Quantum Mechanical Transport Theory. I. Incoherent Processes

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The transport of particles through a scattering medium is studied. A generalization of a technique due to Placzek and Wick is used to handle sums over states of excitation of the medium. The collision processes which occur are classified as "inelastic," "elastic," and "quasi-elastic" and correspond to different orderings of the Placzek-Wick series. The inelastic scatterings are described by an essentially classical transport equation and the elastic scatterings by assigning a refractive index to the medium. The "quasi-elastic" scattering involves the excitation of low-lying states of the scattering system. The coherent interference of waves scattered from nearby scatterers is important in this case and depends upon the structure of the medium. In this paper the general theory is developed in terms of a systematic sequence of approximations, of which the first gives just the classical form of transport theory. The correction terms then appear as quantum-mechanical corrections to the classical transport problem.

I. INTRODUCTION

WE consider the transport of particles through a medium of scatterers under conditions such that a quantum mechanical treatment is required. It is assumed that the density of scattered particles is sufficiently low that their mutual interactions may be neglected. Thus we need consider only the interaction of the scattered particles with the medium. We shall simplify the problem by assuming that the energy of the *scattered* particles is large compared to the binding energy of the *scattering* particles (hereafter, referred to as "scatterers") within the medium. Also, the scattering mean-free-path within the medium is supposed large compared to the de Broglie wavelength of the scattered particles.

Under these conditions the discussion of multiple interactions proves to be relatively simple if the properties of the medium are understood. In this connection, sums over states of excitation of the medium are handled by a generalization of the technique of Placzek[1] and Wick.[2]

Because the "orbits" of the scattered particles are described by waves, it is necessary to distinguish waves which interfere with each other (that is, are coherent) from those which do not (are incoherent). For this purpose we classify the scattering as of three kinds: First, inelastic (and incoherent) scattering is described by a "classical" transport equation. Elastic scattering is described by assigning a refractive index to the medium (more generally, an optical model potential). Finally, we call scattering "quasi-elastic" if only states of the medium having very low energy are excited. Each of these types of scattering will be discussed qualitatively by means of simple models in this section—the general theory being given in later sections.

As a particular example, we have in mind the scattering of fast particles by atomic nuclei. Then the inelastic scattering will be found to lead to a generalization of the Goldberger transport theory.[3] The optical model has been used frequently for describing elastic scattering. Our detailed handling of quasi-elastic scattering will be given in Part II.

A. Scattering Medium

The medium in which the particles are scattered consists of N identical "scatterers."[4] (We shall refer to the "*scattering particles*" as "scatterers" to distinguish them from the *scattered particles*, which will be called just "particles.") The eigenstates of the medium are described by a set "γ" of quantum numbers. The eigenenergies will be written as W_γ and the corresponding eigenfunctions as g_γ.[5] Before scattering has occurred, we shall suppose the medium to be in its lowest state $\gamma=0$.[6] We assume N to be a large number and that the scattering medium occupies a volume $\mathcal{V}$ large compared to the range of the force between particle and scatterer.

If $\mathbf{Z}_\alpha(\alpha=1, 2, \cdots N)$ represents the position vector of the αth scatterer, then

$$P(\mathbf{Z}_\alpha)\equiv \int |g_0|^2 \prod_{\beta\neq\alpha} d^3Z_\beta \tag{1}$$

is the probability of finding particle α at $\mathbf{Z}_\alpha$. [If the scatterers have spins, a sum over spin states is implied in Eq. (1). In the interest of keeping our notation simple we shall not explicitly write such spin sums.] The density of scatterers is

$$\rho(\mathbf{r})=NP(\mathbf{r}). \tag{1a}$$

It will be assumed that $\rho(r)\simeq$constant within the medium,[7] where

$$P(r)=\mathcal{O}(1/\mathcal{V}).$$

[1] G. Placzek, Phys. Rev. **86**, 377 (1952).
[2] G. C. Wick, Phys. Rev. **94**, 1228 (1954).
[3] M. L. Goldberger, Phys. Rev. **74**, 1269 (1948).
[4] Our results are easily extended to systems containing several kinds of scatterers.
[5] K. M. Watson, Phys. Rev. **105**, 1388 (1957). The notation developed here will be followed in our present discussion.
[6] Actually, there is no difficulty in the development of our theory if we consider $\gamma=0$ to be an arbitrary state and eventually average over a statistical ensemble of states "$\gamma=0$."
[7] Strictly speaking, for our later applications it is necessary to assume only that $\rho\simeq$constant over distances large compared with

Indeed, when ρ=constant within the medium,

$$P(r)=1/\mathcal{V}, \quad (2)$$

for $\mathbf{r}$ within $\mathcal{V}$. Otherwise $P(r)=0$.

The joint probability of finding scatterer α at $\mathbf{Z}_\alpha$ and scatterer β at $\mathbf{Z}_\beta$ is

$$P(\mathbf{Z}_\alpha,\mathbf{Z}_\beta)=\int |g_0|^2 \prod_{\mu(\neq\alpha,\beta)=1}^{N} d^3Z_\mu. \quad (3)$$

This may be re-written in terms of the "pair correlation" function $G(x)$ as

$$P(\mathbf{Z}_\alpha,\mathbf{Z}_\beta)\equiv P(\mathbf{Z}_\alpha)P(\mathbf{Z}_\beta)[1+G(\mathbf{Z}_\alpha-\mathbf{Z}_\beta)] \simeq(1/\mathcal{V}^2)[1+G(\mathbf{Z}_\alpha-\mathbf{Z}_\beta)]. \quad (4)$$

(This last form is, of course, valid only when $\mathbf{Z}_\alpha$ and $\mathbf{Z}_\beta$ both lie within $\mathcal{V}$.) It will be assumed that $G(x)$ is of order unity only for $x<R_c$, where R_c is referred to as the "range of correlation." It is important for our purposes to assume that

$$R_c\ll\mathcal{V}^{\frac{1}{3}}. \quad (5)$$

This means that our medium is "liquid-like," rather than crystalline.

If we use the second form of Eq. (4) and Eq. (5), the condition that $\int P(\mathbf{Z}_1\cdot\mathbf{Z}_2)d^3Z_1d^3Z_2=1$ implies that

$$\int G(r)d^3r=0. \quad (6)$$

This means, for instance, that a tendency for two particles to cluster at short distances must be compensated by a *decreased* probability of finding the two particles at *large* distances from each other. Condition (6) may be formally met by writing

$$G(r)=G_S(r)+G_L(r),$$

where G_S is the "short-range" ($\simeq R_c$) part and G_L the "long-range" part of G. If we set

$$\bar{G}\equiv\int G_S d^3r, \quad \bar{G}_L\mathcal{V}\equiv\int G_L d^3r,$$

then Eq. (6) implies that

$$\bar{G}_L=-\bar{G}/\mathcal{V}. \quad (8)$$

It will often suffice to write

$$G_L=-\bar{G}P(r). \quad (9)$$

Higher order probability functions may be defined in an analogous manner. We mention only

$$P(\mathbf{Z}_1,\cdots\mathbf{Z}_N)=|g_0|^2 =(1/\mathcal{V}^N)(1+\text{multiple correlation functions}). \quad (10)$$

the interparticle spacing in the medium and large compared with the wavelength of the scattered particles.

(An average over possible spins is implied here, we recall.) The last form above is valid, of course, only if all the $\mathbf{Z}$'s lie within $\mathcal{V}$. We assume that the range of the "multiple correlation functions" is not greater than $\mathcal{O}(R_c)$, so they *factor* into G's when only *pairs* of particles are close together.

When one of the "particles" is scattered by a particular "scatterer," the medium will in general be excited. We suppose the spacing of available excited states in such a collision to be $\simeq\Delta W_M$ and that

$$\Delta W_M\ll\epsilon_0, \quad (\text{A})$$

where ϵ_0 is the initial energy of the scattered particles. Assumption (A) will be called the "loose binding" assumption. It will be interpreted as implying that ϵ_0 is large compared to the energy required to "knock" a scatterer from the medium.

B. The Scattering Cross Section

The scattered particles are described by plane wave functions

$$\lambda^k=S^\nu e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (11)$$

Here $\mathbf{k}$ is the momentum vector[8] and ν is the orientation of the spin (if the particles have a spin). The initial momentum (within the medium) before scattering is assumed to be $\mathbf{k}_0$. We shall also write m for the mass of the particles and M for the mass of the scatterers.

The scattering is assumed to take place by a sequence of encounters of a particle with single scatterers. For the qualitative arguments of this section we shall suppose $m\ll M$ and that the scattering amplitude for a scattering through the angle θ is $f(\cos\theta)$.

Then for a single scattering encounter at $\mathbf{Z}_\alpha$ with the αth scatterer, the scattered wave has the familiar form

$$\psi_{sc}(\alpha)=(e^{ik_0R_\alpha}/R_\alpha)f(\mathbf{n}_\alpha\cdot\hat{k}_0)\psi_{\text{inc}}(\mathbf{Z}_\alpha). \quad (12)$$

Here

$$\mathbf{R}_\alpha\equiv\mathbf{x}-\mathbf{Z}_\alpha, \qquad \mathbf{n}_\alpha\equiv(\mathbf{R}_\alpha/R_\alpha). \quad (13)$$

Also, $\psi_{\text{inc}}(\mathbf{Z}_\alpha)$ is the incident wave at the position of scatterer α. Except for an inconsequential phase factor this is

$$\psi_{\text{inc}}=\exp(i\mathbf{k}_0\cdot\mathbf{Z}_\alpha)\times(\text{amplitude factor}). \quad (14)$$

In writing Eq. (12) in its asymptotic form, we have assumed

$$k_0R_\alpha\gg1. \quad (\text{B})$$

This forms our second fundamental assumption: namely that the wave travels far enough before a subsequent scattering that condition (B) is satisfied.

The phases of Eq. (12) have a simple interpretation. The phase of the incident wave just at scattering is given by $\exp(i\mathbf{k}_0\cdot\mathbf{Z}_\alpha)$. On travelling on to the point x, the change in phase is given by $\exp(ik_0R_\alpha)$. When x is

[8] We shall ordinarily use units such that $\hbar=1$.

sufficiently large that $x \gg Z_\alpha$, Eq. (12) takes the form

$$\psi_{sc}(\alpha)=(e^{ik_0x}/x)f_\alpha,$$
$$f_\alpha \equiv \exp[-i(\mathbf{k}-\mathbf{k}_0)\cdot \mathbf{Z}_\alpha]f \quad (15)$$

with $\mathbf{k}=k_0\hat{x}$. The dependence on $\mathbf{Z}_\alpha$ of the scattering amplitude f_α is given by a simple phase factor.

