Lemmer, Nuovo cimento **5**, 1564 (1957).
[10] K. A. Brueckner, *Proceedings of the International Conference on the Nuclear Optical Model* (Florida State University, Tallahassee, 1959), p. 145.

II. SINGLE-PARTICLE MOTION IN A REALISTIC NONSPHERICAL POTENTIAL WELL

In this section we give an outline of the calculations of single-particle binding states and wave functions in a realistic nonspherical force field that is appropriate for strongly deformed nuclei. As usual we assume that the adiabatic approximation holds, i.e., that the intrinsic motion is not appreciably altered by the rotation of the nucleus as a whole. The problem then reduces to a study of nucleon motion in the average nonspherical field $V(\mathbf{r})$ of the nucleus which is regarded as fixed in space. As a further approximation we also ignore all residual interactions and assume that each nucleon moves independently of its neighbors in the collective nuclear field. Actually, this last assumption turns out to be partially justified in the light of recent work on the nuclear many-body problem[6,11] and is also supported by the approximate validity of the extreme shell model[12] and optical model[13] of the nucleus. Hence, one could expect that a good starting point would be to assume a single-particle wave equation like

$$(\mathbf{P}^2/2m_0)u(\mathbf{r})+V(\mathbf{r})u(\mathbf{r})=Eu(\mathbf{r}), \tag{1}$$

for a particle of mass m_0 and energy E moving in the collective field $V(\mathbf{r})$ of the nucleus. However, the self-consistent field methods of Brueckner and collaborators[6,11] which take into account the strong correlations in the nuclear wave functions, show that the extreme independent particle picture represented by the above wave equation is actually only reasonable if $V(\mathbf{r})$ has a nonlocal nature, and the second term on the left-hand side of Eq. (1) must be replaced by the interaction

$$\int V(\mathbf{r},\mathbf{r}')u(\mathbf{r}')d\mathbf{r}', \tag{2}$$

where the integration extends over all space; $V(\mathbf{r},\mathbf{r}')$ is a nonlocal potential which can in principle be self-consistently determined from a knowledge of the individual nuclear two-body forces alone. The immense computational difficulties involved in calculations of this type[10] however make such a procedure unsuitable for the present purposes. Rather we adopt an approximate phenomenological approach and simply assume a physically reasonable form for the nonlocal potential appearing in Eq. (2). Bearing in mind that (i) the interaction represented by (2) must be Hermitian which means that $V(\mathbf{r},\mathbf{r}')$ must be symmetric in its dependence on $\mathbf{r}$ and $\mathbf{r}'$ if it is real, (ii) that the approximate validity of calculations employing local potentials indicates that nonlocal effects should not be too large, and (iii) that Eq. (2) must be translationally invariant

[11] K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).
[12] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (J. Wiley & Sons, Inc., New York, 1955).
[13] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1955).

for infinite nuclear matter, one is lead to the modified wave equation (see reference 9 for details)

$$\frac{1}{4}\left(\frac{1}{2m}\mathbf{P}^2+2\mathbf{P}\cdot\frac{1}{2m}\mathbf{P}+\mathbf{P}^2\frac{1}{2m}\right)u(\mathbf{r})+V(\mathbf{r})u(\mathbf{r})=Eu(\mathbf{r}), \quad (3)$$

in the place of Eq. (1) as a first approximation to describe the nonlocal nature of the nuclear field. Here m denotes a spatially variable effective nuclear mass

$$m=m_0[1-(a^2m_0/2\hbar^2)V(\mathbf{r})]^{-1}, \quad (4)$$

where a is a parameter characterizing the degree of nonlocality in the potential. This parameter is so defined that $a=0$ corresponds to the local case. We take the modified Eq. (3) to be the wave equation describing the single-particle motion and proceed to study how this modification effects the motion of individual nucleus in a realistic nonspherical potential well. The solutions of a similar wave equation have already been extensively studied by Green *et al.*[8,14] for various spherical potentials.

To pursue the case where $V(\mathbf{r})$ is nonspherical we confine ourselves to small deformations of a spheroidal nature only. Accordingly, one may then approximate the potential as[15]

$$V(\mathbf{r})=V(r)-\epsilon(r\partial V/\partial r)P_2(\cos\theta)+\tfrac{1}{2}\epsilon^2(r\partial V/\partial r), \quad (5)$$

where θ is the angle between the particle's radius vector and the nuclear symmetry axis, and ϵ is a measure of the deformation. Following the procedure used by Gottfried,[15] we treat all ϵ-dependent terms which arise when the expression (5) is substituted into Eq. (3) as perturbations, and expand the wave functions u in terms of the basic set of wave functions provided by the solutions of the spherically symmetric wave equation

$$[T_{\rm sym}+V(r)+V_{\rm so}(r)]\psi(nlj\Omega)=E_0(nlj)\psi(nlj\Omega), \quad (6)$$

where $T_{\rm sym}$ is an abbreviation for the symmetrized kinetic energy operator appearing in Eq. (3) and $nlj\Omega$ denote the usual set of radial and angular momentum quantum numbers required to describe a single nucleon state in a spherical potential well.[15] Equation (6) is identical with (3) for $\epsilon=0$ apart from an additional Thomas-type spin-orbit force $V_{\rm so}$ which is necessary if one wishes to obtain a reasonable set of spherical energy levels. We ignore here the influence of deformation and nonlocality on the spin-orbit force since effects of this type are expected to be small and would fall well within the limits of uncertainty involved in the choice of potential shape in (6) anyway.

