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T, P, C Symmetries in the π^0 Decay

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An analysis is given of the decay of the π^0 in which allowance is made for possible breakdowns in T, P, and C symmetries. It is shown that experiments, until now, have demonstrated only that the two-photon state is an eigenstate of TP, but not of T and P separately. A discussion of experiments which may verify T and P symmetry for the two-photon state is given.

I. INTRODUCTION

In the course of surveying the published literature on the decays of the neutral bosons, we were struck by the fact that there exists no published discussion of the π^0 decay which is sufficiently general to include the possibility that the T, P, and C symmetries may break down in the process. Since there is no experimental evidence for the decay $\pi^0\rightarrow 3\gamma$, we restrict our attention mostly to the main decay mode $\pi^0\rightarrow 2\gamma$. As is well known, the π^0 is an eigenstate of C with eigenvalue $C=1$. Therefore, we are lead to discuss the behavior of the 2γ state under P and T. Yang,[1] who was first to discuss the determination of the parity of the π^0 from measurements on the two-photon state, supposed that the two photons were either in a scalar or a pseudoscalar state, but not in a mixture. He proposed an experiment, based on measuring the correlation in angle between the planes of pairs produced in coincidence by the two photons, which would distinguish between these possibilities. In fact, if we call $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ the polarization vectors of the two photons and $\mathbf{K}$ their relative momentum vector, then the two-photon amplitude, $\langle\gamma\gamma|S|\pi^0\rangle$, is given by $\alpha\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'+\beta\boldsymbol{\varepsilon}\times\boldsymbol{\varepsilon}'\cdot\mathbf{K}/K$ and the correlation formulas which Yang gives distinguish between the cases $\alpha=0$ or $\beta=0$. Here S is the S matrix for the decay.

In a more recent paper Bernstein and Johnson[2] generalized the analysis to the situation where $\alpha\beta\neq 0$ but both $\alpha=\alpha^*$ and $\beta=\beta^*$. As we shall see there is, in general, no justification for this reality assumption. On the contrary, a necessary and sufficient condition that the two-photon state be an eigenstate of time reversal is that $\mathrm{Re}(\alpha\beta^*)=0$. Hence we have been led to give an analysis of the π^0 decay in which α and β are arbitrary complex numbers. It will turn out that the circular polarization of each photon in the decay is given by $\mathrm{Im}(\alpha\beta^*)$. Hence the recent experiment of Garwin *et al.*[3] which shows, within experimental error, that $\mathrm{Im}(\alpha\beta^*)=0$, does not prove that the two-photon state is an eigenstate of T and P separately, but rather of TP. Hence, since it is an eigenstate of C, it is also an eigenstate of CTP. The fact that the experiment of Garwin *et al.* does not separate T and P symmetries is somewhat similar to the fact that the nonexistence of an electric dipole moment for the neutron does not prove invariance under P, since T invariance leads to the same results.[4] In the last section of the paper, we discuss experiments which may serve to characterize the two-photon state completely.

A final remark may be in order about the spirit of this work. It is certainly the belief of most physicists, including the present authors, that in a production process like $\gamma+N\rightarrow\pi^0+N$ or in a decay process like $\pi^0\rightarrow 2\gamma$, which both proceed through strong and electromagnetic couplings, T and P symmetries are preserved. However, we feel that it is very important to have clearly in mind what has actually *been* proved by experiment and what one *expects* experiment to prove.