The total cross section for scattering from "α" is now

$$\sigma_\tau=\int d\Omega_{n\alpha}|f|^2.$$

The average distance which the particle will travel before a subsequent scattering is

$$\lambda=(\rho\sigma_\tau)^{-1}. \quad (16)$$

When $k\lambda \gg 1$, we may consider that our assumption (B) is satisfied.

The uncertainty in energy $\Delta\epsilon$ of the scattered particle, if its speed is v, is

$$\Delta\epsilon \simeq \hbar/(\lambda/v), \quad (17)$$

when it has travelled a distance λ. An alternative form for assumption (B) is then

$$\Delta\epsilon \ll \epsilon_0, \quad (18)$$

where ϵ_0 is the kinetic energy of the scattered particle.

We have assumed that the medium is "loosely bound." A much stronger condition would be

$$\Delta\epsilon \gg \Delta W, \quad (19)$$

where ΔW is the excitation energy given to the medium in the course of a scattering encounter. We shall not assume (19) to be necessarily true, although it is frequently satisfied (consider, for instance, the scattering of visible light by the molecules of a gas).

When the inequality (19) is satisfied, scattered wavelets may interfere coherently with each other even if they would eventually be associated with different states of excitation of the medium. This is of course a direct consequence of the indeterminacy principle. In Part II we shall develop a detailed theory for this case.

For our final transport equation of Sec. III it will be helpful to assume that

$$R_c \ll \lambda. \quad (C)$$

(We recall that R_c is the "range of correlation.") This means that the medium may be considered as locally undisturbed (by previous scatterings) at the position of a given scattering before that scattering has taken place.

C. A Simple Example

Before treating the general problem, we consider a simple example illustrating the interference of scattered wavelets. Although well known, this serves as an introduction to related phenomena associated with the transport problem.

The scattering medium is now considered small enough that the scattered particle is unlikely to scatter more than once. Then if $m \ll M$, Eq. (15) gives for the total scattered wave

$$\psi_{sc}=\sum_{\alpha=1}^{N}\psi_{sc}(\alpha)=\frac{e^{ik_0x}}{x}[\sum \exp(-i\Delta\mathbf{k}\cdot\mathbf{Z}_\alpha)f],$$
$$\Delta\mathbf{k}=k_0\hat{x}-\mathbf{k}_0. \quad (20)$$

The scattering amplitude for excitation of the medium to the state γ is thus

$$F_\gamma=(g_\gamma, \textstyle\sum_\alpha \exp(-i\Delta\mathbf{k}\cdot\mathbf{Z}_\alpha)fg_0). \quad (21)$$

When f is independent of spin (or when the spin dependent part of f averages out in evaluating the matrix element), then f factors out of the integrals in Eq. (21). In this case the ratio of excitation probabilities for different γ is independent of f and thus the type of particle causing the transition. This result, which is not true in general, will be discussed in detail in Part II.

If the excitation energy of the medium can be neglected (a condition which will be given quantitative consideration in later sections), the differential scattering cross section is

$$\sigma_M(\theta)=\sum_\gamma |F_\gamma|^2=(g_0, \{N|f|^2+\sum_{\alpha\neq\beta}|f|^2 \times\exp[-i\Delta\mathbf{k}\cdot(\mathbf{Z}_\alpha-\mathbf{Z}_\beta)]\}g_0). \quad (22)$$

Using Eq. (4), this may be written as

$$\sigma_M=\sigma_e(\theta)+\sigma_{in}+\sigma_{qe}, \quad (23)$$

where

$$\sigma_e(\theta)=N^2|f|^2[c(\theta)]^2,$$
$$\sigma_{in}=N|f|^2, \quad (24)$$
$$\sigma_{qe}=N|f|^2\rho C(\theta).$$

Here (θ is the scattering angle)

$$c(\theta)=\int d^3Z\, P(Z)\exp(-i\Delta\mathbf{k}\cdot\mathbf{Z}), \quad (25)$$

and

$$C(\theta)=\int G(r)e^{-i\Delta\mathbf{k}\cdot\mathbf{r}}d^3r. \quad (26)$$

The quantity σ_{in} represents the *incoherent* sum of the individual cross sections. For the general case to be treated in Sec. III, σ_{in} must be treated by a transport equation. The elastic scattering is given by $\sigma_e(\theta)$. In the general theory this is described by the *optical model*.[5] The quantity σ_{qe} will be referred to as the *quasi-elastic* cross section[9] when it is suitably generalized in Part II.

One may well object that σ_{qe} as just defined is not necessarily positive. This difficulty will be remedied in Part II when a more complete discussion is given.

[9] T. K. Fowler (to be published), has recently given a discussion of the *quasi-elastic* scattering from atomic nuclei.

We observe, however, that when σ_{qe} is sufficiently large in magnitude to be of importance, it will usually be positive. First,

$$G(r) \geq -1, \tag{27}$$

as is evident from Eq. (4). A large $|G|$ then means that $G>0$ (or that the scatterers tend to "cluster" together).

$C(\theta)$ may be simplified if we make use of Eqs. (7), (8), and (9). Define

$$\bar{G}C_0^2(\theta) \equiv \int G_S(r) e^{-i\Delta\mathbf{k}\cdot\mathbf{r}} d^3r, \tag{28}$$

so $C_0^2(0)=1$. With Eqs. (7) and (9), we obtain

$$C(\theta) \simeq \bar{G}C(\theta)[1-c(\theta)]. \tag{29}$$

Here we have assumed that $c(\theta)\rightarrow 0$ much more rapidly with increasing θ than does $C_0^2(\theta)$. Indeed, we expect $c(\theta)$ to become small for

$$\theta > 1/k_0R_0, \tag{30}$$

where $R_0 \simeq \sqrt[3]{\mathcal{V}}$ is the radius of the medium. $C_0^2(\theta)$ will be expected to become small for

$$\theta > 1/k_0R_C. \tag{31}$$

For the case that $|f|^2$ is independent of θ we have plotted in Fig. 1 the expected form of the cross sections (24).

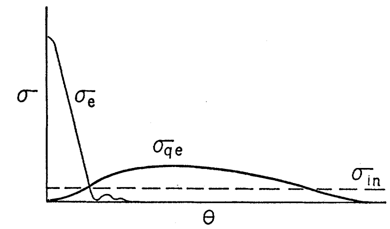

FIG. 1. A schematic plot of the "elastic," "inelastic," and "quasi-elastic" scattering, as given by Eqs. (24), etc.

II. GENERAL THEORY

The detailed discussion of the transport of particles through the scattering medium will be based on the multiple scattering equations developed in reference 5. There it was shown that the exact solution of the Schrödinger equation for the scattering of a particle by a scattering medium is given by

$$\begin{aligned}\Psi(x) &= \Phi_C(x) + \frac{1}{d}\sum_{\alpha=1}^{N} P_0 t_\alpha \Psi_\alpha(x),\\ \Psi_\alpha(x) &= \Phi_C(x) + \frac{1}{d}\sum_{\beta\neq\alpha} P_0 t_\beta \Psi_\beta(x)\\ \Phi_C(x) &= e^{i\mathbf{k}_0\cdot\mathbf{x}} g_0 \equiv \phi_C g_0.\end{aligned} \tag{32}$$

Here ϕ_C represents the incident wave, once it has *entered* the scattering medium. The wave number k_0 is related to the wave number q_0 of the particle *before* it entered the medium by the refractive index n[10]:

$$k_0 = nq_0. \tag{33a}$$

Alternatively, k_0 and q_0 are related in terms of the optical model potential v_C:

$$k_0^2 + 2mv_C = q_0^2. \tag{33b}$$

In writing ϕ_C in the form (32) we have neglected reflection and refraction at the boundary of the medium—these effects not being of importance for our argument.

The quantities t_α are "two-body" scattering operators for the scattering of the given particle by the αth scatterer in the medium. They are taken as momentum-conserving matrices:

$$t_\alpha = (\mathbf{k},\mathbf{Q}_\alpha|t|\mathbf{k}',\mathbf{Q}_\alpha')\delta(\mathbf{k}+\mathbf{Q}_\alpha-\mathbf{k}'-\mathbf{Q}_\alpha'). \tag{34}$$

Here $\mathbf{Q}_\alpha$ is the momentum of the αth scatterer. The t_α may also depend on the spin (if any) of particle or scatterer. In the interest of simplicity (that is, to avoid the complication of handling polarized beams), we shall assume either that the t_α's are spin independent or that *polarization* effects average out due to azimuthal symmetries in the distribution function of the scattered particles. Thus spin labels need not be kept on our wave functions and scattering operators. In spite of this simplifying assumption, our methods are directly applicable to situations involving polarization in the scattering. (More detailed discussions of the t-operators have been given previously.[5,11,12]

The "energy denominators" d in Eqs. (32) are (η is a positive, infinitesimal number, as usual in scattering theory)

$$\begin{aligned}d &\equiv \epsilon_0 + i\eta - \epsilon(k) + (W_0 - W_\lambda),\\ \epsilon(k) &= k^2/2m + v_C(k),\\ \epsilon_0 &= q_0^2/2m.\end{aligned} \tag{35}$$

The optical model potential v_C has been introduced in connection with Eq. (33b). The zeros of d give just the wave numbers k of the scattered wave. In particular, for $W_\lambda = W_0$,

$$\epsilon(k_0) = \epsilon_0,$$

which is equivalent to Eq. (33b). [For relativistic particles, we replace $k^2/2m$ by $c(k^2+m^2c^2)$, etc., in Eqs. (35).]

Finally, the symbol "P_0" in Eqs. (32) represents a counting operator which forbids repetition of states "γ" of the medium during the course of successive scatterings. This means, first of all, that the initial state "0" must never re-occur. It also implies that every scattering must be inelastic, or lead to a change in the state of the medium (since elastic scatterings have

[10] S. Fernhach, R. Serber, and T. B. Taylor, Phys. Rev. 75, 1352 (1949).

[11] G. R. Chew and M. L. Goldberger, Phys. Rev. 87, 778 (1952).

[12] K. M. Watson, Phys. Rev. 89, 575 (1953); N. C. Francis and K. M. Watson, Phys. Rev. 91, 291 (1953).

already been taken into account in v_C). Aside from these two rather simple effects, we may in general ignore P_0 when Assumption C is valid.

Let us now recall that we proposed to decompose the scattering into elastic, inelastic, and quasi-elastic contributions. Because of the properties of P_0, as just described, the elastic part of Ψ is just Φ_C. We do not intend to consider the elastic scatterings in more detail here, since this has been done previously.[12,5,10] The distribution between inelastic and quasi-elastic scattering will be based on two different schemes for evaluating the "propagators" $1/d$. The inelastic scattering will be discussed first, since this seems conceptually simpler.

