this experiment. Terms similar to those neglected in Eq. (A1) have also been neglected here.

If we let $w_{xs}(\lambda)d\lambda = w(E_1,E_2)dE_2$, corresponding to an electron energy E_1 before radiation and E_2 after radiation, then the probability that an electron of initial energy E_0 will have an energy E' after radiating, scattering, and again radiating will be

$$v(E_0,E')dE' = dE' \int_{E'}^{E_0}\int_{E'}^{E_0} w(E_0,E_1) \times\sigma(E_1,E_2)w'(E_2,E')dE_1dE_2, \quad (A5)$$

where $\sigma(E_1,E_2)$ is the theoretical scattering cross section for electrons of initial energy E_1 and final energy E_2 and w and w' are the probabilities for radiation before and after scattering, respectively.

For an elastic cross section, $\sigma(E_1,E_2)$ is a delta function and the integrals of Eq. (A5) can be evaluated approximately to yield Eq. (1) of the text if E_4 in that equation is replaced by E_0'. Again, terms of the same order as those neglected in Eq. (A1) were neglected in Eq. (1), in addition to terms depending on (E_0-E_3) but which were considerably smaller than those that were retained.

If $\sigma(E_1,E_2)$, as a theoretical inelastic cross section, is considered to be a series of many delta functions (elastic cross sections), Eq. (2) of the text results, where the summation has been replaced by the integral sign of that expression. In deducing Eq. (2), it was assumed that the shape of $\sigma(E_1,E_2)$ as a function of E_2 with fixed E_1 does not change with E_1. This assumption gives adequate accuracy for this work.

PHYSICAL REVIEW VOLUME 118, NUMBER 3 MAY 1, 1960

High-Energy Behavior in Quantum Field Theory*

STEVEN WEINBERG†
Department of Physics, Columbia University, New York, New York
(Received May 21, 1959)

An attack is made on the problem of determining the asymptotic behavior at high energies and momenta of the Green's functions of quantum field theory, using new mathematical methods from the theory of real variables. We define a class A_n of functions of n real variables, whose asymptotic behavior may be specified in a certain manner by means of certain "asymptotic coefficients." The Feynman integrands of perturbation theory (with energies taken imaginary) belong to such classes. We then prove that if certain conditions on the asymptotic coefficients are satisfied then an integral over k of the variables converges, and belongs to the class A_{n-k} with new asymptotic coefficients simply related to the old ones. When applied to perturbation theory this theorem validates the renormalization procedure of Dyson and Salam, proving that the renormalized integrals actually do always converge, and provides a simple rule for calculating the asymptotic behavior of any Green's function to any order of perturbation theory.

I. INTRODUCTION

In many respects, the central formal problem of the modern quantum theory of fields is the determination of the asymptotic behavior at high energies and momenta of the Green's functions of the theory, the vacuum expectation values of time-ordered products. Complete knowledge of the asymptotic properties of these functions would allow us to test the renormalizability of a given Lagrangian, to count the number of subtractions that must be performed in dispersion theory, etc. We shall attack this problem from a rather new direction, which allows a solution in perturbation theory, and which provides an analytic tool that may prove useful in solving the problem in the exact theory.

One might hope to find a solution either kinematically, using only assumptions of covariance, causality, etc., or dynamically, by using the field equations that actually determine the Green's functions. The first method has been successfully applied to the 2-field functions, the particle propagators, and yields the result that the true propagators are asymptotically "larger" than the bare propagators.[1] However, because the theory of several complex variables is so difficult and incomplete, this approach seems unpromising for expectation values of three or more fields. For this reason, and also because we would eventually like to obtain renormalizability conditions on the Lagrangian, we propose to attack the problem on the dynamical level.

Now, what are the equations that, in principle, would determine the Green's functions. In perturbation theory we know that the Green's functions appear as multiple integrals, the integrand being constructed according to the Feynman rules. In a nonperturbative approach the Green's functions are again given by multiple integrals, but with integrands that themselves depend on the

* Supported in part by the United States Atomic Energy Commission.
† Present address: Lawrence Radiation Laboratory, University of California, Berkeley, California.

[1] H. Lehmann, Nuovo cimento 11, 342 (1954).