By substituting the last two terms in the expression (5) into the wave equation (3) we obtain the perturbation interaction

$$U=U_1+U_2, \quad (7)$$

with

$$U_1=-\epsilon(rV')P_2+\tfrac{1}{2}\epsilon^2(rV'),$$

and

$$U_2=-\tfrac{1}{16}\epsilon a^2[(rV')P_2\nabla^2+2\nabla\cdot(rV')P_2\nabla+\nabla^2(rV')P_2],$$

where the primes denote differentiation with respect to r. Our perturbed eigenvalue problem then reads

$$(H_0+U)u=Eu, \quad (8)$$

for the energies E and intrinsic wave functions u in the deformed field, where H_0 denotes the spherical Hamiltonian in Eq. (6). Expanding u in terms of the complete set of eigenfunctions $\psi(nlj\Omega)$ of H_0 which belong to the same projection Ω of the particle's angular momentum along the nuclear symmetry axis and have the same parity w since only these quantum numbers are conserved when all perturbations are included, Eq. (8) reduces to the matrix eigenvalue problem

$$\sum_{nlj} C(q,\Omega w;nlj)[E_0(nlj)-E(q,\Omega w)+(n'l'j'\Omega|U|nlj\Omega)]=0, \quad (9)$$

for the energies $E(q,\Omega w)$, and eigenvectors $C(q,\Omega,w;nlj)$ which are at the same time the expansion coefficients for the wave functions u in terms of the $\psi(nlj\Omega)$. The additional quantum number q in the above expressions serves to label the different roots of the matrix (9) belonging to the same Ω and w. A deformed state is then specified by the set (q,Ω,w); due to the assumed axial symmetry in our potential a degeneracy with respect to the sign of Ω in each state still remains.

Now, in principle, the summations in (9) should also include integrations over the continuous positive energy spectrum of H_0, so that the matrices (9) are formally of infinite order. The complete diagonalization of this system of matrices is not possible in practice, and considerable simplifications must be introduced to make the procedure a practical one. So, with Gottfried,[15] we first delete the entire continuous spectrum of H_0 so that the matrix elements of U in (9) refer to bound-state wave functions only. One expects the errors introduced by this approximation to be confined mostly to the particle states lying near the top of the potential well. Further simplifications result from the selection rules for the matrix elements of U appearing in (9). As U is an even operator in space, no parity change is introduced by it, and only even or odd values of l can appear in the summations in (9). Since the eigenvalues of (6) fall into definite shells of even or odd parity well separated from each other in energy, the operation of the parity selection rule makes it sufficient to diagonalize the matrices (9) in the subspace spanned by only those wave functions of H_0 which have the same parity and which belong to the same major shell. The next shell of states having the correct parity to contribute is energetically then too distant to have a

[14] A. E. S. Green and P. J. Wyatt (to be published).
[15] K. Gottfried, Phys. Rev. 103, 1017 (1956).

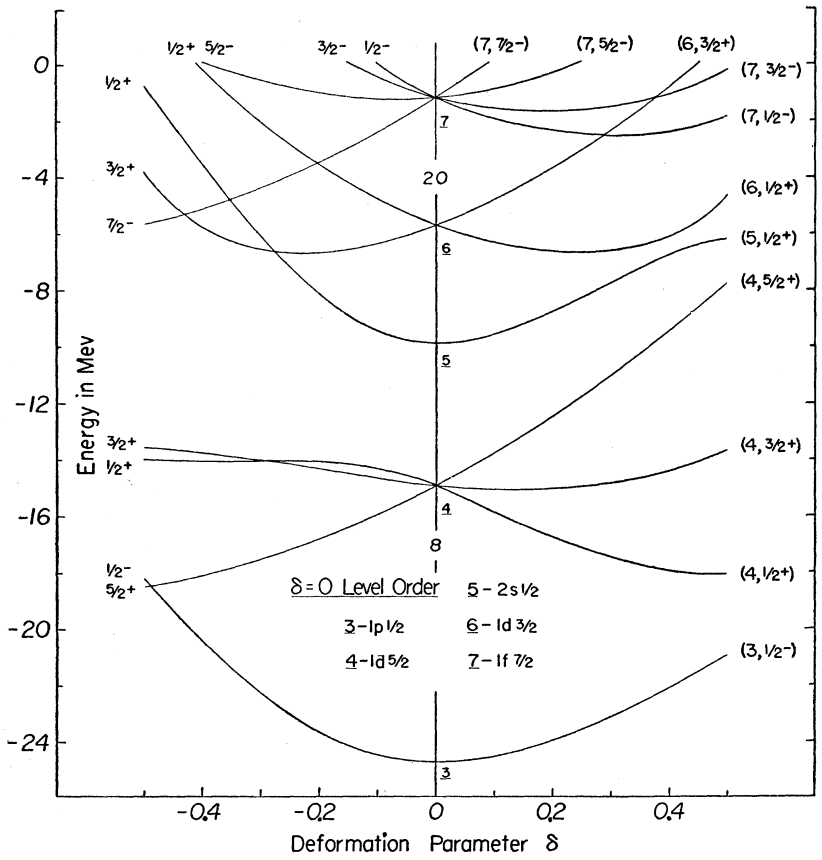

FIG. 1. Energy levels for protons and neutrons from $N=8$ to $N=20$ as a function of the nuclear deformation. Each state is labeled by the quantum numbers $(q,\Omega w)$ defined in the text.

large effect and the contributions of all such distant states can be neglected in the first order.

