Moments of Inertia of Even-Even Rare Earth Nuclei*

J. J. GRIFFIN AND M. RICH
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico

(Received November 16, 1959)

Moments of inertia of even-even nuclei are computed using the Nilsson model for deformed nuclei and the moment formula derived from the superconductor theory of nuclei. Values for the energy gap and the deformation are obtained from appropriate experimental data. Good agreement is found between the computed and observed energies of the first excited states of twenty-six rare earth nuclei. This success lends strong support to the superconductor theory of the nucleus.

I. INTRODUCTION

THE problem of calculating moments of inertia of spheroidal nuclei is one which has received considerable attention[1–3] since the inception of the collective model of the nucleus. In the early treatments[4,5] of collective motions, it was found convenient to regard the nucleus as a droplet of incompressible irrotational fluid. Inertial parameters were ascribed to the action of surface waves on the nuclear droplet, which involved the flow of only a small fraction of the nuclear fluid. The inadequacy of such a picture became evident as experimental information established that the actual moments are approximately five times larger than it implies.[6]

The irrotational fluid model of nuclear moments was superseded by the "cranking model"[1] in which the nucleus was treated as an aggregate of nucleus moving independently in a deformed average potential well, and the effect of an externally driven rotation of the well was taken into account via perturbation theory. Excitation of nucleons near the top of the Fermi sea, due to the rotational Coriolis perturbation, led to rigid body values for the moments of inertia. These are two or three times larger than the observed moments.[7]

A possible solution to this dilemma was offered when Bohr and Mottelson[2] proposed that residual two-body interactions could be responsible for moments of inertia smaller than the rigid value. If nucleons in conjugate states[8] of the average spheroidal potential interacted strongly, an additional pairing energy D would be required to excite such a paired nucleon to an unpaired state above the Fermi sea. As a result, the "cranking model" formula would be modified by the addition of the pairing energy to the energy denominators:

$$\mathfrak{F}=2\hbar^2 \sum_{i(\text{unfilled})} \sum_{k(\text{filled})} \frac{|\langle i|j_x|k\rangle|^2}{E_i-E_k+D}. \tag{1}$$

These authors estimated that a pairing energy equal to about one-third of the characteristic independent particle level spacing would bring the moments calculated with Eq. (1) into approximate agreement with observation. Detailed calculations using this formula[9] have shown that it is capable of giving correct magnitudes and many of the general features of the structure in the moments in the rare earth region, although only when the pairing energies are assumed to be about twice those observed experimentally.

This approach is based on the assumption that the principal effect of the residual two-body interactions is to modify the single particle energies without significant changes of the eigenfunctions. However, consistent perturbation treatments[10,11] of two-body interactions in large systems of Fermi particles have shown that the modification of the wave functions is such as to cancel exactly the effect of the energy shifts on the moment of inertia. Thus, serious doubt is cast upon the validity of merely augmenting the energy denominators by a pairing term as in Eq. (1), provided that pairing corrections to the nuclear moments can actually be obtained by direct application of perturbation theory. There is some basis for questioning whether such perturbative expansions are applicable to nuclei.

Following a suggestion of Bohr, Mottelson, and Pines[12] that the systematics of the excitations in even-even nuclei indicated the possible existence of an energy gap in the nuclear level structure, Belyaev[13] has recently applied the formalism of the theory of super-

* Work performed under the auspices of the U. S. Atomic Energy Commission.

1 D. Inglis, Phys. Rev. **96**, 1059 (1954).

2 A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1 (1955).

3 R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. (London) **A70**, 381 (1957); J. Yoccoz, **A70**, 388 (1957).

4 A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1954).

5 A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

6 K. Ford, Phys. Rev. **95**, 1250 (1954).

7 This model is quite successful, however, in describing the magnitude of the *differences* between moments of odd mass and neighboring even-even nuclei. See O. Prior, Arkiv Fysik (to be published), cited in reference 22.