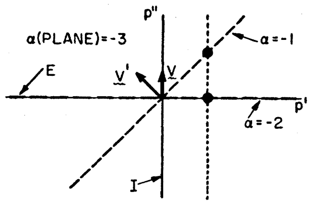

FIG. 1. Diagram of the asymptotic behavior of the function $f(p',p'')$ used as an example in Sec. II. The coefficient α is -3 for the entire plane, and -3 for every line in the plane except the two dashed lines. The "self-energy" $\Sigma(p')$ is calculated by integrating $f(p',p'')$ along the vertical dotted line. It is the intersections of this line with all other lines in the plane, and with the plane itself, that determines the asymptotic behavior of $\Sigma(p')$ according to (III-12).

Green's functions. In either case, it is clear that we need a theorem that would give general rules for the asymptotic behavior of functions defined as multiple integrals. Furthermore, for application to the nonperturbative integral equations, we would like these rules to be given solely in terms of the asymptotic behavior of the integrand, and no other properties.

Such a theorem is presented in Sec. III, and proven rigorously in Sec. IV. The theorem states that if the asymptotic behavior of a function of n real variables may be described in a certain manner by a set of "asymptotic coefficients," then the integral over any subspace of these variables may be similarly described, with new asymptotic coefficients that may be determined as functions of the old ones.

The theorem proven turns out to be extremely useful in perturbation theory. It gives a simple rule that allows one to read off the asymptotic properties of any diagram, by noting the "connectivity" properties of the graph, providing that we can rotate energy contours in the usual way. The asymptotic behavior so obtained is just right to guarantee the basic assumption of the Dyson-Salam renormalization method, that a multiple integral converges if it is "superficially" convergent, and if *all* sub-integrations converge. This theorem, therefore, completes the proof of renormalizability in perturbation theory, for it tells us that the subtractions made according to the Salam prescription[2,3] which can be shown equivalent to a renormalization of masses and coupling constants,[4] actually do give finite remainders in all orders. Of course, it remains an open question whether the perturbation series converges, and whether the asymptotic behavior determined in perturbation theory bears any relation to the actual asymptotic behavior of the complete Green's function.

The application of this theorem to the nonperturbative case and to dispersion theory will be reserved for a future publication.

[2] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[3] A. Salam, Phys. Rev. **82**, 217 (1951); Phys. Rev. **84**, 426 (1951).
[4] P. T. Matthews and A. Salam, Phys. Rev. **94**, 185 (1954).

II. A SIMPLE EXAMPLE

Our task will be to define precisely what we mean by the asymptotic behavior of functions of several real variables, and to show how the convergence and asymptotic properties of multiple integrals may be determined directly from the asymptotic behavior of their integrands. Before proceeding to the general case, we shall discuss a simple example that will illustrate the problems to be faced and our approach in solving them.

Consider the integral $\Sigma(p')$, defined by

$$\Sigma(p')=\int_{-\infty}^{\infty} dp''\, f(p',p''), \tag{1}$$

$$f(p',p'')=\frac{p''}{(p''^2+m^2)[(p''-p')^2+\mu^2]}. \tag{2}$$

Except for the fact that p', p'' are one-dimensional real variables, instead of four-vectors, $\Sigma(p')$ is just the lowest order "fermion" self-energy insertion, in a theory with "fermions" of mass m, "bosons" of mass μ, and interaction $\psi^\dagger\psi\phi$; p' is the fermion momentum, and p'' and $p''-p'$ the momenta of the virtual fermion and boson lines.

Now, of course, the function $f(p',p'')$ is so simple that one sees immediately that $\Sigma(p')$ converges, and can compute,

$$\Sigma(p')=\frac{\pi p'[p'^2+(\mu-m)^2]}{\mu[p'^4+2p'^2(\mu^2+m^2)+(\mu^2-m^2)^2]}, \tag{3}$$

so that

$$\Sigma(p')=O\{p'^{-1}\} \quad \text{as} \quad p'\rightarrow\infty. \tag{4}$$

Usually we are not able to proceed so directly; we may have to deal with complicated functions of many real variables which may not even be entirely known. Therefore, we wish to find some way of characterizing the asymptotic behavior of $f(p',p'')$ so that, *with no further information*, we may obtain the asymptotic behavior (4) of its integral.