II. THE ANALYSIS

In order to fix the notation, we begin by a discussion of the state of one photon. A photon may be characterized with the help of three complex vectors; $\mathbf{k}$, the photon momentum, and $\boldsymbol{\varepsilon}_\pm(\mathbf{k})$, the circular polarization vectors. We have the usual orthonormality relations

$$\begin{gathered}\boldsymbol{\varepsilon}_\pm(\mathbf{k})\cdot\mathbf{k}=0,\\ \boldsymbol{\varepsilon}_\pm(\mathbf{k})\cdot\boldsymbol{\varepsilon}_\mp(\mathbf{k})=0,\\ \boldsymbol{\varepsilon}_\pm(\mathbf{k})\cdot\boldsymbol{\varepsilon}_\pm(\mathbf{k})=1,\end{gathered}\tag{1}$$

where the dot is understood in the sense of the Hermitian inner product. For the operations T, P, and TP one has

$$\begin{gathered}T\boldsymbol{\varepsilon}_\pm(\mathbf{k})=\boldsymbol{\varepsilon}_\pm(-\mathbf{k}),\\ P\boldsymbol{\varepsilon}_\pm(\mathbf{k})=\boldsymbol{\varepsilon}_\mp(-\mathbf{k}),\\ TP\boldsymbol{\varepsilon}_\pm(\mathbf{k})=PT\boldsymbol{\varepsilon}_\pm(\mathbf{k})=\boldsymbol{\varepsilon}_\mp(\mathbf{k}).\end{gathered}\tag{2}$$

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[1] C. N. Yang, Phys. Rev. **77**, 722 (1950). A similar experiment for the positronium annihilation had been proposed by J. A. Wheeler, Ann. N. Y. Acad. Sci. **48**, 219 (1946) and discussed in detail by M. H. L. Pryce and J. C. Ward, Nature **160**, 435 (1947) and H. S. Snyder, S. Pasternack, and J. Hornbostel, Phys. Rev. **73**, 440 (1948). Since the quanta in the positronium annihilation carry off energies which are of the order of the electron rest mass, it is feasible to use coincidence Compton scattering to measure correlations in the photon polarization vectors, and this is what these authors discuss.

[2] J. Bernstein and K. A. Johnson, Phys. Rev. **109**, 189 (1958).

[3] R. L. Garwin, G. Gidal, L. M. Lederman, and M. Weinrich, Phys. Rev. **108**, 1589 (1957).

[4] L. Landau, Nuclear Phys. **3**, 127 (1957).

In our future work we shall usually regard $\mathbf{k}$ as fixed and shall not make explicit reference to it.

It is useful to consider the correspondence

$$\boldsymbol{\varepsilon}_{\pm} \rightarrow |\boldsymbol{\varepsilon}_{\pm}\rangle, \tag{3}$$

where $|\boldsymbol{\varepsilon}_{\pm}\rangle$ are ket vectors in a Hilbert space. This correspondence may be shown to be linear in the sense that if the general pure state of polarization is represented by

$$\boldsymbol{\varepsilon}=C_{+}\boldsymbol{\varepsilon}_{+}+C_{-}\boldsymbol{\varepsilon}_{-}, \tag{4}$$

with

$$|C_{+}|^2+|C_{-}|^2=1, \tag{5}$$

then the corresponding ket is

$$|\boldsymbol{\varepsilon}\rangle=C_{+}|\boldsymbol{\varepsilon}_{+}\rangle+C_{-}|\boldsymbol{\varepsilon}_{-}\rangle, \tag{6}$$

with the same coefficients $C_{\pm}$.

We may construct the density matrix $|\boldsymbol{\varepsilon}\rangle\langle\boldsymbol{\varepsilon}|$ for the polarization of the one-photon state. This will be a 2×2 Hermitian matrix and hence may be expressed in terms of the Pauli matrices $\boldsymbol{\tau}$. Using the usual representation of these matrices, we have the following relations:

$$\begin{aligned} |\boldsymbol{\varepsilon}_{\pm}\rangle\langle\boldsymbol{\varepsilon}_{\pm}| &= \tfrac{1}{2}(1\pm\tau_3), \\ |\boldsymbol{\varepsilon}_{\pm}\rangle\langle\boldsymbol{\varepsilon}_{\mp}| &= \tfrac{1}{2}(\tau_1\pm i\tau_2), \end{aligned} \tag{7}$$

which lead to

$$|\boldsymbol{\varepsilon}\rangle\langle\boldsymbol{\varepsilon}|=\tfrac{1}{2}(1+\boldsymbol{\zeta}\cdot\boldsymbol{\tau}), \tag{8}$$

where

$$\boldsymbol{\zeta}=\mathrm{Tr}(|\boldsymbol{\varepsilon}\rangle\langle\boldsymbol{\varepsilon}|\boldsymbol{\tau}) \tag{9}$$

is the so-called Stokes vector.[5]

