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Stability of Uniform Plasmas with Respect to Longitudinal Oscillations

PETER D. NOERDLINGER*
Norman Bridge Laboratory, California Institute of Technology, Pasadena, California
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It is possible to relate the dispersion formula for longitudinal oscillations in an infinite, uniform, collision-free plasma with no magnetic field to the complex potential of a line charge distribution on the real axis of the phase velocity ($u=\omega/k$) plane. If the initial velocity distribution integrated over directions orthogonal to the direction of propagation is $f_0(v)$ the plasma is stable if and only if

$$U(u)=P\int_{-\infty}^{\infty}\frac{f_0'(v)dv}{v-u}$$

is negative at the minima of $f_0(v)$ on the real axis, with unimportant exceptions. In particular it is shown that single-peaked distributions are stable, while those with very sharp (e.g., nondifferentiable) minima or with a zero of f_0 between two peaks are not. The charge analogy yields information on the wavelengths for which oscillations can grow and on rates of growth. Examples are given, including the case of two identical inter-penetrating hot plasmas. A limited generalization to transverse oscillations is given.

I. THE PROBLEM AND ITS SIGNIFICANCE

We shall consider the stability of an infinite, uniform plasma neglecting collisions and with no magnetic field by studying whether there can be growing linearized oscillations in it. At first we shall deal with longitudinal waves; our use of the terms "stable" and "unstable" below must be regarded as qualified by the phrase "with respect to longitudinal oscillations." Self-excited transverse electromagnetic waves have also been found[1] and in Sec. VII our results will be extended to such waves, but under more restrictive assumptions. Many authors[2–6] have studied the initial value problem and dispersion relations for waves in such a plasma, but none has given necessary and sufficient conditions for stability. It has been shown[2] that there is instability in the presence of a beam of high-energy particles with small spread in velocity, which is of interest in explaining the large electron scattering in plasmas.[7] It has also been proposed that colliding clouds of plasma may accelerate particles to high energy in a process involving the growth of plasma oscillations,[8] and that plasma oscillations may provide a mechanism for resistivity at high temperatures.[9]

II. REDUCTION TO THE ELECTROSTATICS PROBLEM

We shall at first regard the ions as fixed, but shall indicate below how this restriction is easily removed. They will be uniformly distributed with number density n_0. The initial electron density will be n_0, and the velocity distribution $n_0f_0(\mathbf{v})$. We shall consider linearized oscillations with coordinate-velocity distribution

$$f(\mathbf{r},\mathbf{v},t)=n_0f_0(\mathbf{v})+f_1(\mathbf{v})\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)], \qquad (1)$$

with real wave number $\mathbf{k}$ and complex frequency ω. We shall say the plasma is unstable if there can exist waves for which $\mathrm{Im}(\omega)>0$.

Following Case[6] rather closely, we define $v_{\parallel}$ and $\mathbf{v}_{\perp}$ to be the components of $\mathbf{v}$ along and perpendicular to $\mathbf{k}$ and

$$f_0(v_{\parallel})=\int f_0(v_{\parallel},\mathbf{v}_{\perp})d\mathbf{v}_{\perp}. \qquad (2)$$

Thus Case's $\eta(v_{\parallel})=-(\omega_p^2/k^2)(\partial/\partial v_{\parallel})\bar{f}_0(v_{\parallel})$ where ω_p^2

* Howard Hughes Fellow; work done in part on National Science Foundation Summer Fellowship.
[1] E. S. Weibel, Phys. Rev. Letters **2**, 83 (1959).
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[7] I. Langmuir, Phys. Rev. **26**, 585 (1925).
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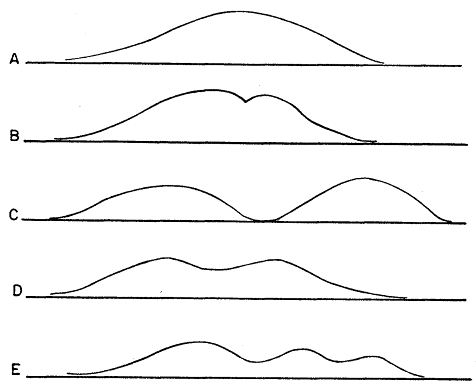

FIG. 1. Initial velocity distributions $f_0(v)$ versus v.

$=4\pi n_0 e^2/m_e$. Henceforth we write $f_0(v)$ for $\bar{f}_0(v_{\parallel})$, remembering that different vectors **k** may yield different functions $f_0(v)$ so that our stability criterion should be applied for each **k**. Denote $\partial f_0(v)/\partial v = f_0'(v)$.