It is very convenient to introduce a vector notation, writing

$$f(\mathbf{P})=\frac{\mathbf{P}\cdot\mathbf{V}}{[(\mathbf{P}\cdot\mathbf{V})^2+m^2][(\mathbf{P}\cdot\mathbf{V}')^2+\mu^2]}, \tag{5}$$

where

$$\mathbf{P}=(p',p''),\quad \mathbf{V}=(0,1),\quad \mathbf{V}'=(-1,+1). \tag{6}$$

Suppose we consider the behavior of $f(\mathbf{P})$ as $\mathbf{P}$ tends to infinity along some fixed line. It is apparent that this behavior depends strongly on the direction of the line. If we let $\mathbf{P}=\mathbf{L}\eta+\mathbf{C}$, where $\mathbf{L}$ and $\mathbf{C}$ are fixed vectors, and $\eta\rightarrow\infty$, then

$$f(\mathbf{L}\eta+\mathbf{C})=O\{\eta^{\alpha(\mathbf{L})}\}, \tag{7}$$

where

$$\alpha(\mathbf{L})=\begin{cases}-3 & \text{if}\quad \mathbf{L}\cdot\mathbf{V}\neq0,\quad \mathbf{L}\cdot\mathbf{V}'\neq0\\ -2 & \text{if}\quad \mathbf{L}\cdot\mathbf{V}=0,\quad \mathbf{L}\cdot\mathbf{V}'\neq0\\ -1 & \text{if}\quad \mathbf{L}\cdot\mathbf{V}\neq0,\quad \mathbf{L}\cdot\mathbf{V}'=0.\end{cases} \tag{8}$$

This behavior is indicated in Fig. 1. To be a little more precise, if we confine $\mathbf{C}$ to a finite region W in the $(p'p'')$ plane, then for any $\mathbf{L}$ there exist positive numbers $b(\mathbf{L},W)$ and $M(\mathbf{L},W)$, such that

$$|f(\mathbf{L}\eta+\mathbf{C})|\leqslant M(\mathbf{L},W)\eta^{\alpha(\mathbf{L})}, \tag{9}$$

for $\mathbf{C}$ in W and $\eta\geq b(\mathbf{L},W)$.

Now it is unfortunately the case that simply stating (8) and (9) would not allow us to tell anything about the asymptotic behavior of $\Sigma(p')$. It is necessary also to give some information about the behavior of $b(\mathbf{L},W)$. This may be most easily accomplished, if we introduce *two* positive variables, η_1 and η_2, which will be allowed to go to infinity independently. Let us set

$$\mathbf{P}=\mathbf{L}_1\eta_1\eta_2+\mathbf{L}_2\eta_2+\mathbf{C}, \tag{10}$$

where $\mathbf{L}_1$ and $\mathbf{L}_2$ are fixed and independent, and $\mathbf{C}$ is again confined to a finite region W in the (p',p'') plane.

It is easy to see that then

$$f(\mathbf{L}_1\eta_1\eta_2+\mathbf{L}_2\eta_2+\mathbf{C})=O\{\eta_1^{\alpha(\mathbf{L}_1)}\eta_2^{-3}\}, \tag{11}$$

or, in other words, there exist positive numbers $b_1(\mathbf{L}_1,\mathbf{L}_2,W)$, $b_2(\mathbf{L}_1,\mathbf{L}_2,W)$, $M(\mathbf{L}_1,\mathbf{L}_2,W)$ such that

$$|f(\mathbf{L}_1\eta_1\eta_2+\mathbf{L}_2\eta_2+\mathbf{C})|\leqslant M(\mathbf{L}_1,\mathbf{L}_2\mathbf{C})\eta_1^{\alpha(\mathbf{L}_1)}\eta_2^{-3}, \tag{12}$$

whenever $\mathbf{C}$ is in W, and

$$\eta_1\geq b_1(\mathbf{L}_1,\mathbf{L}_2,W) \quad \eta_2\geq b_2(\mathbf{L}_1,\mathbf{L}_2,W). \tag{13}$$