Referring to Eqs. (32) it appears that our first problem is to simplify the expression

$$I \equiv P_0(1/d)t_\beta\Psi_\beta. \tag{36}$$

This is just the wave scattered from scatterer "β".

The wave Ψ_β may be expanded as

$$\Psi_\beta(x) = \sum_{\gamma'} g_{\gamma'} \int d^3k' \frac{e^{i\mathbf{k}'\cdot\mathbf{x}}}{(2\pi)^{\frac{3}{2}}} \psi_\beta(\mathbf{k}',\gamma'). \tag{37}$$

We also introduce a partial Fourier expansion of the medium wave function $g_{\gamma'}$:

$$g_{\gamma'} = \int \frac{d^3Q'}{(2\pi)^{\frac{3}{2}}} \exp(i\mathbf{Q}'\cdot\mathbf{Z}_\beta) a_{\gamma'}(Q'). \tag{38}$$

($a_{\gamma'}$ is a function of all $\mathbf{Z}$'s except $\mathbf{Z}_\beta$ of course). Then

$$\begin{aligned} I = P_0 \int \frac{d^3k'd^3Q'}{(2\pi)^3} \sum_{\gamma,\gamma'} g_\gamma \Big(g_\gamma, \\ \times \int \frac{d^3k d^3Q e^{i\mathbf{k}\cdot\mathbf{x}} \exp(i\mathbf{Q}\cdot\mathbf{Z}_\beta]}{\epsilon_0 + i\eta - \epsilon(k) + (W_0 - W_\gamma)} \delta(\mathbf{k}+\mathbf{Q}-\mathbf{k}'-\mathbf{Q}') \\ \times (\mathbf{k},\mathbf{Q}|\,t\,|\mathbf{k}',\mathbf{Q}') a_{\gamma'}(\mathbf{Q}')\psi_\beta(\mathbf{k}',\gamma') \Big). \end{aligned} \tag{39}$$

This may be simplified by writing

$$\sum_{\gamma'} a_{\gamma'}(\mathbf{Q}')\psi_\beta(\mathbf{k}',\gamma') \equiv \psi_\beta(\mathbf{k}')a_0(\mathbf{Q}'), \tag{40}$$

where $\psi_\beta(\mathbf{k}')$ is now an operator acting on the coordinates $(\mathbf{Z}_1,\cdots \mathbf{Z}_N)$. Equation (40) just states that $\gamma = 0$ was the initial state of the medium. It will turn out, however, that $\psi_\beta(\mathbf{k}')$ behaves very much as if it were a diagonal operator in a coordinate-space representation.

Now Eq. (39) becomes

$$\begin{aligned} I = P_0 \int \frac{d^3k'd^3Q'}{(2\pi)^3} \int \frac{d^3k e^{i\mathbf{k}\cdot\mathbf{x}}}{\epsilon_0 + i\eta - \epsilon(k) + (W_0 - H_N)} \\ \times \exp(-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{Z}_\beta)(\mathbf{k}|\,t\,|\mathbf{k}',\mathbf{Q}') \\ \times \exp(i\mathbf{Q}'\cdot\mathbf{Z}_\beta)\psi_\beta(\mathbf{k}')a_0(\mathbf{Q}'). \end{aligned} \tag{41}$$

In accordance with the discussion following assumption C,

$$[\exp(i\mathbf{Q}\cdot\mathbf{Z}_\beta), \psi_\beta(\mathbf{k}')] = 0, \tag{42}$$

since we assume the medium "near" Z_β to be undisturbed by previous scatterings. Then, we set

$$\int \exp(i\mathbf{Q}'\cdot\mathbf{Z}_\beta)(\mathbf{k},\mathbf{Q}|\,t\,|\mathbf{k}',\mathbf{Q}')a_0(Q')\frac{d^3Q'}{(2\pi)^{\frac{3}{2}}} = (\mathbf{k}|\,t\,|\mathbf{k}')g_0(\mathbf{Z}_1\cdots\mathbf{Z}_\beta\cdots\mathbf{Z}_N), \text{ etc.} \tag{43}$$

Here $\mathbf{Q}'$ is interpreted as $(1/i)\nabla_{Z\beta}$ in t, so t *operates* on g_0. (We have suppressed the "symbol" $\mathbf{Q}'$ in t in Eq. (43), but do not imply that $\mathbf{Q}'$ is omitted.) Because the scatterer binding energy has been assumed small, a valid first approximation consists of setting $\mathbf{Q}'=0$ in t. This treats the scatterer as if initially as rest. If $\mathbf{Q}'$ is kept in t, then we must eventually average the scattering cross section over the distribution of Q-values (as will be seen later).

To complete the evaluation of I, we shall expand d^{-1} in a manner reminiscent of the expansions of Wick[2] and Placzek[1]:

$$\begin{aligned} d &= d_0 - (H_N - \bar{W}), \\ d_0 &= \epsilon_0 + i\eta - \epsilon(k) - (\bar{W} - W_0), \\ \frac{1}{d} &= \frac{1}{d_0}\left(1 + \frac{1}{d_0}(H_N - \bar{W}) + \cdots\right), \end{aligned} \tag{44}$$

where $\bar{W}$ is an "average excitation energy" which is yet to be evaluated. Then, using the expansion (44) for d^{-1},

$$\begin{aligned} I = P_0 \int \frac{d^3k'}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{d_0} e^{i\mathbf{k}\cdot\mathbf{x}} \\ \times \left(1 + \frac{1}{d_0}(H_N - \bar{W}) + \frac{1}{d_0{}^2}(H_N - \bar{W})^2 + \cdots\right) \\ \times \exp(i\mathbf{Q}_0\cdot\mathbf{Z}_\beta)\, t\psi_\beta(\mathbf{k}')g_0. \end{aligned} \tag{45}$$

Here we have written

$$\mathbf{Q}_0 \equiv \mathbf{k}' - \mathbf{k}, \tag{46}$$

which is the momentum transferred to the scatterer.

Now, because H_N contains a term $(1/2M)\nabla_{Z\beta}{}^2$,

$$\begin{aligned} (H_N - \bar{W})^n \exp(iQ_0\cdot Z_\beta) \\ = (H_N - \bar{W})^{n-1} \exp(i\mathbf{Q}_0\cdot\mathbf{Z}_\beta) \\ \times \left(\frac{Q_0{}^2}{2M} + \frac{\mathbf{Q}_0\cdot\nabla_{Z\beta}}{Mi} - \bar{W} + H_N\right) \end{aligned} \tag{47a}$$

(if the scatterer β is not bound by exchange forces). H_N does not in general commute with either t or ψ_β. To simplify our discussion we shall neglect $[H_N,t]$, even though such terms may be kept and involve no essential difficulty for our presentation.[13] We obviously may not, however, neglect $[H_N,\psi_\beta]$, since ψ_β operating on g_0 describes the excitation of the medium due to previous scatterings. We have assumed that the scattering mean free path is sufficiently great that energy is *very nearly* conserved between collisions. Thus, we anticipate that

$$H_N[\psi_\beta g_0]\simeq W_{\rm ex}[\psi_\beta g_0], \quad (48a)$$

where $W_{\rm ex}$ is an "average" energy of the medium and is just W_0+ "the excitation associated with previous scatterings." On comparing Eqs. (45), (47a), and (48a), we see that it is most reasonable to set

$$\bar{W}\equiv(Q_0^2/2M)+W_{\rm ex}. \quad (48b)$$

Now Eq. (47a) becomes

$$(H_N-\bar{W})^n\exp(i\mathbf{Q}_0\cdot\mathbf{Z}_\beta) =\exp(i\mathbf{Q}_0\cdot\mathbf{Z}_\beta)\left(H_N-W_{\rm ex}+\frac{\mathbf{Q}_0\cdot\nabla_{Z\beta}}{Mi}\right) \times\cdots\left(H_N-W_{\rm ex}+\frac{\mathbf{Q}_0\cdot\nabla_{Z\beta}}{Mi}\right). \quad (47b)$$

Using the expansion (38) for g_0, we have

$$\frac{1}{d}\exp(iQ_0\cdot Z_\beta)\psi_\beta(\mathbf{k}')g_0 =\int\frac{d^3Q'}{(2\pi)^3}\frac{\exp[i(\mathbf{Q}_0+\mathbf{Q}')\cdot\mathbf{Z}_\beta]}{d_0} \times\left(1+\sum_{m=1}^{\infty}\frac{1}{d_0^n}\left(\frac{\mathbf{Q}_0\cdot\mathbf{Q}'}{M}\right)^n +\frac{1}{d_0}\exp(-i\mathbf{Q}'\cdot\mathbf{Z}_\beta)(H_N-W_{\rm ex})\exp(i\mathbf{Q}'\cdot\mathbf{Z}_\beta) +\frac{1}{d_0^2}(i\mathbf{Q}_0\cdot\mathbf{B}_\beta)+\cdots\right)\psi_\beta(\mathbf{k}')a_0(\mathbf{Q}'). \quad (49)$$

Here $\mathbf{B}_\beta$ is the gradient of the binding potential of "β" divided by M. It is defined by

$$[H_N,\mathbf{Q}_0\cdot\nabla_{Z\beta}/M_i]\equiv i\mathbf{Q}_0\cdot\mathbf{B}_\beta. \quad (50)$$

For discussing the *inelastic* scattering it is very convenient to partially re-sum the series (49) using

$$\frac{1}{b}\equiv\frac{1}{d_0}\left\{1+\sum_{m+1}^{\infty}\left(\frac{\mathbf{Q}_0\cdot\mathbf{Q}'}{d_0M}\right)^m\right\}=\frac{1}{d_0-\mathbf{Q}_0\cdot\mathbf{Q}'/M},$$

so

$$b=[\epsilon_0+(Q'^2/2M)+W_0-W_{\rm ex}]+i\eta-\epsilon(k)-Q^2/2M, \quad (51)$$

with

$$\mathbf{Q}\equiv\mathbf{Q}_0+\mathbf{Q}'. \quad (52)$$

The series in Eq. (49) may evidently be summed formally in such a way that all propagators d_0^{-1} are replaced by b^{-1}.