In setting up the appropriate matrices for the deformed field it is convenient to break the basic wave functions $\psi(nlj\Omega)$ down into radial and angular parts by writing

$$\psi(nlj\Omega)=(1/r)G(nlj)i_{lj\Omega}, \tag{10}$$

where the Y's are the well-known eigenfunctions of j^2 and j_z and the G's can be numerically determined once the shape of the spherical potential V (r) in (6) is defined. The matrices for U can then be reduced to simple expressions involving the product of terms depending only on the radial wave functions G and the known[16] matrix elements of $P_2(\cos\theta)$. After some reduction one finds

$$(n'l'j'\Omega|\,U_2\,|nlj\Omega)=-\epsilon\{I_1-\tfrac{1}{4}a^2[I_2+I_3+f(l',l)I_4]\}\times(l'j'\Omega|\,P_2\,|lj\Omega)+\epsilon^2 I_1\delta_{n'n}\delta_{l'l}\delta_{j'j}, \tag{11}$$

where

$$\begin{aligned} f(l',l)&=(l+1)(l+2), && l'=l+2,\\ &=l(l+1)-1, && l'=l,\\ &=l(l-1), && l'=l-2, \end{aligned}$$

and the I's denote the radial matrix elements

$$\begin{aligned} I_1&=(G(\alpha'),rV'G(\alpha)), \quad \alpha=(nlj)\\ I_2&=(dG(\alpha')/dr,rV'dG(\alpha)/dr),\\ I_3&=-(G(\alpha'),rV'''G(\alpha)),\\ I_4&=(G(\alpha'),(1/r)V'G(\alpha)). \end{aligned} \tag{12}$$

It is clear that the values of these matrix elements are very sensitive to the behavior of the radial functions G and the behavior of the potential at the nuclear surface and hence the importance of using realistic nuclear potentials would appear obvious. Since the integrations involved in (12) are confined to the surface, the task of carrying out the indicated integrations numerically for a realistic well shape is considerably reduced.

Further evaluation of (12) now depends on the values of the various nuclear parameters (well depth, nuclear radius constant, etc.) defining the well shape chosen for $V(r)$, which in turn determines the functions G. These parameters have been found approximately in the work of Wyatt, Wills, and Green for a well shape very similar to a Woods-Saxon potential (see reference 14 for details) by looking at both bound and scattering states for spherical nuclei. We will assume that their values are appropriate for our problem too although

[16] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 27, No. 16 (1953).

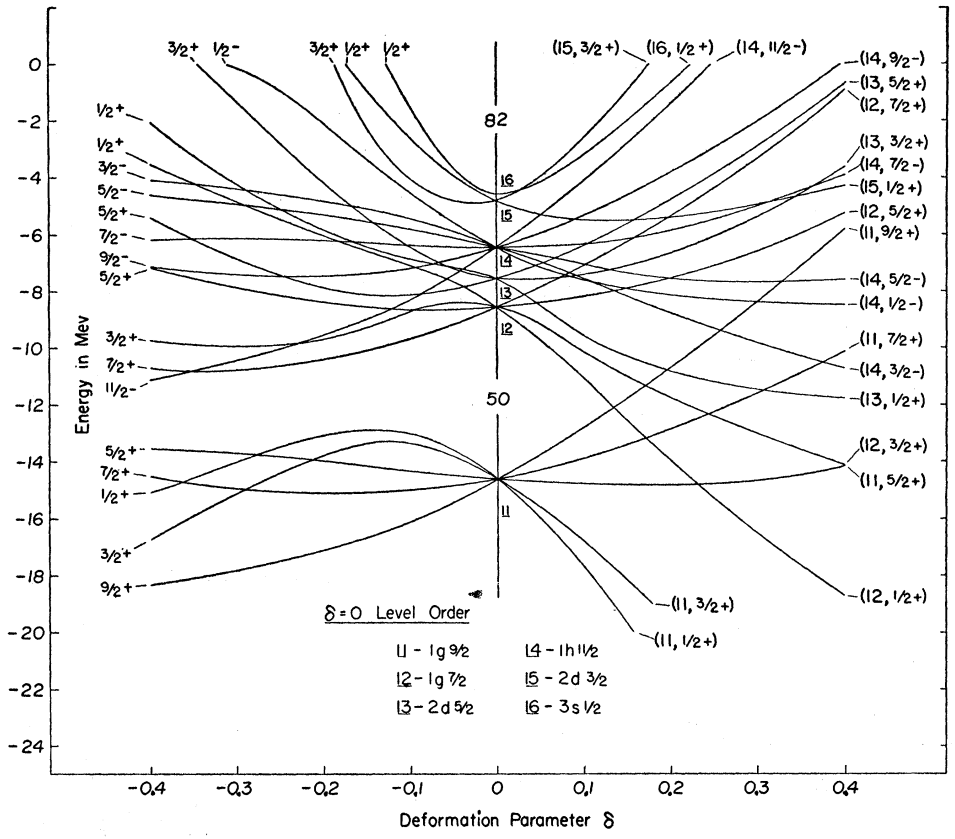

FIG. 2. Energy levels for protons from $A=50$ to $A=82$ as a function of the nuclear deformation. The quantum numbers $(q,\Omega w)$ are given on the right for each state and the connection between q and the relevant spherical levels is shown in the inset.

this need not necessarily be so. This is no serious restriction, however, since it turns out that most of the results of the deformed calculation are primarily dependent only on the over-all shape of the radial wave functions which in turn is rather insensitive to the exact well parameters used in their calculation, as long as the general features of the well are retained.