Proof: A simple calculation shows that if $\mathbf{L}_1\cdot\mathbf{V}\neq0$,

$$\frac{|\mathbf{P}\cdot\mathbf{V}|}{(\mathbf{P}\cdot\mathbf{V})^2+m^2}\leqslant\eta_1^{-1}\eta_2^{-1}\frac{4}{|\mathbf{L}_1\cdot\mathbf{V}|}$$
$$\text{for}\quad \eta_1\geq2[|\mathbf{L}_2\cdot\mathbf{V}|+|\mathbf{C}\cdot\mathbf{V}|]/|\mathbf{L}_1\cdot\mathbf{V}|,\quad \eta_2\geq1; \tag{14}$$

while if $\mathbf{L}_1\cdot\mathbf{V}=0$ (and hence $\mathbf{L}_2\cdot\mathbf{V}\neq0$),

$$\frac{|\mathbf{P}\cdot\mathbf{V}|}{(\mathbf{P}\cdot\mathbf{V})^2+m^2}\leqslant\eta_2^{-1}\frac{4}{|\mathbf{L}_2\cdot\mathbf{V}|}\quad\text{for }\eta_2\geq2|\mathbf{C}\cdot\mathbf{V}|/|\mathbf{L}_2\cdot\mathbf{V}|. \tag{15}$$

Furthermore, if $\mathbf{L}_1\cdot\mathbf{V}'\neq0$,

$$\frac{1}{(\mathbf{P}\cdot\mathbf{V}')^2+\mu^2}\leqslant\eta_1^{-2}\eta_2^{-2}\frac{4}{|\mathbf{L}_1\cdot\mathbf{V}'|^2}$$
$$\text{for}\quad \eta_1\geq2[|\mathbf{L}_2\cdot\mathbf{V}'|+|\mathbf{C}\cdot\mathbf{V}'|]/|\mathbf{L}_1\cdot\mathbf{V}|,\quad \eta_2\geq1; \tag{16}$$

while if $\mathbf{L}_1\cdot\mathbf{V}'=0$ (and hence $\mathbf{L}_2\cdot\mathbf{V}'\neq0$),

$$\frac{1}{(\mathbf{P}\cdot\mathbf{V}')^2+\mu^2}\leqslant\eta_2^{-2}\frac{4}{|\mathbf{L}_2\cdot\mathbf{V}'|^2}\quad\text{for}\quad \eta_2\geq2|\mathbf{C}\cdot\mathbf{V}'|/|\mathbf{L}_2\cdot\mathbf{V}'|. \tag{17}$$

Multiplying (14) or (15) by (16) or (17), and referring back to (8), shows that (11) is correct.

It is the special circumstance summarized in (11) that establishes $f(\mathbf{P})$ as a member of the class (later called A_n in the case of n variables) of functions with which we shall deal, and that allows us to obtain (4). We shall say in this particular case that $\alpha(\mathbf{L}_1)$ is the "asymptotic coefficient" associated with the line $\{\mathbf{L}_1\}$, and that $\alpha(\{\mathbf{L}_1,\mathbf{L}_2\})=-3$ is the coefficient associated with the whole (p',p'') plane $\{\mathbf{L}_1,\mathbf{L}_2\}$. (We use $\{\mathbf{L}_1,\mathbf{L}_2,\cdots\}$ to denote the subspace spanned by the vectors $\mathbf{L}_1$, $\mathbf{L}_2$, $\cdots$).

It is worth emphasizing how much stronger (11) is than the statement (7). According to (7) alone we can easily see that as $\eta_2\to\infty$,

$$f(\mathbf{L}_1\eta_1\eta_2+\mathbf{L}_2\eta_2+\mathbf{C})=O\{\eta_2^{\alpha(\mathbf{L}_1\eta_1+\mathbf{L}_2)}\}, \tag{18}$$

or more fully, that

$$|f(\mathbf{L}_1\eta_1\eta_2+\mathbf{L}_2\eta_2+\mathbf{C})|\leqslant M(\mathbf{L}_1\eta_1+\mathbf{L}_2,W)\eta_2^{\alpha(\mathbf{L}_1\eta_1+\mathbf{L}_2)}, \tag{19}$$