We may now turn to a discussion of the state of two photons in terms of their polarizations $\boldsymbol{\varepsilon}_{\pm}(+\mathbf{k})$, $\boldsymbol{\varepsilon}_{\pm}(-\mathbf{k})$ and momenta $\pm\mathbf{k}$. Once again fixing the momenta, the states of polarization form a four-dimensional Hilbert space. A natural basis for this space is the orthonormal set

$$|\boldsymbol{\varepsilon}_{+}\boldsymbol{\varepsilon}_{+}'\rangle,\quad |\boldsymbol{\varepsilon}_{-}\boldsymbol{\varepsilon}_{-}'\rangle,\quad |\boldsymbol{\varepsilon}_{+}\boldsymbol{\varepsilon}_{-}'\rangle,\quad |\boldsymbol{\varepsilon}_{-}\boldsymbol{\varepsilon}_{+}'\rangle,$$

where it is understood that we have symmetrized these kets with respect to the two photons.

If the total angular momentum of the two-photon state, $|\gamma\gamma\rangle$, is zero, then one can show on the grounds of general invariance principles that the complex number $\langle\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|\gamma\gamma\rangle$ is given by

$$\langle\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'|\gamma\gamma\rangle=\alpha\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}'+\beta\boldsymbol{\varepsilon}\times\boldsymbol{\varepsilon}'\cdot\mathbf{K}, \tag{10}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are any two polarizations and $\mathbf{K}$ is here the relative momentum vector of the two photons. We have specialized to the barycentric system.

Using Eq. (10) we may expand the two-photon state in the orthonormal basis described above, to find (where a, b, are $\pm$)

$$\begin{aligned} |\gamma\gamma\rangle &= \sum_{a,b}|\boldsymbol{\varepsilon}_a\boldsymbol{\varepsilon}_b\rangle\langle\boldsymbol{\varepsilon}_a\boldsymbol{\varepsilon}_b|\gamma\gamma\rangle \\ &= |\boldsymbol{\varepsilon}_{+}\boldsymbol{\varepsilon}_{+}'\rangle(\alpha+i\beta)+|\boldsymbol{\varepsilon}_{-}\boldsymbol{\varepsilon}_{-}'\rangle(\alpha-i\beta). \end{aligned} \tag{11}$$

It is convenient to take

$$\langle\gamma\gamma|\gamma\gamma\rangle=1, \tag{12}$$

which means

$$|\alpha|^2+|\beta|^2=\tfrac{1}{2}. \tag{13}$$

It is now straightforward to examine the effects of T,P, and TP on $|\gamma\gamma\rangle$. To this end we first record the actions of these operations on the vectors $|\boldsymbol{\varepsilon}_{\pm}\boldsymbol{\varepsilon}_{\pm}'\rangle$:

$$\begin{aligned} P|\boldsymbol{\varepsilon}_{\pm}\boldsymbol{\varepsilon}_{\pm}'\rangle &= |\boldsymbol{\varepsilon}_{\mp}\boldsymbol{\varepsilon}_{\mp}'\rangle, \\ T|\boldsymbol{\varepsilon}_{\pm}\boldsymbol{\varepsilon}_{\pm}'\rangle &= |\boldsymbol{\varepsilon}_{\pm}\boldsymbol{\varepsilon}_{\pm}'\rangle, \\ PT|\boldsymbol{\varepsilon}_{\pm}\boldsymbol{\varepsilon}_{\pm}'\rangle &= TP|\boldsymbol{\varepsilon}_{\pm}\boldsymbol{\varepsilon}_{\pm}'\rangle=|\boldsymbol{\varepsilon}_{\mp}\boldsymbol{\varepsilon}_{\mp}'\rangle. \end{aligned} \tag{14}$$