Knowing the radial integrals (12) and spherical energy levels $E_0(nlj)$, it is a simple matter to solve the eigenvalue equation numerically for various values of the deformation parameter. The results of such diagonalizations are shown in Figs. 1–3 where we give the resulting energy level schemes for light and heavy nuclei plotted as a function of the more usual deformation parameter $\delta=0.67\epsilon$. This is used rather than ϵ to allow a direct comparison with other existing calculations,[3] especially those of Nilsson.[17] The quantum number q defined previously in connection with the designation of the deformed levels is given along the vertical axis at $\delta=0$ in each figure and serves to indicate directly the spherical state from which each deformed level originates. The bracketed numbers labeling each state give the set of quantum numbers (q,Ω,w) appropriate to that state. A table of the coefficients C is also available but is too lengthy to reproduce here.[18] In these calculations we have used the Green-Wyatt well shape with depth $V_0=70$ Mev, surface diffuseness $d=2.86$ fermi and a nuclear radius $R=1.20A^{\frac{1}{3}}$ fermi. The nonlocality parameter was fixed at $a=0.82$ fermi, giving a mass reduction of 0.54 in the nucleus [see Eq. (4)], in good agreement with other estimates.[19]

Figure 1 refers to a nucleus with radius corresponding to $A=29$. However, by repeating the calculation for another value of A (i.e., changing the radius parameter R) one can readily see that the main features of the level scheme are preserved, there being only a slight shift in the over-all position of each level. This statement is even more true for the expansion coefficients $C(q,\Omega,w)$ which hardly change at all when the positions of the spherical levels are altered in this way. Thus one can suppose that our level scheme will be valid for range of nuclei in this region. Furthermore, since the Coulomb interaction in light nuclei is small, Fig. 1 can be expected to apply to both neutron and proton states. The deformed level scheme plotted in Fig. 2 is based on the spherical wave functions generated by the Wyatt-Wills-Green code for protons with parameter

[17] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 29, No. 16 (1955).

[18] Tables will be available on microfilm.

[19] See, for instance, K. A. Brueckner, Phys. Rev. 97, 1353 (1955). Note that fixing the mass reduction from Eq. (4) at $r=0$ only determines the combination a^2V_0, but it is only this combination of parameters that enters into the nonlocal perturbation.

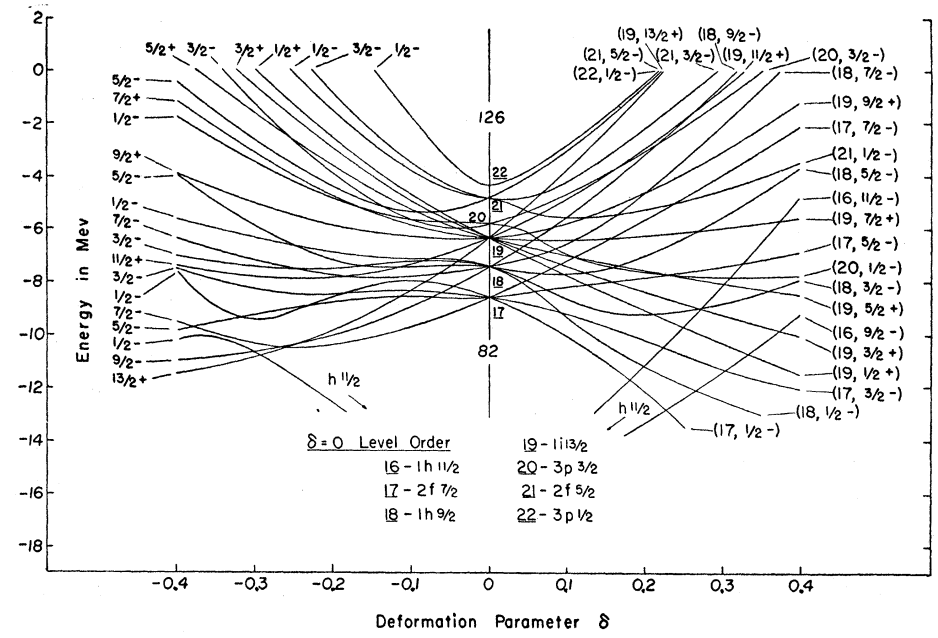

FIG. 3. Energy levels for neutrons from $N=82$ to $N=126$ as a function of the nuclear deformation. The notation is the same as in Fig. 2.

values as above except for Coulomb corrections, for a well radius corresponding to $A=177$. Figure 3 shows a similar computation for neutron states at the same mass number.

III. APPLICATIONS

A comparison of the level schemes presented in the previous section with the diagrams computed by Nilsson using an oscillator potential for the nuclear field shows a very similar behavior in the energy levels as a function of deformation. This might have been expected since the main features of any deformed level scheme will depend primarily on the spherical level ordering. In Nilsson's case this level order is adjusted to reproduce empirical single-particle levels as well as possible. In the present calculation it is determined directly as a result of choosing well parameters which reproduce scattering and boundstate data as well as possible.[14]

Mottelson and Nilsson[2] have recently made an intensive analysis of the ground-state properties of deformed odd A nuclei in terms of the Nilsson level scheme. By way of application we present here a parallel analysis of ground-state spins and magnetic moments based on the present calculations.