for $\mathbf{C}$ restricted to W and

$$\eta_2\geq b(\mathbf{L}_1\eta_1+\mathbf{L}_2,W). \tag{20}$$

Furthermore, it is obvious that for any $\mathbf{L}_1$, $\mathbf{L}_2$, the vector $\mathbf{L}_1\eta_1+\mathbf{L}_2$ will not be orthogonal to $\mathbf{V}$ or $\mathbf{V}'$ for sufficiently large η_1, so that for η_1 large enough,

$$\alpha(\mathbf{L}_1\eta_1+\mathbf{L}_2)=-3. \tag{21}$$

What is *not* obvious, and indeed is not contained in (7), is that we can find numbers $b_1(\mathbf{L}_1,\mathbf{L}_2,W)$, $b_2(\mathbf{L}_1,\mathbf{L}_2,W)$, and $M(\mathbf{L}_1,\mathbf{L}_2,W)$, such that (12) holds, or alternatively, by comparison with (19) that

$$\begin{gathered}M(\mathbf{L}_1\eta_1+\mathbf{L}_2,W)\leqslant M(\mathbf{L}_1,\mathbf{L}_2,W)\eta_1^{\alpha(\mathbf{L}_1)},\\ b(\mathbf{L}_1\eta_1+\mathbf{L}_2,W)\leqslant b_2(\mathbf{L}_1,\mathbf{L}_2,W),\\ \alpha(\mathbf{L}_1\eta_1+\mathbf{L}_2)=-3.\end{gathered} \tag{22}$$

for all $\eta_1\geq b_1(\mathbf{L}_1,\mathbf{L}_2,W)$. The statement (7) alone would of course allow us to determine the convergence of $\Sigma(p')$; however we need (11) [or alternatively (22)] to determine its asymptotic behavior.

The proof that the asymptotic behavior (4) of $\Sigma(p')$ can be directly obtained from (8) and (11) alone, without knowing any other properties of the function $f(p',p'')$, will be reserved until the next section. We shall show there that (4) follows immediately in a simple application of the general asymptotic theorem. [See (III-15).]

III. THE ASYMPTOTIC THEOREM: DEFINITIONS AND STATEMENT

We shall now consider the general case of functions $f(p',p'',\cdots)$ of n real variables p', p'', $\cdots$, which for convenience we unite into a vector $\mathbf{P}$ in the n-dimensional linear vector space R_n. We will define a class A_n of such functions, whose asymptotic behavior for high p may be specified in a certain manner, by means of certain "asymptotic coefficients." (The integrands of covariant perturbation theory, constructed according to the Feynman rules, with the n real variables taken as all components of all internal and external momenta, are shown in Sec. V to belong to A_n, providing that all energy integration contours may be rotated up to the imaginary axis.) The exact definition of the classes A_n and of the "asymptotic coefficients" is chosen in just such a way that we will be able to prove the asymptotic theorem, which says that if a function belongs to A_n, any sufficiently convergent integral over k of its arguments belongs to A_{n-k}, and which provides a rule for calculating the convergence properties and asymptotic coefficients of the integral in terms of the asymptotic coefficients of the integrand.

Definition

The function $f(\mathbf{P})$ is said to belong to the class A_n if to every subspace $S\subset R_n$ there corresponds a pair of coefficients, a "power" $\alpha(S)$ and a "logarithmic power" $\beta(S)$, and for any choice of $m\leqslant n$ independent vectors $\mathbf{L}_1\cdots\mathbf{L}_m$ and finite region W in R_n we have

$$\begin{aligned}f(\mathbf{L}_1\eta_1\eta_2\cdots\eta_m+\mathbf{L}_2\eta_2\cdots\eta_m+\cdots+\eta_m\mathbf{L}_m+\mathbf{C})\\=O\{\eta_1^{\alpha(\mathbf{L}_1)}(\ln\eta_1)^{\beta(\mathbf{L}_1)}\eta_2^{\alpha(\{\mathbf{L}_1,\mathbf{L}_2\})}(\ln\eta_2)^{\beta(\{\mathbf{L}_1,\mathbf{L}_2\})}\\ \times\cdots\eta_m^{\alpha(\{\mathbf{L}_1,\mathbf{L}_2\cdots\mathbf{L}_m\})}(\ln\eta_m)^{\beta(\{\mathbf{L}_1,\mathbf{L}_2,\cdots,\mathbf{L}_m\})}\},\end{aligned}\quad(1)$$