It has been suggested that instability exists if $f_0(v)$ has two peaks at least one of which is quite sharp, but not if there is only one peak or two broad ones.[2] It has been shown there is no instability if $f_0(\mathrm{v})$ is Maxwellian.[10] We shall prove below that if $f_0(v)$ has only one peak, the plasma is indeed stable, but that the situation is more complicated for two peaks. Essentially, if f_0 is smooth enough (e.g., twice differentiable) the above suggestion is correct, but if, for example f_0' exhibits step function behavior, instabilities may appear which persist no matter how broad the peaks or how shallow the dip between them. Referring to Fig. 1, we shall find that plasmas with $f_0(v)$ like A are stable, like B and C unstable, and like D or E stable or unstable according to the depth and sharpness of the minima. We shall obtain a mathematically precise test for the stability of a rather general $f_0(\mathrm{v})$.

We assume $f_0(v)$ has these properties:

(a) $f_0'(v)$ exists and is differentiable at all but a finite number of points, where it has jump discontinuities.
(b) $f_0(v)$ and $f_0'(v)$ are small at least of order $1/v^4$ at infinity.
(c) Of course $f_0 \geqslant 0$.
(d) $\int_{-\infty}^{\infty} f_0(v)dv = \int f_0(\mathrm{v})d\mathrm{v} = 1$ (normalization).
(e) f_0 has a finite number of extrema.

Primarily we have excluded cases where f_0 has discontinuities, or where the energy density is infinite. Conditions (b), (c), and (e) imply that f_0 tends to zero monotonically for v^2 sufficiently large.

Several authors have shown that waves of the form (1) exist when ω and k are related by

$$\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{f_0'(v)dv}{v-u}=1, \qquad (3)$$

where $ku=\omega$. For real phase velocity u, the solutions of greatest physical significance (at least in a plasma which was initially in thermal equilibrium) are found by taking the principal part of the integral.[3] In the case of nonreal u, the integral is unambiguous as the integrand is regular along the real axis. Since (3) is invariant under complex conjugation, growing and damped solutions occur in pairs. *Damped* solutions (the famous Landau damping[5]) may also be built out of a superposition of waves of form (1) with real values of ω,[3] but growing waves can exist only with phase velocities $u=\omega/k$ which satisfy (3).

It will soon appear that the discontinuity of the imaginary part of the integral in (3) at the real axis can be interpreted as the usual multivaluedness of the imaginary part (stream function) of a complex potential in the neighborhood of charges; we shall need to know its value only above the real axis where it is well defined as we shall deal with u henceforth in the (open) upper half-plane.

It is easily shown by a modification of the derivations quoted in the foregoing[3,6] that the motion of the ions may be taken into account by replacing $f_0(v)$ in (3) by $f_{0e}(v)+(m_e/m_i)f_{0i}(v)$ where $f_{0e}(v)$ and $f_{0i}(v)$ are the initial electron and ion distribution functions averaged as in (2). Thus the stability of a two-component plasma may be treated in terms of an effective distribution function $f_0(v)$ for a hypothetical one-component plasma.

Under the foregoing assumptions (a) and (b) we may integrate (3) by parts to obtain

$$-\int_{-\infty}^{\infty}\ln(v-u)f_0''(v)dv \equiv W(u) = k^2/\omega_p^2, \qquad (4)$$

which defines $W(u)$. We interpret any discontinuities in $f_0'(v)$ as δ functions in $f_0''(v)$. $W(u)=U(u)+iV(u)$ is the complex potential of a line charge distribution along the real axis of the u plane of strength $\frac{1}{2}f_0''(v)$. The δ functions in f_0'' correspond to true line charges (logarithmic singularities) and the remainder to a charged sheet. We shall take advantage of the large body of knowledge extant about the complex electrostatic potential by discussing our stability problem from now on in terms of the properties of the charge $\frac{1}{2}f_0''$ and its complex potential[11] W. The symbol u will refer to points in the upper half plane and v to points on the real axis. The plasma is unstable if and only if we can find a point u such that

$$U(u)>0, \quad V(u)=0, \quad \mathrm{Im}(u)>0, \qquad (5)$$

for then we can choose a real number k to fulfill (4). As U is symmetric with respect to the real axis, so are the

[10] I. B. Bernstein, Phys. Rev. **109**, 10 (1958).

[11] The author has recently discovered that a different analogy, which seems less fruitful, between W and the electric *field* of a line charge distribution, due to L. Walker, was described by H. A. Haus in the *Proceedings of the Conference on Plasma Oscillations* (Speedway Research Laboratory, Linde Company, Indianapolis, Indiana, June 8–10, 1959).

lines of force (V=const) and equipotentials (U=const). We list some important properties of the charge distribution and potential.

(i) By integrating f_0'' from minus infinity to v, we see that the total charge to the left of v is $\frac{1}{2}f_0'(v)$ and the total charge of the distribution is zero.