Assuming that the nuclear spin is determined by the spin of the odd particle, it is clear that the deformed states to which the odd nucleon is assigned, in our case on the basis of measured nuclear spin, will almost invariably coincide with the Mottelson-Nilsson assignments. This is to be expected, since the procedure involved here only requires that a deformed state of the appropriate spin be available in the vicinity of the observed nuclear deformation at around the correct mass number. A better indication of any improvements introduced by the more realistic calculation must come from comparing quantities like the predicted magnetic moments with other similar estimates and experiment. This is especially the case with magnetic moments since these quantities only involve the comparison coefficients $C(q,\Omega w)$ which are sensitive to the main features of the potential alone and are not very dependent on the exact parameters of the well. We will not attempt to calculate any collective features like the effective moments of inertia or equilibrium deformations, since such calculations always involve contributions from many particles and the actual level spacing does become important. We do not consider the energy values given in the previous section to be unique enough to make such calculations (which are very tedious to perform) significant.

The magnetic moments of the coupled system divide up into contributions from the collective motion of the nucleons as a whole and from the odd extra particle, and may be expressed as (see reference 1 for details)

$$\boldsymbol{\mu}=g_\Omega\boldsymbol{\Omega}+g_R\mathbf{R}, \tag{13}$$

where g_Ω and g_R are the gyromagnetic ratios associated with the intrinsic and rotational motion, and $\mathbf{R}$ is the angular momentum of rotation. Both g_Ω and g_R depend on details of the intrinsic motion, but especially the former, which may be calculated directly in terms of the expansion coefficients; useful formulas have been given by Gottfried.[15] The g_R on the other hand involve the cooperation of many particles, so we will accept these factors as empirically determined parameters for different nuclei.

Some calculations for deformed odd A nuclei based

TABLE I. Calculated values of intrinsic g factors and magnetic moments for odd A nuclei lying in the region $A=151$ to $A=179$. The ground state spin I_0, measured equilibrium deformations δ and the deformed state occupied by the odd particle according to Figs. 2 and 3 are also given. For comparison the second last column lists the Gottfried values μ_G for the magnetic moment.[a] The empirical data on g_Ω (which can have two possible values unless the sign of $(g_\Omega - g_R)$ is also known) comes from Adler et al.[b] as do the values for I_0 and δ. The experimental values for the μ_{exp} are taken from a recent review article.[c] We take $g_R=0.4$ throughout and the intrinsic magnet moments for the neutron and proton equal to that appropriate for the free nucleon.

Nucleus	I_0	δ	$(q,\Omega w)$	g_Ω	g_Ω, exp	μ	μ_G	μ_{exp}
$_{63}Eu^{151}$	5/2	0.16	14, 5/2−	1.53	?	3.02	3.15	3.4
$_{63}Eu^{153}$	5/2	0.30	12, 5/2+	0.36	0.6, 0.5	0.92	0.69	1.5
$_{65}Tb^{159}$	3/2	0.31	13, 3/2+	2.50	1.6, 0.4	2.50	2.34	1.5 ± 0.4
$_{67}Ho^{165}$	7/2	0.30	14, 7/2−	1.32	1.1, 0.8	3.91	?	3.29± 0.17
$_{71}Lu^{175}$	7/2	0.28	12, 7/2+	9.40	0.9, 9.6	1.41	4.5	1.7
$_{73}Ta^{181}$	7/2	0.23	12, 7/2+	0.39	9.7 only	1.36	1.46	1.9
$_{75}Re^{185}$	5/2	0.19	13, 5/2+	1.90	1.6, 0.9	3.66	3.22	3.14
Gd_{91}^{155}	3/2	0.31	18, 3/2−	−0.71	−0.5, 0.1	−0.48	−0.07	−0.31
Gd_{93}^{157}	3/2	0.31	18, 3/2−	−0.71	0.6, 0.1	−0.48	−0.71	−0.37± 0.4
Dy_{95}^{161}	5/2	0.31	19, 5/2+	−0.30	?	−0.50	?	±0.38± 0.05
Er_{99}^{167}	7/2	0.29	19, 7/2+	−0.30	?	−0.49	−0.95	±0.50+−0.12
Yb_{101}^{171}	1/2	0.29	20, 1/2− 21, 1/2−			0.79 0.67	0.62	0.43± 0.05
Yb_{103}^{173}	5/2	0.29	18, 5/2−	−0.71	?	−0.98	−0.87	−0.67± 0.01
Hf_{105}^{177}	7/2	0.26	17, 7/2−	0.39	0.1, 0.2	1.39	?	0.61± 0.03
Hf_{107}^{179}	9/2	0.27	19, 9/2+	−0.3	−0.2, 0	−0.52	?	0.47± 0.03

[a] See reference 15.
[b] See reference 20.
[c] G. Laukien, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. XXXVIII, Part I, p. 338.

on our wave functions are summarized in Table I. For completeness, we list the observed nuclear spin I_0, empirical deformation δ and the possible state of the odd nucleon consistent with this data according to our level schemes. Computed values of the gyromagnetic ratios g_Ω and magnetic moments μ are compared with available experimental values,[20] taking $g=0.4$ throughout and intrinsic nucleon moments appropriate to the free nucleon. Gottfried's estimates μ_G for magnetic moments based on a local square well potential are also included for comparison.

One notes that a satisfactory agreement results in most cases especially as far as the intrinsic g factors are concerned. Inspection of the table of coefficients[18] shows that, for heavy nuclei, the C's vary slowly and smoothly with deformation, so that the results in Table I are insensitive to the exact value assumed for the nuclear deformation in each case.