Therefore we have:

$$\begin{aligned} P|\gamma\gamma\rangle &= (\alpha-i\beta)|\boldsymbol{\varepsilon}_{+}\boldsymbol{\varepsilon}_{+}'\rangle+(\alpha+i\beta)|\boldsymbol{\varepsilon}_{-}\boldsymbol{\varepsilon}_{-}'\rangle, \\ T|\gamma\gamma\rangle &= (\alpha^*-i\beta)|\boldsymbol{\varepsilon}_{+}\boldsymbol{\varepsilon}_{+}'\rangle+(\alpha^*+i\beta^*)|\boldsymbol{\varepsilon}_{-}\boldsymbol{\varepsilon}_{-}'\rangle, \\ PT|\gamma\gamma\rangle &= TP|\gamma\gamma\rangle \\ &= (\alpha^*+i\beta^*)|\boldsymbol{\varepsilon}_{+}\boldsymbol{\varepsilon}_{+}'\rangle+(\alpha^*-i\beta^*)|\boldsymbol{\varepsilon}_{-}\boldsymbol{\varepsilon}_{-}'\rangle. \end{aligned} \tag{15}$$

The reader is reminded that T and TP are antilinear operations. Strictly speaking, the relations (15) are true only up to a phase.

We may now list the necessary and sufficient conditions for $|\gamma\gamma\rangle$ to be an eigenstate of T,P, and TP:

of P, if and only if $(\alpha-i\beta)/(\alpha+i\beta)=(\alpha+i\beta)/(\alpha-i\beta)$, i.e., if and only if $b=\pm 1$

of T, if and only if

$$(\alpha^*-i\beta^*)/(\alpha+i\beta)=(\alpha^*+i\beta^*)/(\alpha-i\beta),$$

i.e., if and only if $c=0$

of TP, if and only if

$$(\alpha^*+i\beta^*)/(\alpha+i\beta)=(\alpha^*-i\beta^*)/(\alpha-i\beta).$$

i.e., if and only if $d=0$.

Having chosen $a=|\alpha|^2+|\beta|^2=\frac{1}{2}$ we have defined

$$\begin{aligned} b &= 2(|\alpha|^2-|\beta|^2), \\ c &= 4\,\mathrm{Re}(\alpha\beta^*), \\ d &= 4\,\mathrm{Im}(\alpha\beta^*), \end{aligned} \tag{16}$$

and therefore

$$b^2+c^2+d^2=1.$$

Any experiment to measure the two photons will be represented by two Hermitian operators A and A'. For later use, we note that A must be a 2×2 matrix in the photon variables. This is because we have fixed the photon momentum and the photon polarizations span a two-dimensional space. Hence in terms of the Pauli matrices $\boldsymbol{\tau}$ and a real arbitrary vector $\mathbf{u}$, A may

[5] See, for example, U. Fano, J. Opt. Soc. Am. **39**, 859 (1949), where the density matrix for photons is discussed. See also J. M. Jauch and R. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, 1955) for a description of the Stokes vector.

be written as

$$A=r(1+\mathbf{u}\cdot\boldsymbol{\tau}), \tag{17}$$

with $u=|\mathbf{u}|<1$. Here r is a real number.