(ii) For this distribution the total dipole moment vanishes but the quadrupole moment is 1. Asymptotically, then, $W(u)\sim 1/u^2$. Thus there is always a $V=0$ line of force which tends to infinity, asymptotically parallel to the imaginary axis. Any other $V=0$ lines in the open upper half-plane which tend to infinity would have to be asymptotically parallel to the real axis. It can be shown (with some effort) that there are no such lines; but if there were, U would be decreasing toward infinity on them and from the discussion in Sec. III it will be clear that they would have no effect on our conclusions. From the asymptotic form we see that there are two $U=0$ lines in the upper half-plane which tend to infinity at angles $\pm 45°$ with the imaginary axis. For large $|u|$, U is negative between them and positive between them and the real axis.

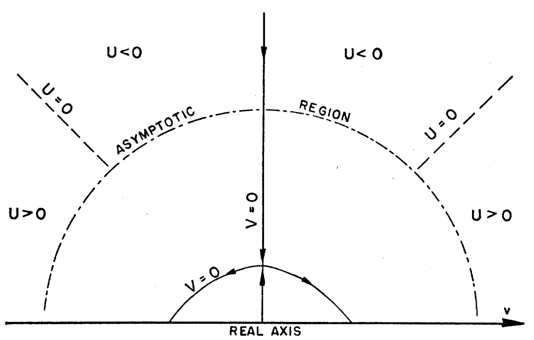

FIG. 2. Net of $V=0$ lines in the complex u plane, for a case where $f_0(v)$ has two symmetrical peaks.

(iii) Just above a point v on the real axis V is nearly $\pi f_0'(v)$ whenever the latter exists, since this is just half the flux from the charge $\frac{1}{2}f_0'(v)$ to the left of v.[12] When f_0' has a jump discontinuity at v, say from value α to value β, a succession of flux lines radiates from the discontinuity (a line charge) with V values ranging from $\pi\alpha$ to $\pi\beta$. Thus a $V=0$ line meets the real axis at v if and only if f_0' changes sign there. This is just the condition that f_0 have on extremum at v. If $f_0'=0$ at a point v but $f_0''\neq 0$ there, we see by looking at the variation of V just above the real axis near v that a single $V=0$ line meets the real axis there; again f_0 has an extremum at v. If f_0'' is also zero at v, or if f_0' is zero in a whole neighborhood of v, several $V=0$ lines may meet the real axis there. Points or intervals where $f_0'=f_0''=0$ and f_0' is of opposite sign on either side will be called "horizontal places of inflection" of f_0. Since these, along with extrema where $f_0''=0$, introduce qualifications into some of our arguments, it will be assumed at first that they do not occur, but we shall give the extension of our method for them later. The particularly simple case where f_0 is zero or an absolute maximum and $f_0'=0$ will be treated explicitly in Sec. III.

We have found that lines $V=0$ meet the real axis at places where f_0' changes sign (extrema of f_0) but should consider one other possibility: If $f_0\equiv 0$, say, for $v>A$, a $V=0$ line runs along the real axis from A to ∞. Is it possible for other $V=0$ lines to meet the real axis in this region? If so, there would be a neutral point ($dW/du=\partial U/\partial v=0$) there, but differentiating the formula

$$U(v)=\int_{-\infty}^{\infty}\frac{f_0'(p)dp}{p-v},$$

with respect to v and integrating by parts (for a point v where $f_0=f_0'=f_0''=0$) we obtain

$$\frac{\partial U}{\partial v}=\int_{-\infty}^{\infty}\frac{f_0(p)dp}{(p-v)^3},$$

which is of constant sign by (c) from A to infinity. Thus no $V=0$ lines can meet the real axis entirely outside the region where $f_0\neq 0$.

III. THE STABILITY THEOREM

In view of (5) above, the lines of flux on which $V=0$ are particularly important, as the plasma is stable if and only if $U<0$ everywhere on them. Since there are no charges in the open upper half-plane all such lines must terminate on the real axis or tend to infinity; we have shown that precisely one tends to $+i\infty$, but $U<0$ on it near infinity [(ii) above]. Since U varies monotonically along lines of force except at neutral points, we can imagine each $V=0$ line to be marked with an arrow in the direction of the electric field, i.e., the direction of decreasing U, illustrated in Figs. 2 and 3. Our problem of checking the sign of U at each point on the net of $V=0$ lines can now be reduced to the following simple

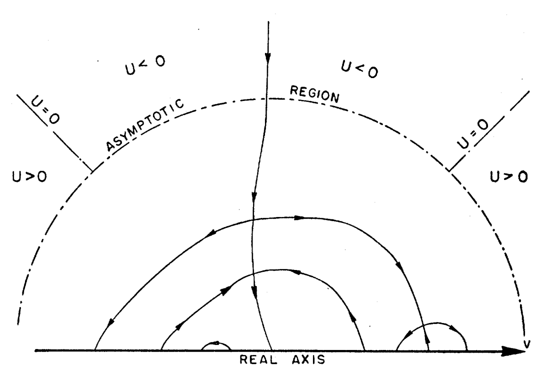

FIG. 3. Net of $V=0$ lines in the complex u plane for a case where $f_0(v)$ has five maxima.