Finally,

$$I=P_0\int\frac{d^3k'd^3Q'}{(2\pi)^3}\int\frac{d^3kd^3Q\, e^{i\mathbf{k}\cdot\mathbf{x}}\exp(i\mathbf{Q}\cdot\mathbf{Z}_\beta)}{b} \times\mathfrak{D}_\beta\delta(\mathbf{k}+\mathbf{Q}-\mathbf{k}'-\mathbf{Q}')(\mathbf{k}|t|\mathbf{k}')\psi_\beta(\mathbf{k}')a_0(\mathbf{Q}'), \quad (53)$$

where

$$\mathfrak{D}_\beta=1+(1/b)\exp(-iQ'\cdot Z_\beta)(H_N-W_{\rm ex}) \times\exp(iQ'\cdot Z_\beta)+(1/b^2)(i\mathbf{Q}_0\cdot\mathbf{B}_\beta)+\cdots. \quad (54)$$

The quantity $(\mathfrak{D}_\beta-1)$ represents purely quantum mechanical binding corrections to the scattering from scatterer "β." We shall choose $W_{\rm ex}$ so as to make the first correction term in Eq. (54) vanish to lowest order in our final equations.

A. Evaluation of I_0

We first set $\mathfrak{D}_\beta=1$ to obtain the zeroth approximation

$$I_0\equiv P_0\int\frac{d^3k'd^3Q'}{(2\pi)^3}\int\frac{d^3P}{b}\exp(i\mathbf{P}\cdot\mathbf{C}+i\boldsymbol{\varrho}\cdot\mathbf{r}_\beta) \times(\mathbf{k}|t|\mathbf{k}')\psi_\beta(\mathbf{k}')a_0(\mathbf{Q}'). \quad (55)$$

Here

$$\mathbf{k}=(\mu/M)\mathbf{P}+\boldsymbol{\varrho},\quad \mathbf{P}=\mathbf{k}'+\mathbf{Q}',$$
$$\mathbf{Q}=(\mu/m^*)\mathbf{P}-\boldsymbol{\varrho},\quad \mu=M[m^*/(M+m^*)], \quad (56)$$
$$\mathbf{r}_\beta=\mathbf{x}-\mathbf{Z}_\beta,\quad \mathbf{C}=(M\mathbf{Z}_\beta+m^*\mathbf{x})(M+m^*)^{-1}.$$

The "effective mass" m^* is defined by

$$1/2m^*=d\epsilon(k)/d(k^2). \quad (57)$$

The "particle velocity" v is defined by (for convenience, we assume that the imaginary part of m^* and v are small)

$$v=\frac{d\epsilon}{dk}=\frac{d\epsilon}{dk^2}\frac{dk^2}{dk}=\frac{k}{m^*}. \quad (58)$$

By Assumption B, the significant contribution to the ρ-integral in Eq. (55) comes from values which make

$$b\simeq0.$$

Near this value, we find, indeed (after a little algebra)

$$b=(\rho_0^2-\rho^2)/2\mu+i\eta+O[(\rho^2-\rho_0^2)^2], \quad (59)$$

where ρ_0 is the root of the equation $b(\rho_0)=0$. The significance of m^* is seen here—it suffices to give b the simple (angle-independent) form (59).

The evaluation of I_0 is now trivial. To appreciate the result, we refer to Fig. 2, which shows the scattering. The initial position of the scatterer is $\mathbf{Z}_\beta^0$. It recoils to

[13] The effect of these commutators has been considered by Fowler (reference 9).

position $\mathbf{Z}_\beta$ when the particle reaches $\mathbf{x}$. We shall see in detail how this diagram follows from the evaluation of I_0.

On evaluating the ρ-integral for $\rho_0 r_\beta \gg 1$, we must set

$$\boldsymbol{\varrho} = \rho_0 \hat{r}_\beta, \quad \hat{r}_\beta \equiv \mathbf{r}_\beta / r_\beta \tag{60}$$

elsewhere in the integrand. This evidently implies that we set [see Fig. 2 and the discussion following Eq. (66) where $\mathbf{Z}_\beta{}^0$ is defined]

$$\mathbf{k} = k_0 \mathbf{R}_\beta / R_\beta, \quad \mathbf{R}_\beta = \mathbf{x} - \mathbf{Z}_\beta{}^0. \tag{61}$$

Now integrating over ρ gives

$$I_0 = P_0 \int \frac{d^3k' d^3Q'}{(2\pi)^3} \exp(i\mathbf{P}\cdot\mathbf{C}) [\exp(i\rho_0 r_\beta)/r_\beta]$$
$$\times \left[-(2\pi)^2 \mu \left(k_0 \frac{\mathbf{R}_\beta}{R_\beta} |t| \mathbf{k}' \right) \right] \psi_\beta(\mathbf{k}') a_0(\mathbf{Q}'). \tag{62}$$

We next evaluate the Q' integral in Eq. (62). To evaluate this integral, we set $\mathbf{Q}' = (1/i) \nabla_{Z_\beta{}^0}$ [see Eq. (66)] everywhere except in oscillating exponentials and in $a_0(\mathbf{Q}')$. Because we have assumed small binding energy in the medium the effect of this dependence on $\nabla_{Z_\beta{}^0}$ is expected to be small. (It may eventually be taken into account by averaging over the spectrum of Q' values.) Since

$$\mathbf{P} = \mathbf{k}' + \mathbf{Q}',$$

we obtain $\exp(i\mathbf{Q}'\cdot\mathbf{C})$ as an exponential factor. From the definition (51) of b it is evident that ρ_0 also depends on $\mathbf{Q}'$ since ρ_0 satisfies $b(\rho_0) = 0$. We therefore write

$$\rho_0 \equiv \rho_\beta + \delta\rho, \tag{63}$$

where $b(\rho_\beta) = 0$ when $\mathbf{Q}' = 0$. We obtain (to *first* order in $\mathbf{Q}'$)

$$\delta\rho = -(\mathbf{Q}'\cdot\mathbf{k}')(\rho_\beta)^{-1} [\mu/(M+m^*)]. \tag{64}$$

[Equation (64) involves a little algebra, but is straightforward.] Thus

$$I_0 = P_0 \int \frac{d^3k' d^3Q'}{(2\pi)^3} \exp(i\mathbf{k}'\cdot\mathbf{C}) [\exp(i\rho_\beta r_\beta)/r_\beta]$$
$$\times \left[-(2\pi)^2 \mu \left(k_\beta \frac{\mathbf{R}_\beta}{R_\beta} |t| \mathbf{k}' \right) \psi_\beta(\mathbf{k}') \right.$$
$$\times \exp i\{\mathbf{Q}'\cdot[\mathbf{C} - \mathbf{k}'(r_\beta/\rho_\beta)(\mu/(M+m^*))]\} a_0(Q') \tag{65}$$
$$= P_0 \int \frac{d^3k'}{(2\pi)^{\frac{3}{2}}} \exp(i\mathbf{k}'\cdot\mathbf{C}) \frac{e^{i\rho_\beta r_\beta}}{r_\beta}$$
$$\times \left[-(2\pi)^2 \mu \left(k_\beta \frac{\mathbf{R}_\beta}{R_\beta} |t| \mathbf{k}' \right) \right] \psi_\beta(k') g_0(\cdots \mathbf{Z}_\beta{}^0 \cdots).$$

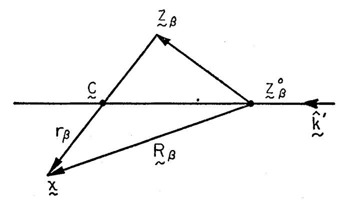

FIG. 2. The kinematics of a single scattering are shown along with the notation used in the text. The "particle" is scattered at position $\mathbf{Z}_\beta{}^0$ by the βth scatterer. When the particle has reached the point $\mathbf{X}$, the scatterer has *recoiled* to the position $\mathbf{Z}_\beta$. The center of mass of the two particles is at $\mathbf{C}$.

Here

$$\mathbf{Z}_\beta{}^0 \equiv \mathbf{C} - \mathbf{k}'(r_\beta/\rho_\beta)\mu/(M+m^*)$$
$$= \mathbf{Z}_\beta + \mathbf{r}_\beta \frac{m^*}{M+m^*} - \mathbf{k}' \left(\frac{r_\beta}{\rho_\beta} \right) \frac{\mu}{M+m^*}, \tag{66}$$

or

$$\mathbf{Z}_\beta - \mathbf{Z}_\beta{}^0 = \mathbf{r}_\beta \frac{m^*}{M+m^*} + \mathbf{k}' \left(\frac{r_\beta}{\rho_\beta} \right) \frac{\mu}{M+m^*}. \tag{66a}$$

The coordinate $\mathbf{Z}_\beta{}^0$ defined by Eq. (66) is just that in Fig. 2. To see this, we introduce the "time" τ since the collision occurred:

$$\tau \equiv (r_\beta/\rho_\beta)\mu. \tag{67}$$

The last term in (66a) is thus

$$\tau[\mathbf{k}'/(M+m^*)] = \mathbf{C} - \mathbf{Z}_\beta{}^0,$$

which is just the velocity of the center of mass multiplied by τ. Also,

$$-\mathbf{r}_\beta[m^*/(M+m^*)] = \mathbf{Z}_\beta - \mathbf{C},$$

as is evident from the diagram [of course, $\mathbf{x} - \mathbf{C} = \mathbf{r}_\beta n/(M+m^*)$].

Therefore, in Eq. (65) the coordinate $\mathbf{Z}_\beta$ appears in g_0 as translated to its "displaced position" to which scatterer "β" has recoiled. The recoil vector $\mathbf{Z}_\beta - \mathbf{Z}_\beta{}^0$ is just that given by classical kinematics. Higher order terms in $\delta\rho$ [see Eq. (64)] give a *dispersion* in the scatterer's position resulting from the spread of Q' values in g_γ[14] (this is just one aspect of the indeterminacy principle, which of course must appear in our problem). Also, the $\mathbf{Q}'$ dependence of t becomes a dependence on $(1/i)\nabla_{Z_\beta{}^0}$ in Eq. (65), as mentioned above.

To further simplify Eq. (65), we write $\varrho_\beta = \rho_\beta \hat{r}_\beta$, so the argument of the exponents becomes

$$\mathbf{k}'\cdot\mathbf{C} + \rho_\beta r_\beta = \mathbf{k}'\cdot\mathbf{C} + \boldsymbol{\varrho}_\beta\cdot\mathbf{r}_\beta$$
$$= k_\beta R_\beta + \mathbf{Q}\cdot(\mathbf{Z}_\beta - \mathbf{Z}_\beta{}^0) + \mathbf{k}'\cdot\mathbf{Z}_\beta{}^0, \tag{68}$$

where *now*

$$\mathbf{Q} = \mathbf{k}' - k_\beta(\mathbf{R}_\beta/R_\beta). \tag{69}$$

Thus $\mathbf{Q}$ is just the momentum imparted to scatterer

[14] In Sec. IV these corrections will be treated in more detail.