8 Single particle states of an axially symmetric deformed potential are doubly degenerate. The degenerate pair have identical quantum numbers except for the signs of the projections of their angular momenta along the symmetry axis. Such a pair of states will be termed conjugate.

9 J. Griffin and M. Rich, Bull. Am. Phys. Soc. **4**, 255 (1959).

10 R. Amado and K. Brueckner, Phys. Rev. **115**, 778 (1959).

11 R. Rockmore, Phys. Rev. **116**, 469 (1959).

12 A. Bohr, B. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958).

13 S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 11 (1959); also *The Many-Body Problem* (John Wiley and Sons, Inc., New York, 1958).

conductors[14–16] to the problem of nuclear matter. He has shown that if the two-body interactions are such as to create a self-consistent field plus residual attractive interactions, effective principally between nucleons in conjugate states, an energy gap in the single particle spectrum will develop. On the basis of this theory, Belyaev has obtained the following result[17,18] for the rotational inertia of deformed even-even nuclei:

$$\mathfrak{F}=\hbar^2\sum_{i,k}\frac{|\langle i|j_x|k\rangle|^2}{E_i+E_k}\left[1-\frac{(\tilde{\epsilon}_i-\lambda)(\tilde{\epsilon}_k-\lambda)+\Delta^2}{E_iE_k}\right], \quad (2)$$

where $\tilde{\epsilon}_i$ are the eigenvalues of the self-consistent field, λ is the "chemical potential" and is approximately equal to the Fermi energy of the system, and E_i are the energies of elementary excitations of the "superconducting" nucleus given by

$$E_i=[(\tilde{\epsilon}_i-\lambda)^2+\Delta^2]^{\frac{1}{2}}. \quad (3)$$

Δ is equal to the "superconducting" energy gap. The sums in Eq. (2) are over all distinct levels of the average potential. In the limit of vanishing energy gap, Eq. (2) reduces to the usual "cranking model" formula for the moment of inertia. This expression exhibits two mechanisms for the reduction of nuclear moments below the rigid value. First, the energy denominators are effectively increased due to the presence of the gap. In addition, the bracketed term in Eq. (2) approaches zero as $\tilde{\epsilon}_i\rightarrow\tilde{\epsilon}_k$ regardless of relative sign of $(\tilde{\epsilon}_i-\lambda)$ and $(\tilde{\epsilon}_k-\lambda)$, thus suppressing contributions from transitions between closely neighboring independent particle states.

In the following sections, we report calculations of nuclear moments of inertia in the rare earth region based on Eq. (2). The relevant parameters of the superconductor theory are discussed in Sec. II, together with the model chosen for the self-consistent field. In Sec. III, the theoretical results are compared with experiment, and some discussion of their sensitivity to the assumptions required is provided.

II. THE NILSSON MODEL AND THE SUPERCONDUCTOR THEORY

In order to apply Belyaev's formal development of the superconductor theory of the nucleus to specific nuclei, it is necessary to select a model for the self-consistent field. We have assumed that the eigenvalues and eigenfunctions of the Nilsson model[19] correspond to the eigenvalues $\tilde{\epsilon}_j$ and eigenfunctions of Belyaev's self-consistent field, with some minor modifications to be discussed below. The quadrupole deformations used are those of the charge distributions as obtained from Coulomb excitation and collective electric quadrupole decay rates.[20]

Neutrons and protons are treated separately. A chemical potential λ is obtained for each by solving the equation[13]

$$\sum_i\left(1-\frac{\tilde{\epsilon}_i-\lambda}{[(\tilde{\epsilon}_i-\lambda)^2+\Delta^2]^{\frac{1}{2}}}\right)=N. \quad (4)$$

The sum here, as in Eq. (2), runs over all distinct neutron (or proton) energies. In practice, the sums have been taken only over states of the partially filled oscillator shells of the Nilsson model. The small contributions to the moment of inertia from completely filled oscillator shells are neglected. Then N is the number of neutrons (or protons) present in the uncompleted shells. The same value of the gap parameter Δ is assumed for both neutrons and protons. Equation (4) gives values for λ_N (or λ_P) very nearly equal to the energy of the last independent particle level which would have been filled if the pairing interaction were zero.