if $\eta_1\cdots\eta_m$ tend independently to infinity with $\mathbf{C}$ confined to W. [Here $\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\})$ and $\beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\})$ are the asymptotic coefficients associated with the subspace $\{\mathbf{L}_1\cdots\mathbf{L}_r\}$ spanned by the vectors $\mathbf{L}_1\cdots\mathbf{L}_r$.] More precisely, there exists a set of numbers $b_1\cdots b_m>1$ and $M>0$ (depending on $\mathbf{L}_1\cdots\mathbf{L}_m$ and W but not of course on the $\eta_1\cdots\eta_m$), such that

$$\begin{aligned}|f(\mathbf{L}_1\eta_1\eta_2\cdots\eta_m+\mathbf{L}_2\eta_2\cdots\eta_m+\cdots+\mathbf{L}_m\eta_m+\mathbf{C})|\\ \leqslant M\eta_1^{\alpha(\mathbf{L}_1)}(\ln\eta_1)^{\beta(\mathbf{L}_1)}\eta_2^{\alpha(\{\mathbf{L}_1,\mathbf{L}_2\})}\\ \times(\ln\eta_2)^{\beta(\{\mathbf{L}_1\mathbf{L}_2\})}\cdots\eta_m^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_m\})}(\ln\eta_m)^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_m\})},\end{aligned}\quad(2)$$

provided that $\mathbf{C}\epsilon W$, and that

$$\eta_1\geq b_1,\ \eta_2\geq b_2,\ \cdots,\ \eta_m\geq b_m. \quad(3)$$

It may readily be observed that $\alpha(S)$, $\beta(S)$ are the values of $\alpha(\mathbf{L})$, $\beta(\mathbf{L})$ for $\mathbf{L}$ a "typical" vector in the subspace S. Suppose we let only η_m go to infinity in (1), keeping $\eta_1\cdots\eta_{m-1}$ fixed and satisfying

$$\eta_1\geq b_1,\ \cdots,\ \eta_{m-1}\geq b_{m-1}. \quad(4)$$

It then follows from (1) that for $S=\{\mathbf{L}_1\mathbf{L}_2\cdots\mathbf{L}_m\}$

$$\begin{aligned}f([\mathbf{L}_1\eta_1\cdots\eta_{m-1}+\mathbf{L}_2\eta_2\cdots\eta_{m-1}\\+\cdots+\mathbf{L}_{m-1}\eta_{m-1}+\mathbf{L}_m]\eta_m+\mathbf{C})\\=O\{\eta_m^{\alpha(S)}(\ln\eta_m)^{\beta(S)}\},\end{aligned}\quad(5)$$

so that we can take

$$\alpha(\mathbf{L}_1\eta_1\cdots\eta_{m-1}+\mathbf{L}_2\eta_2\cdots\eta_{m-1}+\cdots+\mathbf{L}_m)=\alpha(S), \quad(6)$$

and likewise for β. The conditions (4) just ensure that the vector $L=\mathbf{L}_1\eta_1\cdots\eta_{m-1}+\mathbf{L}_2\eta_2\cdots\eta_{m-1}+\cdots+\mathbf{L}_m$ does not take on some special direction for which $\alpha(\mathbf{L})>\alpha(S)$. We shall refer to $\alpha(S)$, $\beta(S)$ therefore as the asymptotic coefficients of $f(\mathbf{P})$ for $\mathbf{P}\to\infty$ along typical directions in S.

In the special example given in Sec. II, we saw that $\alpha(R_2)=-3$, where R_2 was the whole p', p'' plane. Furthermore, this was also the value of $\alpha(\mathbf{L})$ for almost all vectors $\mathbf{L}$ in R_2, the only exceptions being $\mathbf{L}\sim(1,0)$ (with $\alpha=-2$) and $\mathbf{L}\sim(1,1)$ (with $\alpha=-1$), as shown in Fig. 1. By taking $\mathbf{L}=\eta_1\mathbf{L}_1+\mathbf{L}_2$, where $\mathbf{L}_1$ and $\mathbf{L}_2$ are any fixed vectors and η_1 is sufficiently large, we could always avoid these two special directions.