β when its binding is negligible. Now

$$I_0=P_0\int\frac{d^3k'}{(2\pi)^{\frac{3}{2}}}\exp[i\mathbf{k}'\cdot\mathbf{Z}_\beta{}^0]\exp[i\mathbf{Q}\cdot(Z_\beta-Z_\beta{}^0)]$$
$$\times\frac{\exp(ik_\beta R_\beta{}^0)}{R_\beta}\left[-(2\pi)^2\mu\left(\frac{R_\beta}{r_\beta}\right)\left(k\frac{\mathbf{R}_\beta}{R_\beta}|t|\mathbf{k}'\right)\right]$$
$$\times\psi_\beta(\mathbf{k}')g_0(\cdots\mathbf{Z}_\beta{}^0\cdots).\quad(70)$$

To obtain our final form for I_0, we expand

$$\psi_\beta(\mathbf{k}')=\varphi_C(\mathbf{k}')+\sum_{\nu(\neq\beta)=1}^{N}\psi_{\beta\nu}(\mathbf{k}')\equiv\sum_{\nu=0(\neq\beta)}^{N}\psi_{\beta\nu}(\mathbf{k}').\quad(71)$$

Here we abbreviate

$$\varphi_c(\mathbf{k}')\equiv\psi_{\beta 0}(\mathbf{k}').\quad(71\text{a})$$

The quantity $\psi_{\beta\nu}$ is interpreted as the wave scattered from $\mathbf{Z}_\nu{}^0$ to $\mathbf{Z}_\beta{}^0$. The wave number of this wave is k_ν and its direction is

$$\mathbf{n}_{\beta\nu}\equiv(\mathbf{Z}_\beta{}^0-\mathbf{Z}_\nu{}^0)/|\mathbf{Z}_\beta{}^0-\mathbf{Z}_\nu{}^0|.\quad(72)$$

By our Assumption B, $\psi_{\beta\nu}(k')$ must be nonvanishing only for $\mathbf{k}'\simeq\mathbf{k}_\nu\mathbf{n}_{\beta\nu}$.[15] Consequently the k' integral in Eq. (70) may be evaluated to give

$$I_0=\sum_{\nu=0(\neq\beta)}^{N}P_0\frac{e^{ik_\beta R_\beta}}{R_\beta}\exp[i\mathbf{Q}\cdot(\mathbf{Z}_\beta-\mathbf{Z}_\beta{}^0)]$$
$$\times\left[-(2\pi)^2\mu\frac{R_\beta}{r_\beta}\left(k_\beta\frac{\mathbf{R}_\beta}{R_\beta}|t|k_\nu\mathbf{n}_{\beta\nu}\right)\right]$$
$$\times\psi_{\beta\nu}(\mathbf{Z}_\beta{}^0)g_0(\cdots\mathbf{Z}_\beta{}^0\cdots).\quad(73)$$

Here

$$\psi_{\beta\nu}(\mathbf{Z}_\beta{}^0)=(2\pi)^{-\frac{3}{2}}\int d^3k'\exp[i\mathbf{k}'\cdot\mathbf{Z}_\beta{}^0]\psi_{\beta\nu}(\mathbf{k}'),\quad(74)$$

which represents the amplitude of waves at $\mathbf{Z}_\beta{}^0$ which have been scattered from $\mathbf{Z}_\nu{}^0$.

To obtain the scattered intensity, we shall eventually have to integrate over all coordinates $\mathbf{Z}_\alpha(\alpha=1,2,\cdots N)$. The form of Eq. (73) suggests that the $\mathbf{Z}_\alpha{}^0$ are more convenient variables that the $\mathbf{Z}_\alpha$. Thus we extend the definition of $\mathbf{Z}_\alpha{}^0$ by defining

$$\mathbf{Z}_\alpha{}^0=\mathbf{Z}_\alpha,\quad(75)$$

if no scattering has occurred from scatterer "α." If a scattering has occurred, then Eq. (66) defines $\mathbf{Z}_\alpha{}^0$. This terminology requires some care in its application, since for each possible sequence of scatterings we have a *different* set of variables $\mathbf{Z}$. (Because the number of scatterers has been assumed large, we can neglect the possibility that two scatterings may occur from the same scatterer.)

If a scattering has occurred from β, the transformation of volume element from $\mathbf{Z}_\beta$ to $\mathbf{Z}_\beta{}^0$ is, using Eq. (66a),

$$d^3Z_\beta=Jd^3Z_\beta{}^0$$
$$J=\left(\frac{m^*}{\mu}\right)^3\left[1+\frac{k_\nu}{\rho_\beta}\frac{\mu}{M}\hat{r}_\beta\cdot\hat{n}_{\beta\nu}\right]^{-1}.\quad(76)$$

Here "ν" is the scatterer from which the previous scattering has occurred ($\nu=0$ refers to the incident wave, we recall).

It is convenient to introduce $\sqrt{J}$ into I_0, so as to avoid the complication of tracing the historical sequence of scatterings when we transform from the $\mathbf{Z}$'s to the $\mathbf{Z}^0$'s. Thus, we define a "scattering amplitude" f:

$$f\left(k_\beta\frac{\mathbf{R}_\beta}{R_\beta}k_\nu\hat{n}_{\beta\nu}\right)\equiv-(2\pi)^2\mu\frac{R_\beta}{r_\beta}\left(k_\beta\frac{\mathbf{R}_\beta}{R_\beta}|t|k_\nu\hat{n}_{\beta\nu}\right)\sqrt{J}.\quad(77)$$

This is further simplified by noting that the ratio R_β/r_β depends only on the scattering angle and energy. The particle speed v is given by Eq. (58), whereas the relative velocity of particle and scatterer (after the collision) is

$$\mathbf{v}_R\equiv v(\mathbf{R}_\beta/R_\beta)-\mathbf{Q}/M.\quad(78)$$

Then

$$R_\beta/r_\beta=v\tau/v_R\tau=v/v_R,\quad(79)$$

where τ is the time which has elapsed since the collision [see Eq. (67)]. Finally, then

$$f\left(k_\beta\frac{\mathbf{R}_\beta}{R_\beta},k_\nu\hat{n}_{\beta\nu}\right)=-(2\pi)^2\mu\frac{v}{v_R}\left(k_\beta\frac{\mathbf{R}_\beta}{R_\beta}|t|k_\nu\hat{n}_{\beta\nu}\right)\sqrt{J},\quad(77\text{a})$$

and

$$I_0=P_0\sum_{\nu=0(\neq\beta)}^{N}\frac{e^{ik_\beta R_\beta}}{R_\beta}\exp[i\mathbf{Q}\cdot(\mathbf{Z}_\beta-\mathbf{Z}_\beta{}^0)]$$
$$\times f\left(k_\beta\frac{\mathbf{R}_\beta}{R_\beta},k_\nu\hat{n}_{\beta\nu}\right)\psi_{\beta\nu}(Z_\beta{}^0)g_0(\mathbf{Z}_1{}^0\cdots\mathbf{Z}_N{}^0).\quad(80)$$

The expression (80) is now in the form which we shall use. R_β is the distance that the wave has travelled since its last scattering. The velocity of the scattered wave is v, while v_R is the incident velocity. Had no previous scattering occurred, then only the term with $\nu=0$ survives in Eq. (80) and the differential scattering cross section is

$$\sigma_0=(v/v_R)|f|^2$$
$$=|(2\pi)^2\mu t|^2\left(\frac{[(m^*/\mu)(v/v_R)]^3}{1+(k_\nu/\rho_\beta)(\mu/M)\hat{r}_\beta\cdot\hat{n}_{\beta\nu}}\right).\quad(81)$$

The factor $|(2\pi)^2\mu t|^2$ will be recognized as the center-of-mass differential cross section. The second factor will be recognized as the usual coefficient for transforming

[15] This is explicitly demonstrated in connection with Eq. (64) of reference (5). It has been implicitly demonstrated in the process of obtaining the outgoing wave in Eq. (70).

from the center of mass to the laboratory coordinate system.

A more convenient "cross section" for our applications is

$$\sigma(\alpha\beta,\beta\rho)\equiv|f(k_\beta\mathbf{n}_{\alpha\beta},\ k_\nu\mathbf{n}_\beta)|^2$$
$$\sigma(\alpha\beta,\beta 0)\equiv|f(k_\beta\mathbf{n}_{\alpha\beta},\ \mathbf{k}_0)|^2. \qquad (82)$$

In a formal sense these are still *operators*, since they depend upon $\mathbf{Q}'=(1/i)\nabla_{Z_\beta^0}$. It is evident, however, that when we calculate scattered *intensities* (as will be done in Sec. III), the significant consequence of this is that our cross sections must be *averaged* over the spectrum of Q' values. This result is intuitively self-evident.

When $\mathfrak{D}_\beta$ is not set equal to unity in Eq. (53), the "propagator"

$$(1/R_\beta)\exp(ik_\beta R_\beta)\exp[i\mathbf{Q}\cdot(\mathbf{Z}_\beta-\mathbf{Z}_\beta^0)]$$

is replaced by a *generalized propagator* $E_{\beta\nu}(R_\beta)$. Then Eq. (53) still takes the form (80), modified as

$$I=P_0\sum_{\nu=0(\neq\beta)}^{N}E_{\beta\nu}(R_\beta)f(k_\beta(\mathbf{R}_\beta/R_\beta),\ k_\nu\mathbf{n}_{\beta\nu})\times\psi_{\beta\nu}(\mathbf{Z}_\beta^0)g_0(\mathbf{Z}_1^0\cdots\mathbf{Z}_N^0). \qquad (83)$$

The evaluation of $E_{\beta\nu}$ involves a computation of the terms in $\mathfrak{D}_\beta$. [Eqs. (53) and (54).] We shall return to this in Sec. IV.

B. The Coupled Scattering Equations

We return to the second of Eqs. (32) for the $\Psi_\alpha(x)$. In accordance with Eq. (71) we set

$$\Psi_\alpha(x)=[\phi_C(x)+\sum_{\beta(\neq\alpha)=1}^{N}\psi_{\alpha\beta}(x)]g(\mathbf{Z}_1^0\cdots\mathbf{Z}_N^0). \qquad (84)$$

This expression and Eq. (83) for I [I is defined by Eq. (36)] are now substituted into the second Eq. (32) and $\mathbf{x}$ is set equal to $\mathbf{Z}_\alpha^0$. The wave function g_0 "factors" out to give the coupled equations

$$\psi_{\alpha\beta}(\mathbf{Z}_\alpha^0)=P_0[E_{\beta 0}(R_{\alpha\beta})f(\alpha\beta,\beta 0)\phi_C(\mathbf{Z}_\beta^0)+\sum_{\rho(\neq\beta)=1}^{N}E_{\beta\rho}(R_{\alpha\beta})f(\alpha\beta,\beta\rho)\psi_{\beta\rho}(\mathbf{Z}_\beta^0)]. \qquad (85)$$

Here we have introduced the notation

$$f(\alpha\beta,\beta\rho)\equiv f(k_\beta\hat{n}_{\alpha\beta},\ k_\rho\hat{n}_{\beta\rho}), \qquad (86)$$

$$\mathbf{R}_{\alpha\beta}\equiv\mathbf{Z}_\alpha^0-\mathbf{Z}_\beta^0. \qquad (87)$$

When the scattering is spin-dependent, Eq. (85) is easily modified by keeping initial and final spin labels on the f and spin labels on ϕ_C and the $\psi_{\beta\rho}$, etc. and finally summing over spin states.