Evidence for rotational spectra in some light nuclei around $A=25$ have led several recent attempts[21] to apply the strong coupling to this mass number region also, with limited success. By way of illustration we give the decoupling parameter[1] a as a function of deformation in Fig. 4 for a nucleon occupying the $(5, \frac{1}{2}+)$ state (see Fig. 1), which is appropriate for the ground state of Si^{29}, or an excited state of Al^{25}. The measured values of a indicate equilibrium deformation of -0.13 and $+0.4$ for these two nuclei in agreement with previous estimates.[21] However, the rapid variation of the C's with deformation (and hence any quantities dependent on them) make the quantitative significance of such calculations very doubtful in this region, in contract with the situation in heavy nuclei.

IV. DISCUSSION

Before proceeding with a general discussion of the results obtained here, it is useful to review the various approximations made in reducing the original problem to a feasible one, and to indicate the probable shortcomings of the approximations that have been introduced. Since the treatment of the deformed potential depends almost entirely on the solutions of the corresponding spherical well, we discuss this first.

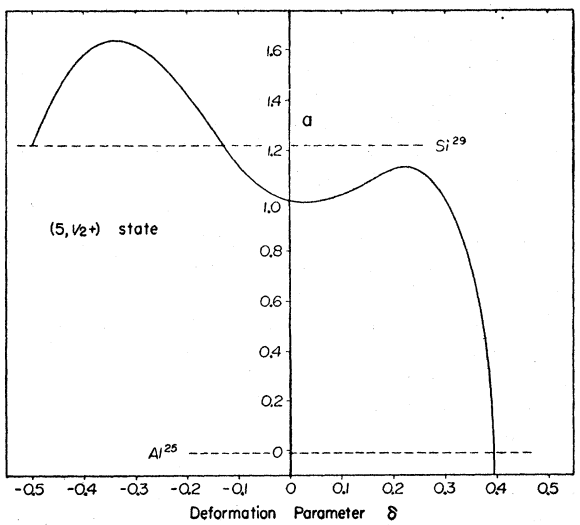

FIG. 4. Decoupling parameter as a function of deformation for the $(5, \frac{1}{2}+)$ state. The experimental values for Si^{29} and an excited state of Al^{25} are indicated.

[20] K. Alder et al., Revs. Modern Phys. 28, 432 (1956).
[21] D. A. Bromley et al., Can. J. Phys. 35, 1042 (1957); G. Rakavy, Nuclear Phys. 4, 375 (1957).

Apart from a more realistic well shape, the principal modification that has been introduced here is a velocity dependence of the average nuclear field. Our argument in this respect was to replace the self-consistent nonlocal potential required by many-body theory by a nonlocal potential which satisfies reasonable physical requirements, but which is in no way self-consistent.[9] By doing so, however, we are able to replace the integro-differential equation for single-particle motion which follows from (1) and (2), by the much simpler differential equation (3) within the limits of applicability of the effective mass approximation. In any event, one can argue that the introduction of an "effective mass" concept for a finite nuclear system to parallel the situation in infinite nuclear matter would have to proceed via some fully symmetrized kinetic energy operator like that in (3) on the basis of Hermiticity requirements alone.

From a semiquantitative point of view our nonlocal interaction appears to be quite reasonable. It can be shown directly[9] that the effective mass equation (3) is essentially equivalent to an additional interaction proportional to the particle's orbital angular momentum $\mathbf{l}^2$, which tends to weaken the effective nuclear potential for higher angular momentum states (see Appendix). This result is independent of the radial shape of $V(r)$ and is a direct manifestation of what is usually referred to as the velocity dependence of the potential in nuclear matter. An effect of this type also emerges clearly from the recent fully self-consistent calculations carried out by Brueckner.[10] In addition, the presence of nonlocal interaction implies a much deeper well than for the corresponding local potential in order to bind the same number of particles and (in view of the above remarks) is distinguishable from the effects coming from the surface diffuseness, which generally influences states with a high radial quantum number and low angular momentum[7] more strongly.

The assumptions made in connection with the deformation perturbation in Sec. II are the usual ones to make[15] to reduce a formidable computational problem to a form from which useful results may be obtained without excessive labor. As stated in the text, there is no way to estimate the error introduced by neglecting the continuum states in the perturbation procedure without explicit knowledge of the continuum wave functions which are not available in any case, so this uncertainty must remain. However, the neglect of the coupling between different shells of the same parity can be shown by actual computation to be a good approximation, so that no appreciable errors come from this assumption.

Furthermore, we have performed calculations for only one nuclear radius in the vicinity of nuclei in which we were most interested. This is an obvious computational limitation, since it is not possible to express our deformed energy levels as a function of mass number, a feature which makes the oscillator potential so useful in work of this nature. However, small variations in R do not effect the spherical wave functions much, and only cause a common shift in the positions of the spherical energy levels. The small change in the spherical wave functions means that the expansion coefficients $C(q,\Omega w)$ hardly change at all in moving through a range of mass numbers as in the rare earths regions for example. Thus, we expect the coefficients to be much more reliable than the actual positions of calculated energy levels at a given deformation, and for this reason have not looked at nuclear properties which depend explicitly on level spacings. Any agreement with experiment found in such computations would not have too much significance since a relatively small change in the positions of the spherical levels will have a corresponding influence upon the relative positions of the deformed states. For similar reasons we have not attempted to calculate equilibrium deformations in the realistic model, since apart from the formidable task of evaluating single-particle kinetic and potential energies in each occupied state for a variety of deformations, the residual uncertainty introduced by picking a constant radius parameter makes the results of such a computation uninteresting.