[12] This can also be seen from $[1/(v-u-i\epsilon)]=P[1/(v-u)]+i\pi\delta(v-u)$.

procedure: By the foregoing assumption (e) and property (iii), the lines $V=0$ meet the real axis at a finite number of points, $v_1, v_2, \cdots v_n$ where f_0' changes sign. Starting at a point v_i we can follow a $V=0$ line into the upper half-plane, and by taking the line immediately to our right away from any neutral point we encounter, return to the real axis at some point v_j (with the one exception that we may end on the line tending to $i\infty$). Since the arrows on lines at a neutral point are directed alternately toward and away from it, U varies monotonically during such a traversal and hence has its extrema on the ends of the lines, being larger at the end where the electric field points away from the axis. Since the normal component of the electric field at the real axis is $\pi f_0''(v)$ we see the points on those $V=0$ lines which are traversed during the above process where U is the largest, are the minima of f_0 on the real axis. From the asymptotic form for W the arrow on the line tending to $i\infty$ is toward the origin, so $U<0$ on it up to the first neutral point. By a well-known theorem of potential theory, the $V=0$ lines cannot enclose any region of the open upper half-plane; therefore we traverse the entire net of $V=0$ lines if we repeat the above process for all i, $1\leq i\leq n$. This proves the theorem: The plasma is stable if and only if $U<0$ at each minimum of f_0. (We have temporarily assumed no places where $f_0'=f_0''=0$.)

In practice, the potential U at a point v on the real axis is calculated from

$$U(v)=\mathrm{Re}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{f_0'(p)dp}{p-v-i\epsilon}=\mathrm{P}\int_{-\infty}^{\infty}\frac{f_0'(p)dp}{p-v}. \quad (6)$$

The principal value sign is not needed if $f_0'(v)=0$. If f_0' is discontinuous at v a line charge is located there and $U\to\pm\infty$ at v according to the sign of the jump discontinuity in f_0'. If v is a nondifferentiable *minimum* [Fig. 1(B)] the sign is "+" and the plasma is unstable, but a non-differentiable maximum gives the opposite sign and introduces no instability. (See Sec. VII, however.)

When $f_0'(v)=0$ we may integrate (6) by parts to obtain

$$U(v)=\int_{-\infty}^{\infty}\frac{f_0(p)-f_0(v)}{(p-v)^2}dp \quad [\text{when } f_0'(v)=0]. \quad (7)$$

Thus if $f_0(v)$ is zero at one of its minima [Fig. 1(C)] the plasma is unstable. At places where $f_0'=f_0''=0$, several $V=0$ lines or none may meet the real axis. At a horizontal place of inflection the sign of V just above the real axis is the same on either side, so an even number of $V=0$ lines meet the axis there. Then if there are any such lines, there is at least one on which U increases away from the real axis. Following this line (in the sense of the procedure outlined in the foregoing for traversing lines) we shall arrive at another point on the real axis where U is surely higher; and hence we need not check the sign of U at the horizontal place of inflection.

If f_0'' is positive on each side of a place where $f_0'=f_0''=0$ (making it a minimum), at least one $V=0$ line must meet the real axis there, as just above the real axis V changes sign. Furthermore by considering the normal component of the electric field right above the real axis, we see that there must be at least one such line on which U increases toward the real axis. Hence we must check the sign of U where such lines meet the axis. If $f_0'=f_0''=0$ only at an isolated point, we simply evaluate U there by means of (6) or (7) but if this holds in a whole interval, the situation becomes more complicated. By studying what patterns of lines of force are possible, one can show that, if $\partial U/\partial v$ has no zero in the interval, we should check the end where U is largest, but otherwise at the zero of $\partial U/\partial v$. Similarly at an isolated maximum point of f_0 there must be at least one $V=0$ line meeting the real axis on which U increases away from the axis, and we need not check the sign of U there. At a maximum, however, where $f_0'=0$ throughout a whole interval, we must check the sign of U at any neutral point ($\partial U/\partial v=0$) therein. In the above discussion, reference was made to finding the zeros of $\partial U/\partial v$ in an interval where $f_0'=0$, a cumbersome procedure for most functions $f_0(v)$. If f_0 is an absolute maximum or is zero there, Eq. (7) shows U is negative or positive throughout the interval, respectively.