The "algebraic" equations (85) provide a complete solution to our problem. Because N is a large number, their general solution is not feasible. For this reason, we shall develop a transport equation from these in the next section.

C. The Quasi-Elastic Scattering

We have just obtained expressions for the scattered wave when the scattering is *inelastic*. By "inelastic" we have meant that the scatterer recoils approximately as if it were not initially bound in the medium. This defined the order of the sequence of terms in Eqs. (53) and (54) for I.

When the scatterer does not recoil with an energy large compared with this binding energy, the propagator b^{-1} is not appropriate for starting a series of approximations. In this case we still use Eq. (44) but evaluate it differently:

$$d_0=\epsilon_0'+i\eta-\epsilon(k),\quad \epsilon_0'\equiv\epsilon_0-(\bar{W}-W_0). \qquad (88)$$

Now we take $\bar{W}$ as the average energy of the medium *before* the scattering of interest has occurred. Thus d_0 is the appropriate *energy denominator* for scattering which is *elastic* (with respect to the medium) and gives a reasonable starting point for studying quasi-elastic scattering.

In this case our first approximation is

$$I^0=P_0\int\frac{e^{i\mathbf{k}\cdot\mathbf{x}}\exp(i\mathbf{Q}\cdot\mathbf{Z}_\beta)}{\epsilon_0'+i\eta-\epsilon(k)}d^3kd^3Q\frac{d^3k'd^3Q'}{(2\pi)^3}\times\delta(\mathbf{k}+\mathbf{Q}-\mathbf{k}'-\mathbf{Q}')(\mathbf{k},\mathbf{Q}|t|\mathbf{k}',\mathbf{Q}')\psi_\beta(\mathbf{k}')a_0(\mathbf{Q}') \qquad (89)$$
$$=P_0\sum_{\nu(\neq\beta)=1}^{N}\frac{\exp(ik_\beta R_\beta)}{R_\beta}f_0\psi_{\beta\nu}(\mathbf{Z}_\beta)g_0(\mathbf{Z}_1\cdot\ \cdots\mathbf{Z}_N).$$

Here k_β is the root of $d_0(k)=0$ and

$$f_0\equiv-(2\pi)^2m^*(k(\mathbf{R}_\beta/R_\beta)|t|k_\nu\mathbf{n}_{\beta\nu}), \qquad (90)$$

and $\mathbf{Z}_\beta^0=\mathbf{Z}_\beta$ whether a scattering has occurred or not from "β." This results since in this case the medium has absorbed the recoil momentum and thus the scatterer behaves as if infinitely heavy.[16]

III. THE TRANSPORT EQUATION

To obtain a transport equation from Eq. (85), we must square it to find the density of scattered particles. At the same time we must average over positions of the scatterers.

The density of scattered particles at $\mathbf{x}$ is [see Eq. (32)]

$$\int|\Psi(x)|^2d^3Z_1\cdots d^3Z_N.$$

Instead of $|\Psi|^2$, we shall find it more convenient to use

[16] We recognize that intermediate situations may arise, in which two or more scatterers recoil together. In this case the equivalent mass of the scatterer would be an integral multiple of the mass M.

instead

$$|\Psi_\alpha(\mathbf{Z}_\alpha)|^2.$$

This is permissible since the number of scatterers is considered to be large.

When we square Eq. (85), it is evident that "cross terms" occur, representing interference of waves scattered from different scatterers. These terms are purely quantum-mechanical and do not occur in a classical theory. They are nonvanishing, however, only because of local "structure" in the scattering medium and are thus expected to involve the correlation function G [as in Eqs. (22)–(26)]. In anticipation of this we define a "mixed density" n by the equation

$$\int \psi_{\alpha\beta}(\mathbf{Z}_\alpha{}^0)g_0g_0{}^\dagger\psi_{\alpha'\beta'}{}^\dagger(\mathbf{Z}_{\beta'}{}^0)\prod_{\nu=1}^N(\nu\neq\alpha,\alpha',\beta,\beta')d^3Z_\nu{}^0$$
$$\equiv n(\alpha\alpha',\beta\beta')P(\mathbf{Z}_\alpha{}^0,\mathbf{Z}_{\alpha'}{}^0)G(\mathbf{Z}_\beta{}^0-\mathbf{Z}_{\beta'}{}^0)\times P(\mathbf{Z}_\rho{}^0)P(\mathbf{Z}_{\beta'}{}^0). \quad (91)$$

(As usual, a sum over possible scatterer spins is implied here.) On multiplying Eq. (85) by its adjoint, we shall obviously obtain a set of coupled equations for the $n(\alpha\alpha',\beta\beta')$.

The notation (α,α'), etc., is used in Eq. (91) to suggest that points $\mathbf{Z}_\alpha{}^0$ and $\mathbf{Z}_{\alpha'}{}^0$ are neighboring positions, as are also $\mathbf{Z}_\beta{}^0$ and $\mathbf{Z}_{\beta'}{}^0$. On the otherhand, $\mathbf{Z}_\alpha{}^0$ and $\mathbf{Z}_\beta{}^0$ are not, being separated on the average by a distance much larger than the correlation range R_C (Assumption C). The justification for these remarks will be given below.

It will be convenient to supplement the definition of G by writing

$$\begin{aligned} G(\mathbf{Z}_\beta{}^0-\mathbf{Z}_{\beta'}{}^0)|_{\beta=\beta'}&=1,\\ P(\mathbf{Z}_\alpha{}^0,\mathbf{Z}_{\alpha'}{}^0)|_{\alpha=\alpha'}&=P(\mathbf{Z}_\alpha{}^0),\\ P(\mathbf{Z}_\beta{}^0)P(\mathbf{Z}_{\beta'}{}^0)|_{\beta=\beta'}&=P(\mathbf{Z}_\beta{}^0),\text{ etc.}\end{aligned}\quad (92)$$

We also define, in analogy to Eq. (91)

$$\begin{aligned} n_C(\beta\beta')&\equiv[\phi_C(\mathbf{Z}_\beta{}^0)\phi^*{}_C(\mathbf{Z}_{\beta'}{}^0)],\\ n_C(\beta)&\equiv n_C(\beta\beta).\end{aligned}\quad (91\text{a})$$

At this point it will be useful to establish a convention for replacing sums over particles by integrations over particle positions, and vice versa. By our assumption C [and the assumption that the correlation range $R_C\ll{}^3\sqrt{\mathcal{V}}$], we are free to write

$$\int|g_0|^2\prod_{\nu(\neq\mu,\mu')}{}' d^3Z_\nu{}^0$$
$$=P(\mathbf{Z}_\mu{}^0,\mathbf{Z}_{\mu'}{}^0)\times\text{factors independent of }(\mathbf{Z}_\mu{}^0,\mathbf{Z}_{\mu'}{}^0).$$

Here the notation $\prod_\nu{}'$ means that an arbitrary number of variables (but small compared to N) is omitted from the integration. This equation is true, of course, only for most positions $\mathbf{Z}_\mu{}^0$ and $\mathbf{Z}_{\mu'}{}^0$—but by assumption C is adequate for our applications.

Then we may write, where $L(\mathbf{Z}_\mu{}^0,\mathbf{Z}_{\mu'}{}^0,\mathbf{Z}_{\alpha1}{}^0,\cdots\mathbf{Z}_{\alpha l}{}^0)$ is some function of the variables indicated, and $l\ll N$,

$$\sum_{\mu,\mu'}\int L(\mathbf{Z}_\mu{}^0,\mathbf{Z}_{\mu'}{}^0,\mathbf{Z}_{\alpha1}{}^0\cdots\mathbf{Z}_{\alpha l}{}^0)|g_0|^2\times\prod_{\nu=1}^N(\nu\neq\alpha_1,\cdots\alpha l)d^3Z_\nu{}^0$$
$$=N^2\int L(\mathbf{Z}_\mu{}^0,\mathbf{Z}_{\mu'}{}^0,\mathbf{Z}_{\alpha1}{}^0\cdots\mathbf{Z}_{\alpha l}{}^0)|g_0|^2\times\prod_{\nu=1}^N(\nu\neq\alpha_1,\cdots\alpha_l)d^3Z_\nu{}^0$$
$$=\sum_{\mu,\mu'}\mathcal{V}^2\int L(\mathbf{Z}_\mu{}^0,\mathbf{Z}_{\mu'}{}^0,\mathbf{Z}_{\alpha1}{}^0\cdots\mathbf{Z}_{\alpha l}{}^0)|g_0|^2\times\sum_{\nu=1}^N(\nu\neq\mu,\mu',\alpha_1\cdots\alpha_l)d^3Z_\nu{}^0. \quad (93)$$

The first equality above is a direct consequence of the equivalence of the N scatterers. The second equality provides a definition for evaluating the sum over particles when this is not accompanied by an integral. For the logical development of the transport equation found below, the second equality in Eq. (93) need never have been introduced. For intuitive clarity it seems very useful, however, to interchange sums over particles and integrations over particle coordinates in this manner.

The particle density at $\mathbf{Z}_\alpha{}^0$ is

$$n(\mathbf{Z}_\alpha{}^0)=\int|\Psi_\alpha(\mathbf{Z}_\alpha{}^0)|^2\prod_{\nu(\neq\alpha)}d^3Z_\nu{}^0 \quad (94)$$

[see Eq. (84)]. In evaluating this and similar expressions, we must pay careful attention to the $\mathbf{Z}_\beta{}^0$'s, since these variables depend upon the past history of the scattering and are thus not uniquely defined. This means that all cross terms will apparently vanish when we substitute the expansion (84). To see this, we introduce the *correlation function*

$$\frac{1}{\mathcal{V}^2}\mathcal{G}(\mathbf{Z}_1{}^0,\mathbf{Z}_2{}^0)\equiv\int\prod_{\nu=3}^N d^3Z_\nu g_0{}^*(\mathbf{Z}_1{}^0,\mathbf{Z}_2,\cdots\mathbf{Z}_N)g_0\times(\mathbf{Z}_1,\mathbf{Z}_2{}^0,\mathbf{Z}^3\cdots\mathbf{Z}_N),$$

where we suppose a scattering to have occurred from particles "1" and "2".