We now consider an integral of $f(\mathbf{P})$, given as

$$f_{\mathbf{L}_1'\cdots\mathbf{L}_k'}(\mathbf{P})=\int_{-\infty}^{\infty}dy_1\cdots\times\int_{-\infty}^{\infty}dy_k\, f(\mathbf{P}+\mathbf{L}_1'y_1+\cdots+\mathbf{L}_k'y_k). \quad(7)$$

In the example of Sec. II, for instance, we had

$$\Sigma(p')=f_{\mathbf{L}'}(\mathbf{L}p'),\quad \mathbf{L}'=(0,1),\quad \mathbf{L}=(1,0). \quad(8)$$

We shall say that the integral (7) "exists" if every subsequent integration converges in the iterated integral

$$\int_{-\infty}^{\infty}dy_1\cdots\int_{-\infty}^{\infty}dy_k|f(\mathbf{P}+\mathbf{L}_1'y_1+\cdots+\mathbf{L}_k'y_k)|. \quad(9)$$

In this case, by a simple application of Fubini's theorem,[5] (7) cannot depend on the particular vectors $\mathbf{L}_1'\cdots\mathbf{L}_k'$ but only on the subspace $I\subset R_n$ that they span. We therefore write in this case

$$\begin{aligned}f_I(\mathbf{P})=f_{\mathbf{L}_1'\cdots\mathbf{L}_k'}(\mathbf{P})\quad\text{for}\quad I=\{\mathbf{L}_1'\cdots\mathbf{L}_k'\}\\=\int_{\mathbf{P}'\epsilon I}d^k\mathbf{P}'f(\mathbf{P}+\mathbf{P}').\end{aligned}\quad(10)$$

Furthermore $f_I(\mathbf{P})$ does not change if we add to $\mathbf{P}$ any vector in I; in other words, $f(\mathbf{P})$ depends only on the projection of $\mathbf{P}$ along the subspace I. It is convenient to choose some particular subspace E such that $R_n=I+E$, with I and E independent, and restrict $\mathbf{P}$ to E. In applications to perturbation theory, a vector in I or E will have as its components the momentum components of the internal or external particle lines. [In Sec. II, $I=\{\mathbf{L}'\}$, $E=\{\mathbf{L}\}$. See (III-8).] We may now state the general asymptotic theorem.

Theorem

If a function $f(\mathbf{P})$ belongs to A_n, with asymptotic coefficients $\alpha(S)$, $\beta(S)$ for S any non-null subspace of R_n, and if $f(\mathbf{P})$ is integrable over any finite region of R_n, then if $D_I<0$, where

$$D_I=\max_{S'\subset I}[\alpha(S')+\dim S'], \quad(11)$$

[5] See, e.g., L. M. Graves, *The Theory of Functions of Real Variables* (McGraw-Hill Book Company, Inc., New York, 1956).

then the following statements hold:

(a) $f_I(\mathbf{P})$ exists;

(b) $f_I(\mathbf{P})\epsilon A_{n-k}$, with asymptotic coefficient $\alpha_I(S)$ given for any $S\subset E$ by

$$\alpha_I(S)=\max_{\Lambda(I)S'=S}[\alpha(S')+\dim S'-\dim S]. \quad (12)$$

Here $S'\subset I$ means that S' is a subspace of I, including the possibility $S'=I$; $\dim S'$ is the dimensionality of S'; $\Lambda(I)S'=S$ means that the projection of S' along I on E is S (this last is discussed in detail in the Appendix). The "max" in (11) and (12) means that we take the maximum over all subspaces S' satisfying $S'\subset I$ and $\Lambda(I)S'=S$, respectively. Actually, by using the Heine-Borel theorem we will be able to show in the next section that only a finite number of S' need be taken into account, so that "max" is finite.