There remains unspecified in Eqs. (2) and (4) only the magnitude of the energy gap, Δ. In principle, this is determined uniquely by the form of the self-consistent field and the elementary nucleon-nucleon interaction. We have chosen in this work to use empirical values of this quantity derived from data on neutron separation energies in the rare earth region[21] (see below).

With this choice, all of the parameters in Eqs. (2) and (4) are determined, and the theoretical value for the moment of inertia can be computed. Actually, there are some ambiguities which arise from (a) the interpretation of the data on the energy gap, (b) the experimental uncertainties in the quadrupole deformation, and (c) certain defects of the Nilsson model indicated by the analysis of the spectra of odd-mass nuclei.[22] The nature of these uncertainties is discussed in more detail below.

Specification of the Energy Gap Parameter

The energy gap parameter, Δ, is approximately equal to one-half the pairing energy, which can be defined[23]

[14] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).
[15] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[16] N. Bogolyubov, Soviet Phys.-JETP **7**, 41 (1958); Nuovo cimento **7**, 794 (1958).
[17] To facilitate the mathematical treatment of the theory, it is necessary[13] to work in a representation which averages over an ensemble of nuclei having a specified mean number of nucleons. Further, a different "energy gap" Δ_i is, in general, associated with each single particle level i. To reduce the theory to a tractable form, Belyaev assumed all gaps to be equal; thus, in a sense, Δ of Eq. (2) is an average of various Δ_i.
[18] The derivation of the rotational moment in reference 13 neglects possible matrix elements arising from transitions of the form $\Omega=\frac{1}{2}\rightarrow\Omega=-\frac{1}{2}$. Inclusion of such effects does not change his final moment formula.

[19] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955).
[20] K. Alder et al., Revs. Modern Phys. **28**, 432 (1956), and references cited therein.
[21] W. H. Johnson and V. B. Bhanot, Phys. Rev. **107**, 1669 (1957).
[22] B. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Skrifter **1**, No. 8 (1959).
[23] This is the definition of reference 21 from which the gap parameters used in the present calculations were derived.

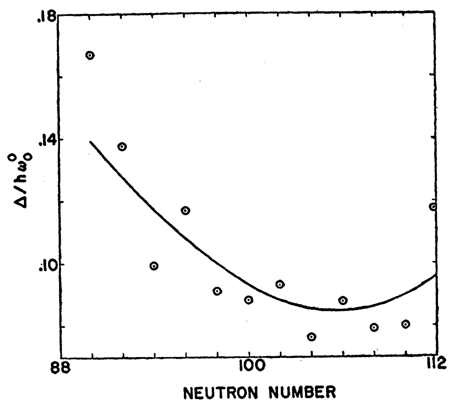

FIG. 1. Values for the energy gap Δ used in the present calculations are plotted as a function of neutron number. Δ is given in units of $\hbar\omega_0{}^0$. The corresponding experimental quantities $(0.5P_N)$ of reference 21 are indicated by circled points.

as the difference between the nucleon separation energies for an even-even and a neighboring odd-even nucleus:

$$\begin{aligned} P_N &= S(Z,N) - S(Z,N-1) \\ &= 2\times M(Z,N-1) - M(Z,N) - M(Z,N-2). \quad (5) \end{aligned}$$

An exact equality between the energy gap and one-half the observed pairing energy does not hold. An estimation of P_N using the expression for the ground state energy[13] of a "superconducting" nucleus and neglecting deformation differences among nuclei indicates that the pairing energy may differ from 2Δ by an amount of the order of one-half the average level spacing in the self-consistent field. Further, P_N contains a rearrangement energy contribution due to differences in deformation between nucleus $(Z,N-1)$ and nucleus (Z,N). We have therefore disregarded the irregular structure in the empirical pairing energies and have instead constructed a smooth curve through the measured points. Values of Δ have been taken from this curve, and the same (neutron) data was used to determine gaps for both neutrons and protons.