If the two photons are described by the density matrix

$$R=|\gamma\gamma\rangle\langle\gamma\gamma|, \tag{18}$$

then the outcome of the experiment is given by s,

$$s=\mathrm{Tr}[R(A\otimes A')], \tag{19}$$

where $\otimes$ means the direct product of two matrices. We may record a few properties of the direct product here, which will help the reader check our computations:

$$(A\otimes B)(C\otimes D)=AC\otimes BD,$$
$$\mathrm{Tr}[(A\otimes B)]=\mathrm{Tr}(A)\ \mathrm{Tr}(B),$$
$$(A\otimes B)^\dagger=A^\dagger\otimes B^\dagger.$$

We shall now write R in terms of the real numbers b, c, and d defined in Eq. (16). Since R is a 4×4 matrix, it can always be expressed in terms of the direct products of Pauli matrices in the following way:

$$R=\tfrac{1}{4}\{(1\otimes1)+\boldsymbol{\zeta}\cdot[(\boldsymbol{\tau}\otimes1)+(1\otimes\boldsymbol{\tau})]+\Gamma_{ij}(\tau_i\otimes\tau_j)\}. \tag{19}$$

We have made use of the identity of the two photons, and of our symmetrical treatment of them, in writing the second term on the right-hand side of Eq. (19). For the same reason we must have $\Gamma_{ij}=\Gamma_{ji}$. The length of the vector $\boldsymbol{\zeta}$ corresponds to the degree of polarization of each photon and the third component of this vector is the quantity determined in the experiment of Garwin *et al.*[3] To find the coefficients in Eq. (19) we simply use Eq. (11) for the explicit computation of $|\gamma\gamma\rangle\langle\gamma\gamma|$ and then compare terms. The reader will be spared the details of the arithmetic; the final answer is

$$R=\tfrac{1}{4}\{(1\otimes1)+d[(\tau_3\otimes1)+(1\otimes\tau_3)]+\Gamma_{ij}(\tau_i\otimes\tau_j)\}. \tag{20}$$

with [in the notation of Eq. (16)]

$$\Gamma_{ij}=\begin{bmatrix} b & -c & 0\\ -c & -b & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{21}$$

It is clear from the definitions that if $\alpha\beta=0$, only a measurement of b can distinguish the pure scalar from the pure pseudoscalar case. In order to characterize such measurements more sharply, we may now prove a theorem which is a generalization of a theorem proved in reference 2, where α and β were taken real. The theorem states: if A and A' are any two apparati which are symmetrical under rotations around the relative photon momentum, i.e., which do not contain an azimuthal angular dependence, then experiments done with A and A' can only measure the lifetime and $d=4\,\mathrm{Im}(\alpha\beta^*)$. The proof is immediate in our formalism; for by the assumption of azimuthal symmetry we must have

$$A=r(1+u_3\tau_3), \tag{22}$$

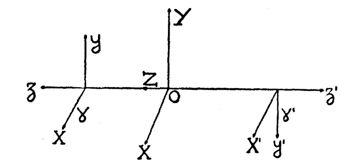

FIG. 1. In this figure the photon coordinate systems are shown. We make use of the system $OXYZ$ in writing Eq. (26).

i.e., A must be of the general form of Eq. (17) and must also commute with rotations about the 3 direction.

In fact, a rotation of angle φ around $\mathbf{K}$ in the two-dimensional Hilbert space, with basis $|\boldsymbol{\varepsilon}_+\rangle, |\boldsymbol{\varepsilon}_-\rangle$ is represented by the diagonal matrix

$$\Omega(\varphi)=\cos(\varphi)-i\tau_3\sin(\varphi). \tag{23}$$

It is important to notice that this rotation induces a transformation on $\mathbf{u}$ or $\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta}\rightarrow\boldsymbol{\zeta}'$ where

$$\begin{aligned}&\boldsymbol{\zeta}'\cdot\boldsymbol{\tau}=\Omega(\varphi)\boldsymbol{\zeta}\cdot\boldsymbol{\tau}\Omega^{-1}(\varphi),\\ &\zeta_3'=\zeta_3,\\ &\zeta_1'=\zeta_1\cos(2\varphi)-\zeta_2\sin(2\varphi),\\ &\zeta_2'=\zeta_1\sin(2\varphi)+\zeta_2\cos(2\varphi).\end{aligned} \tag{24}$$

This is a "rotation" through 2φ.