From the preceding arguments it is clear that a single-peaked distribution is stable: the one extremum is an absolute maximum where either $f_0'=0$ and (7) applies or f_0' is discontinuous and the first form of the stability theorem shows the plasma is stable.

IV. THE "MINIMUM CONDITION"

We have seen above that when $f_0'=0$ throughout an interval a $V=0$ line meets the real axis at any extremum of U in the interval (neutral point). Thus if $U>0$ at such an extremum, the plasma is unstable. This consequence of an extremum in U was first noticed by F. D. Kahn[13] in a graphical analysis of the stability of perfectly cold counterstreaming protons embedded in a cold electron gas. He then carried over the result to hot counterstreaming electrons and ions[14] where it does not apply as the condition $V=0$ is violated at the extrema of U. The method of Kahn, based on the minimum of U, does not seem readily generalizable to other cases as it is the depth and sharpness of the minima of f_0, not the existence of extrema of U which are significant for instability.

V. RATES OF GROWTH. LIMITING WAVELENGTHS FOR GROWING WAVES. EXPANSION OF THE DISPERSION RELATION

We have seen that to each growing wave there corresponds a point u in the upper half of the complex phase velocity plane where (5) holds. From the previous

[13] F. D. Kahn, Revs. Modern Phys. **30**, 1069 (1958).
[14] F. D. Kahn, Astrophys. J. **129**, 468 (1959).

discussion of the $V=0$ lines and the variation of U along them it is clear that these points comprise portions of the $V=0$ lines which are connected to the real axis at those of the points v_i where $U(v_i)>0$. Assume that $f_0''\neq 0$ at each of these points, so that a single $V=0$ meets the real axis at each of them and the normal component of the electric field is nonzero. Consider a point, say $\bar{v}$, where f_0 is a minimum and $U>0$. The electric field points away from the real axis at $\bar{v}$, and if we follow the $V=0$ line away from $\bar{v}$ we shall eventually come to a point $\bar{u}$ where $W=0$, or shall return to the real axis at some point v_j (using the right-hand turn rule at neutral points). In the first case we see from (4) that unstable solutions occur for values of k fulfilling $0<k<k_{\max}$ where

$$k_{\max}=\omega_p[U(\bar{v})]^{\frac{1}{2}} \tag{8}$$

and in the second for $\omega_p[U(v_j)]^{\frac{1}{2}}<k<\omega_p[U(\bar{v})]^{\frac{1}{2}}$. The latter situation seems unlikely in most cases of physical importance. Let $u=u_1+iu_2$ and $\omega=\omega_1+i\omega_2$. Then for the rate of growth of a wave we find

$$\omega_2=ku_2=\omega_p u_2[U(u)]^{\frac{1}{2}}. \tag{9}$$

Thus ω_2 is zero at $\bar{u}$ and $\bar{v}$; in most cases it will have only one maximum in between. This is surely the case when $\bar{u}$ and $\bar{v}$ are close together, as when the distribution differs but little from a stable one.

From (8) we see that if U is bounded near $\bar{v}$ growing waves will occur only for wavelengths λ longer than $\lambda_{\min}=2\pi/k_{\max}$. If, however, f_0' is discontinuous at $\bar{v}$, producing a logarithmic singularity in U there, (8) is no restriction at all and we expect growing waves of arbitrarily short wavelength. One might distrust this result since the derivation of (3) is valid only for sufficiently long wavelengths, when collective motion dominates individual particle effects.[15] If $f_0(\mathrm{v})$ is nearly Maxwellian (3) is valid for $\lambda\gtrsim\lambda_D=(KT/4\pi n_0e^2)^{\frac{1}{2}}$, but if not we can probably use the cutoff distance λ_D' obtained by evaluating λ_D for a Maxwellian plasma with the same particle and energy densities as the one in question. Since an instability usually persists at long wavelengths ($k\rightarrow 0$) our conclusions on instability will not be affected in most cases by the introduction of a minimum wavelength.

Qualitatively, we can see from (8) and (9) that the sharper the minimum in $f_0(v)$ and the steeper its sides the larger the maximum values of k and ω_2 will be for growing waves, as $U(\bar{v})$ will be greater.

We know there are no instabilities for $|u|$ sufficiently large since $W\sim 1/u^2$ asymptotically. On the other hand, U has at worst logarithmic singularities, which are all on the real axis. Therefore the right-hand side of (9) is bounded, and infinite rates of growth do not occur.