Now, when the scattered particle has travelled sufficiently far, $|\mathbf{Z}_1{}^0-\mathbf{Z}_1|$ and $|\mathbf{Z}_2{}^0-\mathbf{Z}_2|$ become arbitrarily large. This means that $\mathcal{G}(\mathbf{Z}_1{}^0,\mathbf{Z}_{2l})=0$, since then the overlap of the wave functions will vanish.

In view of this, Eq. (94) [and using Eq. (93)] is

$$n(\mathbf{Z}_\alpha^0)=n_C(\alpha)+\sum_{\mu(\neq\mu')}\sum_{\mu'} n(\alpha\alpha,\mu\mu')\mathcal{G}(\mathbf{Z}_\mu^0,\mathbf{Z}_{\mu'}^0)+\sum_\mu n(\alpha\alpha,\mu\mu) \quad (95)$$

$$=n_C(\alpha)+\sum_\mu n(\alpha\alpha,\mu\mu).$$

Equation (95) appears to tell us that after sufficient time there is no interference of waves scattered from different scatterers. This, of course, is not in general true—indeed, the error in our conclusions rests in our approximate treatment of the Placzek-Wick[1,2] series in Eq. (55). We shall see, however, that the interference terms are small under the conditions of our problem—and that the evaluation of Sec. II-C is appropriate for obtaining these.

We shall then obtain the first order interference by a more careful reevaluation of Eq. (95). It is evident that waves scattered from two different scatterers will interfere only if the system is left in the same final state after either scattering—which is likely to happen only for very low states of excitation. This suggests that for the interference terms we should use the evaluation (89). Since then $\mathbf{Z}_\beta^0=\mathbf{Z}_\beta$ (all β) interference of scattered waves can occur.

The medium will very likely be left in a low-lying state if $\mathbf{Q}$, the momentum transferred to the scatterer, is not much larger than the width of the spectrum of $\mathbf{Q}'$ values in $a_0(\mathbf{Q}')$. For scattering of sufficiently small angles this condition can always be met.

On re-evaluating Eq. (95), which involves forming the square of Eq. (84), we now obtain

$$n(\mathbf{Z}_\alpha^0)=n_c(\alpha)+\sum_{\mu,\mu'} n(\alpha\alpha,\mu\mu')G(\mathbf{Z}_\mu^0-\mathbf{Z}_{\mu'}^0)\equiv n_c(\alpha)+\sum_\mu \bar{n}(\alpha,\mu). \quad (96)$$

Here we use the convention (92) and the evaluation (83) for the terms with $\mu=\mu'$ in the double sum. For $\mu\neq\mu'$, Eq. (89) is used for the evaluation of the scattered amplitude. Thus $\mathbf{Z}_\mu^0=\mathbf{Z}_\mu$ and $\mathbf{Z}_{\mu'}^0=\mathbf{Z}_{\mu'}$ in G above. The density $\bar{n}(\alpha,\mu)$ is defined by Eq. (96).

One may well ask why it is just the correlated term $(1/\mathcal{V}^2)G(\mathbf{Z}_\mu^0-\mathbf{Z}_{\mu'}^0)$ is kept in Eq. (96) rather than (see Eq. (4)) the full probability $P(\mathbf{Z}_\mu^0, \mathbf{Z}_{\mu'}^0)$. The reason for this is that the P_0 operator instructs us to *discard* elastic scatterings. From Eqs. (24) and (25) we see that the term $P(\mathbf{Z}_\mu^0)P(\mathbf{Z}_{\mu'}^0)$ in $P(\mathbf{Z}_\mu^0, \mathbf{Z}_{\mu'}^0)$ leads to elastic scattering and has *already* been included in the optical model potential.[5]

The transport equation is now very simply obtained. We multiply Eq. (85) by its complex conjugate and use the definitions (91) and (91a). [For the interference terms, we use Eq. (96)]. Finally, the relations (93) permit us to replace the integrations over $\mathbf{Z}_\mu^0$ and $\mathbf{Z}_{\mu'}^0$ by a factor of $\mathcal{V}^2$ in the last term on the right in Eq. (97) below; but this factor of $\mathbf{Z}^2$ makes up for the missing integrals over $\mathbf{Z}_\alpha^0$ and $\mathbf{Z}_{\alpha'}^0$ to give just $n(\beta\beta,\mu\mu')$.

$$\begin{aligned}
&[P(\mathbf{Z}_\beta^0)P(\mathbf{Z}_{\beta'}^0)P(\mathbf{Z}_\alpha^0,\mathbf{Z}_{\alpha'}^0)]^{-1}\\
&\times\int \psi_{\alpha\beta}(\mathbf{Z}_\alpha^0)g_0g_0^\dagger\psi_{\alpha'\beta'}^\dagger(\mathbf{Z}_{\alpha'}^0)\prod_\nu(\nu\neq\alpha\alpha'\beta\beta')d^3Z_\nu^0\\
&\equiv G(\mathbf{Z}_\beta^0-\mathbf{Z}_{\beta'}^0)n(\alpha\alpha',\beta\beta')\\
&=G(\mathbf{Z}_\beta^0-\mathbf{Z}_{\beta'}^0)[E_{\beta0}(R_{\alpha\beta})E_{\beta0}^\dagger(R_{\alpha'\beta'}) \\
&\quad\times\sigma_{\beta\beta'}(\alpha\beta,\beta0)n_C(\beta\beta')\\
&\quad+\sum_{\mu,\mu'}G(\mathbf{Z}_\mu^0-\mathbf{Z}_{\mu'}^0)E_{\beta\rho}(R_{\alpha\beta})E_{\beta\rho}^\dagger(R_{\alpha'\beta'})\\
&\qquad\times\sigma_{\beta\beta'}(\alpha\beta,\beta\mu)n(\beta\beta',\mu\mu')].
\end{aligned}\quad (97)$$

Here we have made the approximation of setting

$$f(k_{\beta'}\mathbf{n}_{\alpha'\beta'},\, k_{\mu'}\mathbf{n}_{\beta'\mu'})=f(k_\beta\mathbf{n}_{\alpha\beta},\, k_\mu\mathbf{n}_{\beta\mu})$$
$$E_{\beta'\rho'}=E_{\beta\rho},$$

etc., in all quantities *except* oscillating exponentials. This is generally justified, as will be seen, since distances such as $(\mathbf{Z}_\beta-\mathbf{Z}_{\beta'})$ are of importance only if less than R_C. Also, we have introduced [see Eqs. (82) and (90)]

$$\begin{aligned}
\sigma_{\beta\beta'}(\alpha\beta,\beta\mu)&\equiv\sigma(\alpha\beta,\beta\mu)\quad\text{for}\quad\beta=\beta'\\
\sigma_{\beta\beta'}(\alpha\beta,\beta\mu)&\equiv|f_0(\alpha\beta,\beta\mu)|^2\quad\text{for}\quad\beta\neq\beta'.
\end{aligned}\quad (98)$$

This is in accordance with our discussion accompanying Eq. (96). Physically, this means that our cross section is that appropriate for an *unbound* scatterer for the *incoherent* terms and is that appropriate to an *infinitely heavy* scatterer for the interference terms.

To further simplify Eq. (97), we observe that

$$E_{\beta\rho}(R_{\alpha\beta})\simeq\frac{\exp(ik_\beta R_{\alpha\beta})}{R_{\alpha\beta}}\times(\text{nonoscillatory terms.})$$

Thus, we set

$$\begin{aligned}
E_{\beta\rho}(R_{\alpha\beta})E_{\beta\rho}^\dagger(R_{\alpha'\beta'})&\simeq|E_{\beta\rho}(R_{\alpha\beta})|^2\exp[ik_\beta(R_{\alpha\beta}-R_{\alpha'\beta'})]\\
&\simeq|E_{\beta\rho}(R_{\alpha\beta})|^2\exp[-i\mathbf{k}_{\alpha\beta}\cdot(\mathbf{y}_\alpha-\mathbf{y}_\beta)],
\end{aligned}\quad (99)$$

where

$$\begin{aligned}
\mathbf{k}_{\alpha\beta}&\equiv k_\beta\mathbf{n}_{\alpha\beta},\\
\mathbf{y}_\alpha&\equiv\mathbf{Z}_{\alpha'}-\mathbf{Z}_\alpha^0,\\
\mathbf{y}_\beta&\equiv\mathbf{Z}_{\beta'}^0-\mathbf{Z}_\beta^0.
\end{aligned}\quad (100)$$

Now Eq. (97) becomes

$$\begin{aligned}
&G(\mathbf{Z}_\beta^0-\mathbf{Z}_{\beta'}^0)n(\alpha\alpha',\beta\beta')\\
&=\exp(-i\mathbf{k}_{\alpha\beta}\cdot\mathbf{y}_\alpha)[G(\mathbf{Z}_\beta^0-\mathbf{Z}_{\beta'}^0)\exp(i\mathbf{k}_{\alpha\beta}\cdot\mathbf{y}_\beta)]\\
&\quad\times[|E_{\beta0}(R_{\alpha\beta})|^2\sigma_{\beta\beta'}(\alpha\beta,\beta0)n_C(\beta\beta')\\
&\quad+\sum_{\mu,\mu'}|E_{\beta\mu}(R_{\alpha\beta})|^2\sigma_{\beta\beta'}(\alpha\beta,\beta\mu)\\
&\qquad\times G(\mathbf{Z}_\mu^0-\mathbf{Z}_{\mu'}^0)n(\beta\beta',\mu\mu')].
\end{aligned}\quad (101)$$

This equation may be "solved" with the Ansatz

$$\exp(-i\mathbf{k}_{\alpha\beta}\cdot\mathbf{y}_\alpha)\bar{n}(\alpha,\beta) = \sum_{\beta'} G(\mathbf{Z}_\beta{}^0-\mathbf{Z}_{\beta'}{}^0)n(\alpha\alpha',\beta\beta') \quad (102)$$

[compare Eq. (96)]. Using Eq. (102), Eq. (101) may be written as

$$\bar{n}(\alpha,\beta)=|E_{\beta 0}(R_{\alpha\beta})|^2\sigma_e(\alpha\beta,\beta 0)n_c(\beta) + \textstyle\sum_\mu |E_{\beta\mu}(R_{\alpha\beta})|^2\sigma_l(\alpha\beta,\beta\mu)\bar{n}(\beta,\mu). \quad (103)$$

We have here introduced the abbreviations [the convention of Eq. (93) is used to set $G(\mathbf{Z}_\beta{}^0-\mathbf{Z}_\beta{}^0)=1$].