Now using Eq. (22) we have,

$$s=\mathrm{Tr}[R(A\otimes A')]=rr'[1+d(u_3+u_3')+u_3u_3'], \tag{25}$$

and all dependence on b and c has vanished. By a similar argument it is easy to see that the same result obtains if only one of the apparati is azimuthally symmetric and the other one arbitrary. Therefore it requires a coincidence experiment with an azimuthal correlation to measure b and c.

We must finish this section with a trite but necessary question of notation. Our explicit use of the conventional representation of the Pauli matrices in an equation like Eq. (7) implies that we have chosen a right-handed coordinate frame for each of the two photons, with the z axes along their momenta. Since the photons have opposite momenta these frames do not coincide (see Fig. 1). Customarily physicists choose a single set of axes to describe the two photons together; for example the set $0XYZ$ of Fig. 1. Therefore let us call, arbitrarily, γ the proton whose momentum is along the positive $0Z$ direction (the other photon will be denoted by γ'). We shall make the convention that all matrices on the left of the $\otimes$ symbol in R refer to this photon (photons are now distinguished by our description of them). We may rewrite R of Eq. (20) in this new coordinate frame simply by changing the sign of the coefficient of the matrices τ_3 and τ_2 on the right of the $\otimes$ symbol. This new matrix we call R' and we have

$$\begin{aligned}R'=\tfrac{1}{4}\{&[(1\otimes1)-(\tau_3\otimes\tau_3)]+d[(\tau_3\otimes1)-(1\otimes\tau_3)]\\ &+b[(\tau_2\otimes\tau_2)+(\tau_3\otimes\tau_3)]\\ &+c[(\tau_1\otimes\tau_2-(\tau_2\otimes\tau_1)]\}.\end{aligned} \tag{26}$$

When $b=-1$, R' reduces to

$$R'=\tfrac{1}{4}[(1\otimes 1)-(\boldsymbol{\tau}\otimes\boldsymbol{\tau})], \tag{27}$$

an expression which has already been given by Fano[6] with applications to positronium annihilation in mind.

In the next section we give a discussion of measurements of b and c.

III. MEASUREMENT OF b AND c

In the last section we gave necessary conditions for measuring the b and c terms. To see how this measurement is to be carried out, we consider coincidence experiments with two apparati A and A' which have been designed to analyze plane polarizations. In our formalism the description of A, for example, is very simple. A is characterized by a "Stokes vector" $\mathbf{u}$ with components

$$\mathbf{u}(u\cos(2\phi_a),u\sin(2\phi_a),0). \tag{28}$$

The number $|u|\leq 1$ gives the efficiency of the apparatus as a plane polarization analyzer and ϕ_a is an azimuth characterizing the "setting" of the apparatus. For example, in Compton scattering ϕ_a may be chosen as the azimuth of the scattering plane, for pair production as the azimuth of the normal to the plane of the pair, or for deuteron photodisintegration, as the azimuth of the plane of the final nucleons. The angle ϕ_a is defined only up to π.

We may also define ϕ_γ to be the azimuth of the electric vector of the photon whose Stokes vector is

$$\boldsymbol{\zeta}=(\zeta\cos(2\phi_\gamma),\zeta\sin(2\phi_\gamma),0). \tag{29}$$

Clearly ϕ_γ is also only defined up to π. We must have $|\zeta|\leq 1$.