A comparison of our orbit assignments given in Table I for the odd particle state in both odd proton and odd neutron nuclei shows no significant differences from or alternatives for the Mottelson-Nilsson assignments.[2] For the reasons given in the text this is an expected result. On the other hand, the calculation of magnetic moments in terms of our wave functions does have an independent significance. The determining factor here is the expansion coefficients themselves which are insensitive to the exact location of the energy states, but directly involve the structure of the radial wave functions G in Eq. (10). These *do* differ considerably from the corresponding oscillator wave functions. It is possible that the main improvement obtained by using a realistic potential is that our wave functions represent a much better picture of the situation for a real nucleus than what oscillator functions are able to do. We observe in this connection that the calculated coefficients[18] usually show considerably less "mixing" of different angular momentum states in our case than the oscillator potential leads one to expect.[17] The reason for this is twofold: (i) Our energy levels are on the average further apart energy-wise due to the nonlocal interaction than in a local well, and (ii) the surface region for the realistic potential is smaller than for an oscillator potential (which is all surface), and the coupling of the nucleons to the nuclear surface is reduced accordingly. To illustrate this, Table II shows the average of rV' (which determines the surface coupling) for several states in a realistic well, and the corresponding values for a nonlocal oscillator potential of the same rms radius. One notes that there is always a reduction, amounting to about 15% for the deeper states. In contrast, the direct effect of the nonlocal

TABLE II. Values of the radial integrals I_1 for neutron states in a realistic well, and in a nonlocal oscillator potential of the same rms radius. We take $\hbar w^* = 82A^{-\frac{1}{3}}$ Mev and $A=177$ for the nonlocal oscillator level spacing.[a]

State	I_1 (Mev)	$I_{1,\,\mathrm{osc}} = (N_0+\frac{3}{2})\hbar w^*$ (Mev)
$1i_{13/2}$	104	109, $N_0=6$
$3p_{1/2}$	75.9	
$3p_{3/2}$	73.9	
$2f_{5/2}$	80.9	94.8 $N_0=5$
$2f_{7/2}$	77.3	
$1h_{9/2}$	82.5	
$1h_{11/2}$	91.2	
$3s_{1/2}$	68.6	
$2d_{3/2}$	70.1	
$2d_{5/2}$	69.2	80.2 $N_0=4$
$1g_{7/2}$	65.3	
$1g_{9/2}$	75.2	

[a] See reference 9.

interaction through the terms proportional to ϵa^2 in Eq. (11) turn out to be very small.

While the satisfactory values obtained for the intrinsic g factors tend to support the point of view that our wave functions are rather good, one must bear in mind that the formulas used in our computations have some theoretical uncertainties.[22] To resolve some of these uncertainties, it would be helpful at this stage to investigate for instance electric dipole transitions in heavy deformed nuclei on the basis of these wave functions.[23]

In summary it appears that the main features of the Nilsson level scheme and the qualitative conclusions based on it, are retained in a more realistic treatment of the average force field governing the independent motion of the nucleons. One observes, however, that the present level schemes have been generated in a rather systematic way, and are at least understandable in physical terms in that the radius, diffuseness and nonlocality, etc., all fall within ranges of values which are reasonable from other independent experimental and theoretical investigations. In this sense there has been no "fitting" of empirical data in any way in the present investigation. We have simply accepted the results based on an attempted unification[14] of bound and scattering states in terms of a realistic potential well, and have pursued the effects of spheroidally deforming this potential. The good over-all agreement with empirical data certainly indicates that the realistic potential used as the basis of these calculations is capable of producing reasonable results within the independent particle picture of the nucleus.

[22] F. Villars, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1957), Vol. 7, p. 185.

[23] D. Strominger, University of California Radiation Laboratory Report UCRL-3374, 1958 (unpublished).

APPENDIX

We present a brief discussion of the structure of the nonlocal potential used as the basis of the present work in this section and indicate a semiquantitative comparison with the nonlocal potential following from Brueckner's calculations. Since $V(\mathbf{r},\mathbf{r}')$ in Eq. (2) is a matrix in coordinate space, it is simpler to integrate over one coordinate first and regard the resulting function of the remaining coordinate as representing in some way an "effective potential" for a single nucleon. To do so, we first split off the angular dependence by writing

$$V(\mathbf{r},\mathbf{r}')=\sum_{l=0}\frac{2l+1}{4}\frac{k_l(r,r')}{rr'}P_l(\mathbf{r},\mathbf{r}'). \tag{14}$$

The function $k_l(r,r')$ may be found by inversion of (14) if $V(\mathbf{r},\mathbf{r}')$ is known. Taking the form[9]

$$V(\mathbf{r},\mathbf{r}')=V((\mathbf{r}+\mathbf{r}')/2)\delta_a(\mathbf{r}-\mathbf{r}'), \tag{15}$$

with $\delta_a(\mathbf{r})=(\pi^{\frac{1}{2}}a)^{-3}\exp(-r^2/a^2)$, one finds

$$k_l(r,r')=4\pi rr'V(r)(\pi^{\frac{1}{2}}a)^{-3} \times\exp[-(r^2+r'^2)/a^2](-1)^l j_l(2irr'/a^2), \tag{16}$$

where j_l is a spherical Bessel function. The parameter a is then identical with the a in Eq. (3). The expression (16) is approximate in that we assume $V(r)$ is a relatively slowly varying function of r over a distance a. Then we may calculate the effective potential