Since the $V=0$ lines are lines of flux the electric field $\mathbf{E}=-(dW/du)^*$ is tangential to them. Having found a point $\bar{v}$ on the real axis where a $V=0$ line meets it and $U>0$, we could in principle trace the instabilities into the upper half-plane by integrating the equation

$$\frac{du}{ds}=\left(\frac{dW}{du}\right)^*\bigg/\left|\frac{dW}{du}\right|,$$

where s is arc length, or simply

$$\frac{du}{dp}=\left(\frac{dW}{du}\right)^*,$$

where p is a parameter. This would yield a parametric form of the $\omega-k$ relation for nonreal ω. If we know $\bar{u}$ and $\bar{v}$ are close together, this can be done approximately by expanding W in a series about $\bar{v}$, if f_0 is sufficiently smooth there. Using the derivatives of W defined by

$$\left.\frac{d^nW^+}{du^n}\right|_v=\lim_{\epsilon\rightarrow 0+}\left(\frac{d^nW}{du^n}\right)_{u=v+i\epsilon},$$

we get a Taylor series which gives W in the upper half-plane and an analytic continuation of it below the real axis. For points near $\bar{v}$, we write

$$\begin{aligned}k^2/\omega_p{}^2=W(u)&\doteq W(\bar{v})+(u-\bar{v})(dW^+/du)_{\bar{v}}\\&\doteq U(\bar{v})+(u_1+iu_2-\bar{v})\\&\qquad\times[(\partial U/\partial v)_{\bar{v}}+i(\partial V/\partial v)_{\bar{v}}].\end{aligned} \tag{10}$$

Taking real and imaginary parts of (10) we find

$$\omega_2=ku_2\doteq\omega_p u_2\left[U-u_2|\nabla U|^2\bigg/\left(\frac{\partial V}{\partial v}\right)\right]^{\frac{1}{2}}, \tag{11}$$

where all quantities (except u_2) are to be evaluated at $\bar{v}$. For the maximum value of ω_2 between $\bar{v}$ and $\bar{u}$ we get

$$\omega_{2\,\max}\doteq\omega_p\frac{2}{\sqrt{3}}\left[\frac{(\partial V/\partial v)}{|\nabla U|^2}U^{\frac{3}{2}}\right]_{u=\bar{v}}. \tag{12}$$

This is valid if $U(\bar{v})$ is small but fails if $(\partial V/\partial v)_{\bar{v}}=f_0''(\bar{v})=0$. This formula cannot yield the Landau damping[5] which is intimately connected with a superposition solution of the initial value problem using only waves with real phase velocity. It shows that at the threshold of instability, when $U(\bar{v})\doteq 0$, the rates of growth increase slowly (like $U^{\frac{3}{2}}$) as U increases.

VI. EXAMPLES

A. Colliding Plasmas

Suppose two plasmas which were initially at the same temperature and density collide. It is reasonable to assume that each has had time beforehand to reach thermal equilibrium but that collisions may be neglected for a short time during interpenetration. We may then study the growth of plasma oscillations in the region of interpenetration by using the effective distri-

[15] D. Pines and D. Bohm, Phys. Rev. 85, 338 (1952).

bution function (see Sec. II)

$$f_0(\mathbf{v})=\tfrac{1}{2}(m_e/2\pi KT)^{\frac{3}{2}}\{\exp(-m_ev^2/2KT)+\exp[-m_e(\mathbf{v}-\mathbf{v}_1)^2/2KT]\}+\tfrac{1}{2}(m_e/m_i)(m_i/2\pi KT)^{\frac{3}{2}}\{\exp(-m_iv^2/2KT)+\exp[-m_i(\mathbf{v}-\mathbf{v}_1)^2/2KT]\}. \quad (13)$$

The direction for $\mathbf{k}$ most likely to yield growing waves is along $\mathbf{v}_1$. Integrating, then, over $\mathbf{v}_\perp$ to find $f_0(v)$ and differentiating with respect to v, we obtain

$$f_0'(v)=-\tfrac{1}{2}(m_e/2\pi KT)^{\frac{1}{2}}[S(v)+S(v-v_1)],$$

where

$$S(v)=(vm_e/KT)\exp(-m_ev^2/2KT)+(m_e/m_i)^{\frac{1}{2}}(vm_i/KT)\exp(-m_iv^2/2KT).$$

We compute U from (6) at the extremum $\bar{v}=\frac{1}{2}v_1$ of $f_0(v)$ by the method of Kahn,[14] who gives

$$\int_{-\infty}^{\infty}\frac{\xi\exp(-\frac{1}{2}\xi^2)d\xi}{\xi-\eta}=(2\pi)^{\frac{1}{2}}\left[1-\eta\exp(-\tfrac{1}{2}\eta^2)\int_0^\eta\exp(\tfrac{1}{2}p^2)dp\right]\equiv(2\pi)^{\frac{1}{2}}h(\eta), \quad (14)$$

where $h(\eta\sqrt{2})$ is tabulated in Unsöld.[16] This gives

$$U(\bar{v})=-(m_e/KT)[h(\tfrac{1}{2}\eta)+h(\tfrac{1}{2}\eta(m_i/m_e)^{\frac{1}{2}})], \quad (15)$$