$$\sigma_l(\alpha\beta,\beta 0)=\sum_{\beta'} G(\mathbf{Z}_\beta{}^0-\mathbf{Z}_{\beta'}{}^0)\exp(i\mathbf{k}_{\alpha\beta}\cdot\mathbf{y}_\beta) \times[n_C(\beta,\beta')/n_C(\beta)]\sigma_{\beta\beta'}(\alpha\beta,\beta 0)$$

$$=\left[1+\frac{\rho}{T}\int d^3y_\beta\, G(\mathbf{y}_\beta)\exp(i\mathbf{k}_{\alpha\beta}\cdot\mathbf{y}_\beta) \times n_C(\beta,\beta')/n_C(\beta)\right]\sigma(\alpha\beta,\beta,0) \quad (104)$$

$$\sigma_e(\alpha\beta,\beta\mu)=\sigma(\alpha\beta,\beta\mu)\left\{1+(\rho/T)\int d^3y\, G/\mathbf{y}) \times\exp[i(\mathbf{k}_{\alpha\beta}-\mathbf{k}_{\beta\mu})\cdot\mathbf{y}]\right\}$$

$$T\equiv|f|^2/|f_0|^2. \quad (105)$$

[The cross section σ_l is essentially that appearing in Eq. (22)]. The occurrence of T_{-1} in Eqs. (104) expresses the fact that the kinematics is different for the quasi-elastic than for the inelastic collisions. Also, the energy of the particles scattered quasi-elastically is not equal to that of inelastically scattered particles. Thus, even though Eq. (103) is formally correct, one may have to label separately the quas-elastic scatterings.[17] In most cases, however, we do not anticipate this complication, since the quasi-elastic scattering is expected to be important for small angle scatterings—and here the kinematics are the same for the two "types" of scattering.

Equation (103) is our final transport equation. Recalling that

$$|E|^2\simeq[1/(R_{\alpha\beta})^2]\exp[-(1/\lambda)R_{\alpha\beta}],$$

the structure of this equation is rather obvious. It represents an inhomogeneous equation for the $\bar{n}$ (α,β), since $n_C(\beta)$ is considered as *known* (it is just the initial beam intensity).

Depending upon the information which one desires, there are a variety of special forms for (103). For example, we may define

$$\delta\epsilon\bar{n}(\alpha;\epsilon)=\sum_{\beta(\text{in }\delta\epsilon)}\bar{n}(\alpha,\beta) \quad (106)$$

[17] We shall give a more detailed treatment of quasi-elastic scattering in Part II.

as the density of particles at α having energy between ϵ and $\epsilon+\delta e$. The sum in Eq. (106) is carried out over only those scatterers "β" which scatter particles into the correct energy range.

Again,

$$\delta\epsilon d\Omega_{\mathbf{k}}\bar{n}(\alpha;\epsilon,\mathbf{k})=\simeq_\beta\bar{n}(\alpha,\beta) \quad (107)$$

represents the density of particles at α having energy ϵ and momentum parallel to $\mathbf{k}$ (within the solid angle $d\Omega_{\mathbf{k}}$). The sum over "β" runs over those scatterings which satisfy these conditions.

When the scattered particles are sufficiently light that their energy loss on scattering may be neglected, a very simple transport theory is obtained. We then define

$$n(\mathbf{Z}_\alpha{}^0,\mathbf{k})\equiv n_C(\mathbf{Z}_\alpha{}^0)\delta(\Omega_{\mathbf{k}}-\Omega_{\mathbf{k}0})+\sum_\beta\bar{n}(\alpha,\beta)(1/\delta\Omega_{\mathbf{k}}) = n_C(\mathbf{Z}_\alpha{}^0)\delta(\Omega_{\mathbf{k}}-\Omega_{\mathbf{k}0})+\rho\int R_{\alpha\beta}{}^2dR_{\alpha\beta}\,\bar{n}(\alpha,\beta). \quad (108)$$

Here the $\sum_\beta$ is carried out so that $(\mathbf{Z}_\beta{}^0-\mathbf{Z}_\alpha{}^0)$ lies within $\delta\Omega_{\mathbf{k}}$ of the direction of $\mathbf{k}$.

On summing both sides of Eq. (103) over β, subject to the above condition, we finally find

$$n(\mathbf{x},\mathbf{k})=n_C(\mathbf{x})\delta(\Omega_{\mathbf{k}}-\Omega_{\mathbf{k}0})+\rho\int R^2dR\int d\Omega_{\mathbf{k}'} \times|E_{\mathbf{k}'}(R)|^2\sigma_e(\mathbf{k},\mathbf{k}')n(\mathbf{x}',\mathbf{k}'). \quad (109)$$

For simplicity we have replaced $\mathbf{Z}_\alpha{}^0$ by $\mathbf{x}$, etc., and have set

$$\mathbf{R}\equiv\mathbf{x}'[\mathbf{x}=-R(\mathbf{k}/k). \quad (110)$$

Equation (109) thus gives us the number of particles in a unit volume at $\mathbf{x}$ travelling within unit solid angle of the direction $\mathbf{k}$. The incident density $n_C(\mathbf{x})$ contains only particles travelling in the direction $\mathbf{k}_0$. Were we to take

$$|E(R)|^2=(1/R^2)e^{-(1/\lambda)R},$$

Eq. (109) could have been written down intuitively (using Eq. (22)).

As a final comment covering Eqs. (103), we recall that $\sigma_e(\alpha\beta,\beta\mu)$ in a strict sense depends upon the momentum $\mathbf{Q}'$ of particle β *before* it was struck. We must suppose then that the σ_e actually used in Eq. (103) represents an average over Q'-values. It is also true that $|E_{\beta\mu}|^2$ is an average over the medium wave function. The manner in which these averages are evaluated will depend on the particular problem at hand.

IV. FURTHER DISCUSSION OF THE TRANSPORT EQUATION

First, we should like to describe in more detail the evaluation of the "propagator" $E_{\beta\rho}$ of Eq. (83). This is defined implicitly on comparing Eqs. (53) and (83).

In lowest order

$$E_{\beta\rho}=(e^{ik_\beta R_\beta}/R_\beta)\exp[i\mathbf{Q}\cdot(\mathbf{Z}_\beta-\mathbf{Z}_\beta^0)], \qquad (111)$$

as evaluated in Sec. II. The corrections to Eq. (111) arise from two sources. First, the exact root of $b(\rho)=0$ is not given by Eq. (64). Second, the correction terms to $\mathfrak{D}_\beta$ (Eq. 54) do not in general vanish.

The corrections may be treated in a straightforward and systematic way. The expansion (54) is explicit and the terms of the series may be evaluated as expectation values over the state $g_{\gamma 0}$. Also, the root $b(\rho_0)=0$ may be expanded as

$$\rho_0=\rho_\beta+\delta\rho+\delta\rho_2+\cdots$$

$$\delta\rho_2=-\frac{1}{2}\left(\frac{\mu}{M+m^*}\right)^2\frac{(\mathbf{k}'\cdot\mathbf{Q}')^2}{\rho_\beta^3}+\frac{1}{2}\frac{\mu m^*}{M(M+m^*)}\frac{Q'^2}{\rho_\beta}, \qquad (112)$$

etc. Here $\delta\rho$ is given by Eq. (64). To the next order, Eq. (65) gets a factor

$$\exp(iR_\beta\delta\rho_2)\simeq(1+iR_\beta\delta\rho_2). \qquad (113)$$

After a little reduction, we are led to

$$|E_{\beta\rho}|^2\simeq(1/R_\beta^2)e^{-R_\beta/\lambda_s}\langle 0|1+i[R_\beta,\delta\rho_2]|\gamma\rangle. \qquad (114)$$

Here we have set

$$i(k_\beta-k_\beta^*)\equiv-1/\lambda_s. \qquad (115)$$

If the distribution of $\mathbf{Q}'$ is spherically symmetric, then the commutator in Eq. (114) goes to zero as R_β becomes large. Thus to this order $|E|^2$ is not modified.

It is important to observe that $|E|^2$ is in general much simpler than E itself. This follows, since

$$E\simeq(1/R)\exp[ik(\gamma)R]$$

for excitation of a state "γ" of the medium. When we form $|E|^2$, the oscillating phase factor drops out. We can see in detail how this occurs for the correction (113). Hence we have to evaluate such quantities as

$$(a_0(Q''),\exp[-i\rho_0(Q'')R]\exp[i\rho_0(Q')R]a_0(Q')).$$

Now, $\mathbf{Q}''\sim\mathbf{Q}'+\Delta k$, where Δk is the uncertainty in the momentum of the scattered particle. But $\Delta k\simeq\hbar/R$, so there is not an effect which increases with R.

Returning to Eq. (54) for $\mathfrak{D}_\beta$, we recall that W_{ex} is to be chosen to make the expectation value of the first correction term vanish. Also, the leading term in the Placzek-Wick expansion does not occur for us. This term was

$$\left(1+(\tfrac{1}{3})\left(\frac{m^*}{M}\right)\frac{[(1/2M)\langle Q'^2\rangle]}{\epsilon_0}\right), \qquad (116)$$

which is just $\langle v/v_R\rangle$ in Eq. (81). This factor does not appear in our cross section σ[Eq. (82)], however. The next Placzek-Wick term does occur. It arises from the $i\mathbf{Q}_0\cdot\mathbf{B}_\beta$ term in Eq. (54). One is led to

$$|E_{\beta\nu}|^2\approx\frac{1}{R_\beta^2}\exp\left(\frac{-R_\beta}{\lambda_s}\right)\left[1-\frac{1}{96}\left(\frac{M+m^*}{M}\right)^2\times\left(\frac{m^*}{M^2}\right)\frac{\langle\gamma_0|\nabla z_\beta P_\beta^2|\gamma_0\rangle}{\epsilon_0^3}+\cdots\right]. \qquad (117)$$

Here P_β is the binding potential of particle "β" in the medium.

The evaluation of the correction terms to $|E|^2$ seems straightforward in principle (in that they are reduced to expectation values with respect to the ground state $\gamma=0$). They also become rapidly small as ϵ_0 becomes large compared to the binding interaction of the scatterers. Actual evaluation of $|E|^2$ is rather tedious; however, some further discussion of this will be given in Part II where some simplifying approximations are demonstrated.

When the particles scattered are indistinguishable from those in the scattering medium, our formalism requires little modification.[18] The scattering matrices t must of course be properly symmetrized. If the scattering medium is a degenerate Fermi gas, then one must exclude states for the recoil particle which violate the Pauli principle.

[18] A detailed study of this case has been made by G. Takeda and K. M. Watson, Phys. Rev. **97**, 1336 (1955).