The rate of any experiment done with A is

$$\begin{aligned}\sigma&=\mathrm{Tr}[\tfrac{1}{2}(1+\boldsymbol{\zeta}\cdot\boldsymbol{\tau})r(1+\mathbf{u}\cdot\boldsymbol{\tau})]=r(1+\boldsymbol{\zeta}\cdot\mathbf{u})\\&=r[1+\zeta u\cos(2\phi)],\end{aligned} \tag{30}$$

where $\phi=\phi_\gamma-\phi_a$. For unpolarized photons, $\zeta=0$ and $\sigma=r$. We may remark, parenthetically, that in many computations given in the literature[7] for totally plane polarized photons, i.e., $\zeta=1$, the reader will find the cross section given in the form

$$\sigma=\sigma_0[1+\eta\cos^2(\phi)]. \tag{31}$$

To go from this form to Eq. (30) is simply a matter of making the substitutions,

$$r=\sigma_0(1+\eta/2),$$
$$u=\eta/(2+\eta).$$

In terms of the ideas and notations already presented, a coincidence measurement on the two γ's from the π^0 decay using two apparati characterized by r, $\mathbf{u}(u,2\phi_a)$ and r', $\mathbf{u}'(u',2\phi_a')$ is given by

$$\begin{aligned}\mathrm{Tr}[R'(A\otimes A')]&=rr'(1+b\mathbf{u}\cdot\mathbf{u}'+c\mathbf{u}\times\mathbf{u}'\cdot\mathbf{K})\\&=rr'\{1+uu'[b\cos(2\omega)\\&\qquad+c\sin(2\omega)]\},\end{aligned} \tag{30}$$

where

$$\omega=\phi_a'-\phi_a.$$

In any actual experiment one would probably choose the two measuring instruments, A and A', as nearly identical as possible. In this case Eq. (32) reduces to;

$$s=r^2\{1+u^2[b\cos(2\omega)+c\sin(2\omega)]\}.$$

If only relative counts are made, such a coincidence experiment is characterized by a single pure number u^2 (with $0\leq u^2\leq 1$) which is fixed by the experimental geometry. We may give a few examples which are relevant for the π^0 decay and which have been taken from the literature.

Yang, in his original work,[1] takes an idealized case of two infinitesimal counters counting pairs in coincidence and finds $u^2=\frac{1}{16}$. In a recent paper,[8] Karlson has studied a less idealized situation in which all pairs produced by both photons are observed, for instance in a bubble chamber. Here u^2 is considerably reduced; he finds $u^2=0.0091$. On the other hand, Kroll and Wada[7] have noted that the angular correlation between pairs produced by internal conversion of the virtual photons from the π^0 decay can also serve to measure the parameters of the virtual two-photon state. The important photons in this process are nearly real and our formulas can be taken over from the real photon case. Kroll and Wada give a branching ratio for $2(e^++e^-)/2\gamma$ of 3.47×10^{-5} and find $u^2=0.18$, i.e., the distribution of angle between the two planes of the pairs is $s=1+0.18[b\cos(2\omega)+c\sin(2\omega)]$.

An experiment to measure this correlation is in process.[9] It is clear from Eq. (33) that any experiment designed to measure b can also be used to measure c, and a complete determination of the two-photon state requires the measurement of the three numbers b, c, and d. In Eq. (16) we have stated the relation $\rho\equiv b^2+c^2+d^2=1$. This relation should also be tested empirically since in the case that the π^0 is not a pure state we would have $0\leq\rho<1$.

Finally, we would like to remind the reader of a few facts about charge conjugation. If $S|\pi_0\rangle$ should not be an eigenstate of C, the component corresponding to the eigenvalue $c=-1$ could not, anyway, appear in the decay mode $\pi^0\rightarrow e^++e^-$ (since an angular momentum 0, e^++e^- state has $c=+1$). Hence, if we take the point of view that the branching ratios among the different possible decay modes of the π^0 (for the same value of c) may be related to each other by quantum electrodynamics, the fastest decay into a $c=-1$ state is, in principle, $\pi^0\rightarrow 3\gamma$. Indeed, it is easily shown that

[6] U. Fano, Revs. Modern Phys. **29**, 74 (1957).
[7] See, for example, N. Kroll and W. Wada, Phys. Rev. **98**, 1355 (1955).
[8] E. Karlson, Arkiv Fysik, **13**, No. 1 (1958).
[9] J. Steinberger (private communication.)

any graph in which the π^0 decays into a single photon, real or virtual, vanishes on the grounds of current conservation, quite apart from other symmetry principles. At present, the experimental limit on the 3γ decay mode is not strong evidence for or against such a $c=-1$ component.