where $\eta=v_1(m_e/KT)^{\frac{1}{2}}$. The only zero of (15) is at $\eta=2.64$ where the second (ion) term is negligible. This supports the view that the electrons come to equilibrium rapidly (say in a few electron plasma periods) followed more slowly by the ions.[8,13] For $v_1<2.64(KT/m_e)^{\frac{1}{2}}$ the plasma is stable and collisions are the principal thermalizing process. The transition to the unstable case $v_1>2.64(KT/m_e)^{\frac{1}{2}}$ where plasma oscillations are important is smooth, however, for the quantity $U(\bar{v})$ in (12) increases smoothly with increasing v_1, implying very small rates of growth at the threshold of instability.

B. Counterstreaming Electrons and Ions

Our method can be used to calculate the value of relative velocity between ions and electrons at which instability appears in a hot plasma [using twice the second and third terms of (13)]. The results agree with those obtained by Jackson[17] and by Buneman[9] using different analyses: the plasma is unstable if the electron translational kinetic energy in the center of mass frame exceeds $0.87KT$, i.e., if the relative velocity v_1 exceeds 1.35 times the electron thermal speed $(KT/m_e)^{\frac{1}{2}}$. Jackson gives a plot of $k_{\max}$ vs v_1. We note that as the relative velocity is increased beyond the threshold of instability, the growing waves occur first at very long wavelengths, as $U(\bar{v})$ is small. This suggests that it may be difficult to achieve Buneman's initial conditions where the relative velocity greatly exceeds the thermal velocity, so that the hydromagnetic approximation may be used to get the wavelength L of the most rapidly amplified waves, but there are as yet no waves larger than thermal fluctuations.[9]

[16] A. Unsöld, *Physik der Sternatmosphären* (Julius Springer, Berlin, 1938), p. 163.

[17] J. D. Jackson, "Plasma Oscillations," Report No. GM-TR-0165-00535, Physical Research Laboratory, Space Technology Laboratories, December, 1958 (unpublished).

VII. TRANSVERSE WAVES

Weibel[1] has given a dispersion relation for linearized transverse plane waves in an infinite, uniform, plasma assuming the ions are fixed. We shall adapt our previous analysis to his formula in the case of no initial magnetic field. Using Weibel's notation $v_0^2=v_x^2+v_y^2$ and the assumption $f_0(\mathbf{v})=F(v_0,v_z)$, we may write his equation (4), the dispersion relation for waves propagating parallel to the z axis as

$$k^2c^2-\omega^2=\omega_p^2\pi\int_0^\infty v_0^2dv_0\int_{-\infty}^\infty dv_z\left[\frac{\partial F}{\partial v_0}-\frac{kv_0}{kv_z-\omega}\frac{\partial F}{\partial v_z}\right], \quad (16)$$

where we have changed numerical factors and the sign of ω to agree with our units, normalization, and sign convention. Denoting $\pi\int_0^\infty v_0^3F(v_0,v_z)dv_0$ by $\varphi(v_z)$ and using the normalization condition $\int Fd\mathbf{v}=1$ on the first term, we find

$$k^2/\omega_p^2=\left(1+\int_{-\infty}^\infty\frac{\varphi'(v)dv}{v-u}\right)\Big/(u^2-c^2)\quad\text{where } u=\omega/k$$
$$\equiv(1+\bar{U}+i\bar{V})/(u^2-c^2)\equiv R(u)+iS(u)=T(u), \quad (17)$$