The experimental data of Johnson and Bhanot[21] is shown in Fig. 1, together with the curve used to specify Δ in our calculations. The high values of P_N at $N=90$ and $N=112$ have not been given full weight in constructing the smoothed curve, because they are presumably associated with the sudden decrease of the nuclear quadrupole deformation which occurs characteristically as one approaches closed shells, and it is desirable to eliminate such spurious rearrangement effects. Moreover, the formula used in computing the moments of inertia already represents an average over several neighboring even-even nuclei,[13] and the gap itself an average over gaps corresponding to various single particle levels within these nuclei. These are additional reasons for eliminating sharp structure of this kind.

Although it does not seem possible to specify Δ uniquely from the data presently available, Fig. 1 suggests that the smooth curve is probably accurate[24] to about $\pm 0.01\hbar\omega_0{}^0$ on the average. We have, therefore, computed the percentage change in the rotational energy which would result from such a variation in Δ, to provide a measure of the uncertainty arising from this ambiguity. Column 8 of Table I shows this uncertainty to be at most about 10%.

Modification of the Nilsson Model

Certain modifications of the relative positions of the model eigenvalues within a given major shell were made in order to remove obvious inconsistencies between the Nilsson model and the data on low-lying spectra of odd-mass nuclei.[22] The $h_{11/2}$ proton eigenvalues were decreased by $0.2\hbar\omega_0{}^0$, and the remainder of the $N=5$ proton eigenvalues were increased by $0.2\hbar\omega_0{}^0$ to make the model consistent with the spin sequence of odd-proton nuclei. Also, all $N=6$ neutron eigenvalues except those of the $i_{13/2}$ subshell were increased by $0.1\hbar\omega_0{}^0$. This change makes the model consistent with the fact that the level[25] $\frac{1}{2}+[651]$ is not observed in rare earth spectra below 0.5 Mev. Finally, the eigenvalues of neutron level $\frac{1}{2}-[521]$ were increased by $0.04\hbar\omega_0{}^0$ to give agreement with the observed ground-state spins of nuclei with 99 and 101 neutrons.

These changes affect numerous levels which are not actually filled in the rare earth region. However, the contributions of such levels to the computed moments

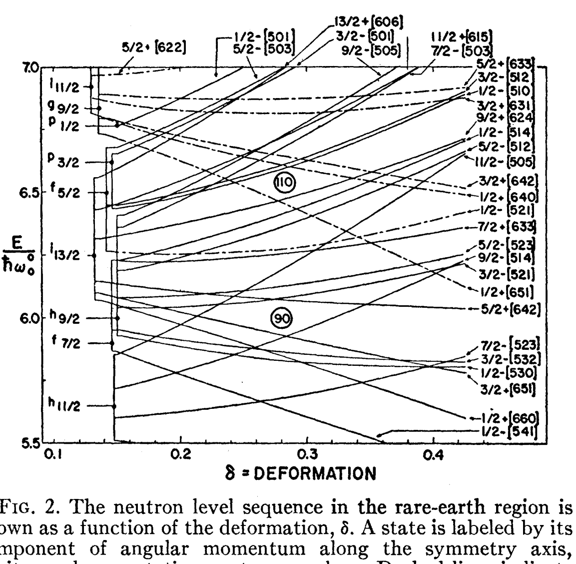

FIG. 2. The neutron level sequence in the rare-earth region is shown as a function of the deformation, δ. A state is labeled by its component of angular momentum along the symmetry axis, parity, and asymptotic quantum numbers. Dashed lines indicate levels which have been shifted from the original assignments of reference 19 (see text).