$$f_l(r)=\int_0^\infty k_l(r,r')dr', \tag{17}$$

which is also given in Brueckner's work.[10] Evaluating (17) when k_l is given by (16), one finds

$$f_l(r)=V(r)\frac{\Gamma((l+2)/2)}{\Gamma(l+\frac{3}{2})}\left(\frac{r}{a}\right)^{l+1} \times F\left(\frac{l+1}{2};l+\frac{3}{2};-\frac{r^2}{a^2}\right), \tag{18}$$

where F is a confluent hypergeometric function.[24] For $l=0$ one has

$$f_0(r)=V(r)\Phi(r/a), \tag{19}$$

where Φ is the error function[24] but for larger l values we have not been able to express f_l in terms of tabulated functions. However, its general behavior may easily be established by examining the limiting cases when (r/a) is much smaller and much larger than unity. In the first case

$$f_l(r)=\frac{\Gamma((l+2)/2)}{\Gamma(l+\frac{3}{2})}V(r)\left(\frac{r}{a}\right)^{l+1}, \quad r\ll a, \tag{20}$$

[24] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954).

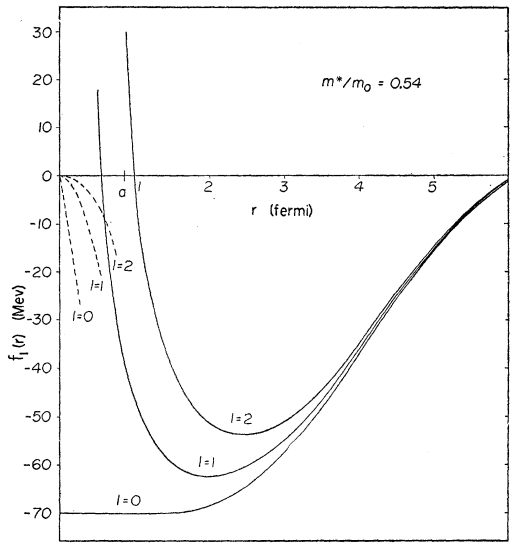

FIG. 5. The effective mass approximation function $f_l(r)$ for different angular momentum states, using the G. W. well shape and parameters, reference 14. The broken curves show the exact behavior near the origin.

so that f_l approaches the origin with increasing curvature as the angular momentum increases. The other limit is

$$f_l(r)=V(r)[1-\tfrac{1}{4}a^2l(l+1)/r^2], \quad r\gg a, \qquad (21)$$

which is identical with the expression which results if we evaluate f_l directly from the effective mass equation (3) [apart from a small term involving the second derivative of V which would also have appeared in (21) had the integration leading to (16) been carried out exactly]. The important point is that, while the small r behavior will be very sensitive to the detailed structure of δ_a in (15), the large r behavior given by the expression (18) is not, and supports our original point of view that the expanded form of the nonlocal interaction [which leads to the effective mass equation (3)] is the most meaningful approach to use in the present case. Now one also sees more clearly just what approximation the "effective mass approximation" leading to Eq. (3) introduces. In this approximation the exact form of the function f_l (which always goes to zero at the origin) is replaced by the approximate form (21) for all r. However, the singularity thus introduced at the origin is not too serious as far as an estimate of the energy shifts due to the nonlocal interaction is concerned, since the radial wave functions vanish strongly in this region.

The behavior of f_l as given by (21) is shown in Fig. 5 for several angular momentum states, using the Green-Wyatt well shape and parameters[14] for $V(r)$, and $a=0.82$ fermi which corresponds to a mass reduction of 0.54. The broken curves indicate the actual behavior f_l should have near the origin [Eq. (20)].

Comparison with Brueckner's results[10] for Ca^{41} shows that our f_l has the same qualitative behavior as shown by the self-consistent calculation of this quantity, especially with respect to the angular momentum dependence of the effective potential which is exhibited by (21) in a very simple way. Brueckner also points out that the small r behavior of the self-consistently determined f_l is sensitive to many details of the two-body force used in its computation, but that the behavior for large r is mainly determined by the exchange nature of these forces. Since the large r behavior of f_l given by (21) agrees qualitatively with the exact calculations, it is tempting to speculate that our calculations describe in some approximate way the nonlocality in the average nuclear potential arising from this aspect of the two-body force only. This conclusion is exactly in the spirit of the effective mass approximation, which has been shown to be independent of the detailed structure of our nonlocal potential.

ACKNOWLEDGMENTS

The programming and numerical calculations associated with this investigation were begun at Oak Ridge National Laboratory. The kind assistance of Dr. H. C. Griffith and Dr. T. Hildebrandt during this period is gratefully acknowledged. Recoding the problem for the IBM 704 and modifying it to yield wave functions and other numerical outputs necessary for the spheroidal nucleus problem were carried out at Los Alamos Scientific Laboratory. The help of John Wills and Dr. P. J. Wyatt was indispensable to this phase of the effort. Most of the remainder of the work was carried out at the Florida State University with the kind support of a grant from the U. S. Atomic Energy Commission. During this latter period, some calculations were carried out on the IBM 704 at Eglin Air Force Base through arrangement made by the Air Force Office of Scientific Research. Ulmer Stabler was particularly helpful at this phase. The matrix element evaluations and matrix diagonalization were finally completed on the Florida State University IBM 650 with the kind guidance and assistance of Dr. H. C. Griffith.