which defines $\bar{W}=\bar{U}+i\bar{V}$ and $R+iS=T$. The plasma is unstable with respect to these transverse waves if there are points in the upper half-plane where $R>0$, $S=0$. (The integral is along the real axis.) Clearly $\bar{W}$ is analytic in the upper half-plane and for physically reasonable functions $F(v_0,v_z)$ goes to zero as $u\to\infty$. Thus T is analytic in the upper half-plane and $T\sim 1/u^2$ as $|u|\to\infty$. Since T is analytic in the upper half-plane, most of our discussion of the network of $V=0$ lines and the variation of U along them carries over to R and S. The charge distribution now has two point dipole sources, at $u=\pm c$. The plasma is unstable if $R>0$ at any point where an $S=0$ line meets the real axis, but again we need check only those where the "electric field" $\mathbf{E}=-(dT/du)^*$ points into the upper half-plane. The $S=0$ lines tending to infinity are of no importance, as before. The points where $S=0$ lines meet the real axis are those where $\bar{V}=0$ [i.e. $\varphi'(v)$ changes sign] and possibly the points $v=\pm c$. If either of the point dipoles at $\pm c$ points toward the upper half-plane, an $S=0$ line with $R\to+\infty$ near the dipole will emanate upwards from it, implying instability. For other orientations no $S=0$ lines with **E** directed into the upper half-plane emanate from the dipoles. The phase of the numerator in (17) determines the dipole orientations. Since $u+c>0$ at $u=c$ and $u-c<0$ at $u=-c$, the dipoles both point directly away from the origin if $\bar{W}(\pm c)$ is real. Since the derivation of (16) involved a nonrelativistic form of the Boltzmann equation, we should assume most particle velocities are much less than c; then $\varphi'(v)$ and $\bar{W}(v)$ are both small for $|v|\doteq c$. Then the condition for no unstable roots near $u=\pm c$ is that $\lim_{\epsilon\to 0}\bar{V}(c+i\epsilon)=\varphi'(c)<0$ and similarly $\varphi'(-c)>0$. From the definition of φ we see that if these conditions did not hold, the number density would have to be an increasing function of $|v_z|$ for $|v_z|\doteq c$, a case which should be excluded in a nonrelativistic treatment. From the resemblance of the role of φ' to that of f_0' in the previous discussion, we see that, so long as the extrema of φ are all for $|v|<c$, **E** is into the upper half-plane at the *maxima* of φ. In particular, at nondifferentiable maxima $\bar{U}\to -\infty$ so $R\to +\infty$ and the plasma is unstable.

For the distribution function

$$f_0(\mathbf{v})=\frac{1}{u_0^2u_3(2\pi)^{\frac{3}{2}}}\exp\left[-\frac{v_0^2}{2u_0^2}-\frac{v_z^2}{2u_3^2}\right],$$

of Weibel[1] we find $(2\pi)^{\frac{1}{2}}\varphi'(v)=(-vu_0^2/u_3^2)\exp(-v^2/2u_3^2)$. There is one zero of φ' at $v=0$, where $c^2T=(u_0/u_3)^2-1$. Thus the plasma is unstable if $u_0>u_3$, recovering the result of Fried[18] who used Nyquist's criterion. We cannot conclude that for $u_0<u_3$ the plasma is stable, as we have taken only waves propagating along the z axis. Using (13) and (14) we can readily find that colliding plasmas are stable with respect to transverse oscillations propagating in the direction of relative motion, no matter how large the relative velocity.[19] Remembering, however, that $u_0>u_3$ in Weibel's case corresponds to greater particle kinetic energy parallel to the wave front than perpendicular to it, we might suspect that transverse waves moving perpendicular to the relative velocity vector of counterstreaming particles would grow.

Since (17) is not invariant under a Galilean transformation $u\to u+\Delta u$, $v\to v+\Delta u$, as was (3), such a transformation can change the properties of waves found from it. If there are few particles near $|v|=c$ and no irregularities in $\varphi(v)$ there, the dipole singularities dominate and there will be no growing waves with $u\doteq c$. For example, in the one case computed above, $\mathrm{Re}(u)$ was zero for all growing waves. If we are sure that $|u|\ll c$, performing a Galilean transformation on (17) will change c^2T by terms only of order $u\Delta u/c^2$, a small discrepancy which is the price of combining the nonrelativistic Boltzmann equation with electrodynamics.

[18] B. D. Fried, Report No. TN-59-0000-00235, Physical Research Laboratory, Space Technology Laboratories, February, 1959 (unpublished).

[19] It can be shown that the trick for including ion motion which was given in Sec. II is valid here too.

VIII. CONCLUSION

The problems discussed in this paper have centered on the determination of whether a function $W(u)$ which is analytic in the upper half-plane and whose imaginary part is known along the real axis is real and positive anywhere in the upper half-plane. While Nyquist's criterion may be used when the function is regular along the real axis and tends to zero at infinity, our new method works for a wider variety of functions and also offers intuitive understanding of the problem by phrasing it in terms of charges and fields. The labor can be much less than in the Nyquist method, especially when the Nyquist diagram divides the W plane into many parts, as we need only to find the sign of the real part of W at a few points. This saving of labor is particularly great when $\mathrm{Re}(W)$ is not easily computed all along the real axis of the u plane.

We have found that discontinuities in $f_0'(v)$ and zeros in $f_0(v)$ may produce instability, which suggests a reevaluation of the theory[2] that trapping of particles moving near the phase velocity of the wave is the cause of instability. In particular, for a case like Fig. 1(C), there are no particles moving at the phase velocity of a slowly growing wave. In fact, for a symmetric velocity distribution f_0 with two peaks and an extended region where $f_0=0$ between them, there are no particles anywhere near the wave phase velocity for a whole family of growing waves. (A $V=0$ line meets the real axis at the origin as there is a neutral point there.)

While the strange velocity distributions described here may not occur in nature, they are still meaningful, and any physical interpretation or understanding of plasma stability should encompass these types of plasma.

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