[24] Except where otherwise stated, energies are given in units of $\hbar\omega_0{}^0$ and quadrupole deformations in terms of the parameter δ, in conformity with the presentation of Nilsson.[19] For a nucleus of mass number A, $\hbar\omega_0{}^0 \simeq 41\times A^{-\frac{1}{3}}$ Mev.

[25] Energy levels are designated by the spin, parity, and the asymptotic quantum numbers $[N,n_z,\Lambda]$ appropriate in the limit of large deformation (see reference 22).

TABLE I. Rotational energies of even-even nuclei. The first column lists the nuclei for which moments of inertia have been calculated. Columns 2 and 3 give the deformation and energy gap used in calculating the energy of the lowest (2+) rotation excitation. The experimental and theoretical energies are listed in columns 4 and 5, and the percentage error given in column 6. Columns 7 and 8 provide a measure of the uncertainties in the calculated rotational energies due to inadequate knowledge of the parameters δ and Δ. The percentage change in the computed energy resulting from an increase in δ by 0.01 is given in column 7, and the percentage change due to an increase of Δ by $0.01\hbar\omega_0^0$ ($\sim$70 kev) is shown in column 8. The final column lists the percentage change in the computed rotational energy resulting from the shift in the $N=6$ neutron levels as compared to calculations with these states unshifted.

Nucleus	δ_{Coulomb}	$\Delta/\hbar\omega_0^0$	E_{exp}	E_{theor}	Percent error	$\frac{1}{E}\left(\frac{\partial E}{\partial \delta}\right)$	$\frac{\hbar\omega_0^0}{E}\left(\frac{\partial E}{\partial \Delta}\right)$	Percent change from level shift
$_{60}Nd_{90}^{150}$	0.24	0.139	130	139.5	+7.3	−5.8	11	−0.1
$_{62}Sm_{90}^{152}$	0.27	0.139	122	123.0	+0.8	−3.7	10	−0.1
Sm_{92}^{154}	0.31	0.128	83	99.7	+20.1	−2.1	8	−0.7
$_{64}Gd_{90}^{154}$	0.28	0.139	123	120.0	−2.4	−3.2	9	−0.2
Gd_{92}^{156}	0.39	0.128	89	85.6	−3.8	−3.1	6	−0.4
Gd_{94}^{158}	0.44	0.117	79	68.3	−13.5	−2.0	5	−2.1
Gd_{96}^{160}	0.45	0.108	76	66.1	−13.0	−1.3	5	−1.9
$_{66}Dy_{94}^{160}$	0.33	0.117	86	92.1	+7.1	−1.3	7	−1.3
Dy_{96}^{162}	0.34	0.108	82	84.4	+2.9	−1.2	7	−4.5
Dy_{98}^{164}	0.39	0.100	73	70.0	−4.1	−1.7	6	−3.6
$_{68}Er_{96}^{164}$	0.31	0.108	90	88.3	−1.9	−1.3	8	−1.6
Er_{98}^{166}	0.31[a]	0.100	80	85.9	+7.4	−1.4	7	−6.4
Er_{100}^{168}		0.093	80	83.7	+4.6	−2.3	7	−7.8
Er_{102}^{170}		0.088	79	77.7	−1.6	−3.5	7	−11.4
$_{70}Yb_{100}^{170}$	0.28	0.093	84	86.1	+2.5	−1.2	7	−18.7
Yb_{102}^{172}	0.29[a]	0.088	78	80.1	+2.7	−1.8	8	−5.8
Yb_{104}^{174}		0.085	76	77.1	+1.4	−3.6	8	−13.4
Yb_{106}^{176}		0.085	82	74.7	−8.9	−6.6	8	−15.8
$_{72}Hf_{104}^{176}$	0.28	0.085	89	83.8	−5.8	−2.4	8	−10.9
Hf_{106}^{178}	0.29	0.085	91	78.5	−13.7	−7.2	8	−16.5
Hf_{108}^{180}	0.26	0.086	93	90.3	−2.9	−4.6	8	−16.9
$_{74}W_{108}^{182}$	0.25	0.086	100	101.9	+1.9	−3.2	8	−15.0
W_{110}^{184}	0.23	0.090	112	110.9	−0.9	−4.2	9	−14.5
W_{112}^{186}	0.23	0.096	124	118.3	−4.6	−8.4	9	−20.1
$_{76}Os_{110}^{186}$	0.19	0.090	137	130.9	−4.4	−2.8	11	−4.1
$_{76}Os_{112}^{188}$	0.17	0.096	155	166.8	+7.6	−4.1	11	−3.7