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Energy Renormalization in Ordinary Wave Mechanics*

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A very simple, exactly soluble compound-particle model, proposed by Wigner and Weisskopf in 1930, is briefly re-examined from the standpoint of renormalization. It consists of postulating, in the center-of-mass system, the wave equations

$$[i(\partial/\partial t)+(1/2m)\nabla^2]\psi(\mathbf{x},t)=F(\mathbf{x})\chi(t),$$
$$[i(d/dt)-\mu]\chi(t)=\int d^3x\, F(\mathbf{x})\psi(\mathbf{x},t)$$

for two particles of separation $\mathbf{x}$ and reduced mass m, interacting through the formation and decay of an intermediate particle with a real form factor F. The analytic behavior of the S matrix is discussed in the local case $F(\mathbf{x})=C\delta(\mathbf{x})$.

I. INTRODUCTION

THE purpose of this note is to point out a very simple example, involving neither second quantization nor relativity, of a theory with energy renormalization and virtual particles. This model, which is soluble exactly, was proposed by Wigner and Weisskopf[1] in 1930, and studied again, independently and from a different point of view, by Moshinsky[2] in 1951. It will be briefly re-examined in this paper from the standpoint of renormalization. It then turns out to be closely related to the so-called one-particle sector of the Lee model.[3]

We consider the following two systems, which can decay into each other: (a) a motionless particle, located at the origin, and whose wave function[4] $\chi(t)$ depends only on time; (b) a moving particle of mass m, whose wave function $\psi(\mathbf{x},t)$ depends also on the position $\mathbf{x}$. Thus the state vector can be represented in Fock space by two components:

$$|t\rangle=|\psi(\mathbf{x},t),\chi(t)\rangle. \tag{1.1}$$

The scalar product is

$$\langle\psi_1,\chi_1|\psi_2,\chi_2\rangle=\int d^3x\,\psi_1{}^*\psi_2+\chi_1{}^*\chi_2. \tag{1.2}$$

The postulated equations of motion are

$$[i(\partial/\partial t)+(1/2m)\nabla^2]\psi(\mathbf{x},t)=F(\mathbf{x})\chi(t), \tag{1.3}$$

$$[i(d/dt)-\mu]\chi(t)=\int d^3x\, F(\mathbf{x})\psi(\mathbf{x},t). \tag{1.4}$$

They have the following features: (a) the only interaction consists of each particle acting as a source for the other; (b) F is a given real form factor and μ a given real energy, so that time-reversal invariance holds; (c) the Hamiltonian H, defined by

$$i(d/dt)|t\rangle=H|t\rangle \tag{1.5}$$

is Hermitian under the scalar product (1.2), so that probability is conserved.

The lack of translational invariance is not an essential restriction. The model equivalently deals with the formation and decay of a compound particle (considered as elementary) in the center-of-mass system, m being the reduced mass. The case of main interest is the local limit

$$F(\mathbf{x})\rightarrow C\delta(\mathbf{x}) \tag{1.6}$$

for a real coupling constant C. For simplicity we assume F to be spherically symmetric about the origin.

2. STATIONARY SCATTERING STATES AND THE S MATRIX

We define Fourier transforms by

$$F(\mathbf{x})=(2\pi)^{-3}\int d^3p\, G(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{2.1}$$

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* Most of this work was carried out at Brandeis University, Waltham, Massachusetts, and supported by the Office of Naval Research.

[1] E. P. Wigner and V. Weisskopf, Z. Physik **63**, 62 (1930).

[2] M. Moshinsky, Phys. Rev. **81**, 347 (1951); **84**, 525 (1951).

[3] G. Sandri has derived the present model as the lowest sector of a Lee Model with nonrelativistic mesons (private communication).

[4] The Schrödinger picture is used throughout.