[a] These values were obtained from measurements on unseparated isotopes.

are associated with large energy denominators so that their shift should involve only very minor changes in the moment of inertia. The levels which are important for the moments, on the other hand, are just those whose empirical behavior demands the modifications listed above.

To test the sensitivity of the moments to such level shifts, we have computed for each nucleus the percentage change in the computed rotational energy which results when the shift of the $N=6$ neutron levels is omitted. These percentages are listed in column 9 of Table I. They should not be interpreted as probable errors associated with lack of empirical information on the levels in question. The analysis of reference 22 establishes the qualitative fact that these levels lie somewhat higher than Nilsson's original model predicts. The shift used in the present calculations ($0.1\hbar\omega_0^0 \sim 700$ kev) is as small in magnitude as is consistent with the available data on low-lying spectra.[22] The levels may, in fact, lie somewhat higher than assumed here. Thus, the sign of any actual error associated with an improper placement of $N=6$ neutron levels will probably be opposite to that listed in Table I.

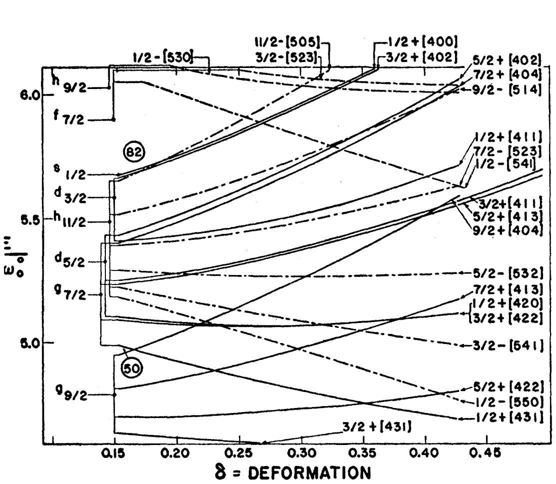

FIG. 3. This figure is similar to Fig. 2, representing the proton energy levels.

The proton modifications involve levels whose positions are better defined by experiment than those of the neutron states. Moreover, the protons contribute only about one-third of the total moment of inertia. Thus, the changes associated with their shifts should be at most of the same order as those in Table I. The neutron and proton eigenvalues used in the calculations are plotted against the quadrupole deformation in Figs. 2 and 3.

Changes in the wave functions associated with the

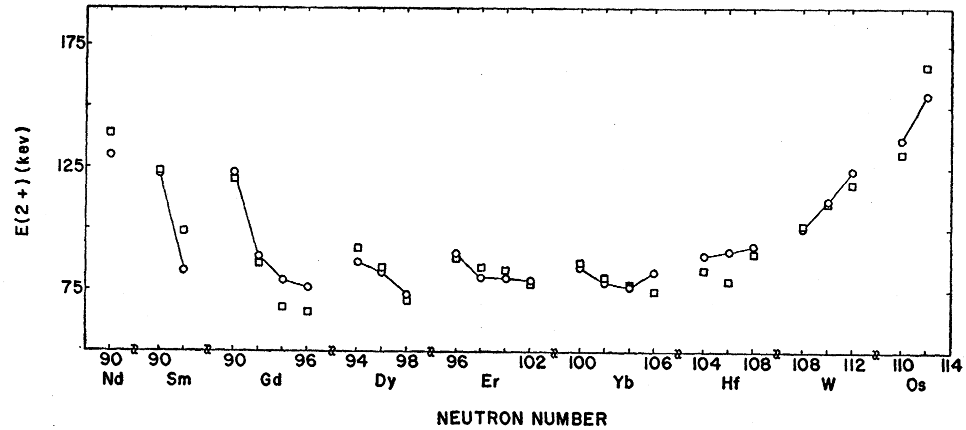

FIG. 4. The experimental and theoretical energies of the first rotational excitations of the nuclei considered in this paper are plotted versus neutron number. Empirical energies are denoted by circles and calculated values as squares.

eigenvalue modifications have been neglected. This neglect should cause only a small error in the computed moments because the level ordering within a shell is not modified by the shifts. Thus, the general character of the wave functions (and, in particular, the asymptotic behavior for large deformations) is unchanged.

To allow convenient calculation for a complete range of deformations, the eigenvalues and eigenfunction coefficients of Nilsson were interpolated parabolically for positive deformations. The same parabolas were also used to extrapolate to deformations beyond the range of reference 19. This procedure seems justified by the generally parabolic character of the eigenvalues and the smooth variation of the eigenfunction coefficients in the region of large deformations.

III. RESULTS AND CONCLUSIONS

Throughout this work, it has been assumed that Eq. (2) can be applied to individual nuclei and that the averaging process involved in its derivation does not smear out the variation of the calculated moments from nucleus to nucleus.[26] With this assumption in mind, the results of the computations are given in Table I. Listed are the nuclei, the deformations and the energy gap parameters assumed in the calculation, and the theoretical and observed[20,27] energies for the first rotational state ($E_{2+}=3\hbar^2/\mathfrak{F}$). Also indicated are the percentage differences between theory and experiment. The experimental and theoretical energies of Table I are plotted in Fig. 4.

The agreement is remarkably good, with an average error of only about 6% and a maximum error of 20% in the theory. These errors are quite consistent with the uncertainties arising from the limited accuracy of the experimental parameters and from the substitution for the actual nuclear level structure of a model substantiated only partially by experimental data.

The last three columns of Table I provide rough measures of the three major uncertainties. They give the approximate percentage change which results in the calculated energy $E(2+)$ from (a) a change in the quadrupole deformation by 0.01, (b) a change in the energy gap parameter by $0.01\hbar\omega_0^0$ ($\sim$70 kev), and (c) the omission of the shift of $N=6$ neutron levels as discussed previously.

It is clear from columns 7 and 8 of Table I that suitable small variations in the gap Δ and deformation δ could bring the calculated moments into exact agreement with experiment. However, such a procedure would shed no further light on the ability of the superconductor theory to reproduce experimental data.

At the present time, the inadequate knowledge of the relevant parameters precludes a more stringent test of the superconductor theory by calculations of rotational moments. However, the present high degree of consistency between predicted and experimental moments of inertia, which so far remains undemonstrated for other proposed descriptions, can be considered to offer strong support for the basic elements of the superconductor theory.

ACKNOWLEDGMENTS

The authors would gratefully like to acknowledge helpful discussions with Dr. Ben R. Mottelson and Dr. Lawrence Wilets.

Note added in proof.—A. B. Migdal [Nuclear Phys. **13**, 655 (1959)] has recently obtained similar results to those reported in this paper for moments of inertia of deformed nuclei, base on a Green's function treatment of nuclear superfluidity.

[26] This assumption is supported by the generally slow change of moments calculated for a sequence of neutron (or proton) numbers, keeping Δ and δ fixed, and the fact that the distribution over which Eq. (2) averages is not very wide.

[27] J. W. Mihelich, B. Harmatz, and T. H. Handley, Phys. Rev. **108**, 989 (1957).