Let us now see how this theorem may be applied to the example of Sec. II. There I was one-dimensional, so that the only $S'\subset I$ is $S'=I$. Thus

$$D_I=\alpha(I)+\dim I=-3+1=-2<0, \quad (13)$$

so that

(a) $\Sigma(p')$ exists;

(b) $\Sigma(p')\epsilon A_1$, which means that we can write

$$\Sigma(p')=O\{p'^{\alpha_I(E)}(\log p')^{\beta_I(E)}\}. \quad (14)$$

Note that $\Lambda(I)S'=E$ is satisfied for $S'=R_2=I+E$, and for S' any line in R_2 except I. Thus by (12) and (II-8),

$$\begin{aligned}\alpha_I(E)=\max[&-3+2-1, \quad (S'=R_2)\\ &-1+1-1, \quad (S'=\{(1,1)\})\\ &-2+1-1, \quad (S'=\{(1,0)\})\\ &-3+1-1] \quad (S' \text{ other lines})\\ =-1,&\end{aligned} \quad (15)$$

which agrees with (II-4).

It is interesting to compare statement (a) of the asymptotic theorem with what we should expect if we attempted to estimate the degree of divergence of integrals by just "counting powers" of the arguments p', p'', $\cdots$. Naively, we should expect the degree of divergence of the integral $f_{I'}$ of f over any subspace I', to be given by the asymptotic power $\alpha(I')$ of $f(\mathbf{P})$ for $\mathbf{P}\to\infty$ along typical directions in I', plus one unit for each integration performed. We shall call this quantity

$$\mathfrak{D}_{I'}\equiv\alpha(I')+\dim I', \quad (16)$$

the *superficial divergence* of the integral of f over I'. According to part (a) of the asymptotic theorem, however, the sufficient condition for existence of the integral of f over I is not just that the integral converge superficially, (i.e., $\mathfrak{D}_I<0$), but that *all* subintegrations also converge superficially (i.e., $\mathfrak{D}_{S'}<0$ for $S'\subset I$), since according to (11),

$$D_I=\max_{S'\subset I}\mathfrak{D}_{S'}. \quad (17)$$

It must be stressed that by all subintegrations we mean all iterated integrals for all possible linear recombinations of the integration variables. This is the mathematical foundation of the perturbative renormalization theory, to be discussed in Sec. V.

IV. THE ASYMPTOTIC THEOREM: PROOF AND POSSIBLE EXTENSIONS

Our proof is by strong mathematical induction, and divided into the following steps:

(A) We prove by purely geometrical reasoning that if the theorem holds whenever $\dim I\leqslant k$ (where $k\geq 1$) then it also holds for $\dim I=k+1$, so that it is only necessary to prove the theorem in the case $\dim I=1$.

(B) We consider the case $I=\{\mathbf{L}\}$ (where L is some vector ϵR_n) and show (trivially) that $f_I(P)$ converges absolutely if $D_I<0$.

(C) We describe a method of covering the infinite interval of integration of f_I with a finite number of subintervals J.

(D) We show that if $D_I<0$ that the sum of the integrals over the intervals J belongs to A_{n-1}, with α_I given by part (b) of the asymptotic theorem.

(A): Assume the theorem holds whenever the subspace of integration has dimensionality $\leqslant k$. Let I be a $(k+1)$-dimensional subspace of R_n. We decompose I into

$$I=S_1+S_2, \quad (1)$$

where S_1 and S_2 are some (non-null) independent subspaces of R_n, with dimensions k_1, k_2 necessarily $\leqslant k$. (We could always choose S_1 or S_2 to be one-dimensional, but the proof is then less illuminating.) Let $f\epsilon A_n$. The integral f_I can be written $f_I=(f_{S_2})_{S_1}$, or in other words,

$$f_I(\mathbf{P})=\int_{\mathbf{P}'\epsilon S_1}d^{k_1}\mathbf{P}'f_{S_2}(\mathbf{P}+\mathbf{P}'). \quad (2)$$

By our induction hypothesis we can apply the asymptotic theorem to both the S_2 and S_1 integrations, obtaining the following results:

(a1) f_{S_2} converges absolutely if $D_{S_2}(f)<0$, where

$$D_{S_2}(f)=\max_{S''\subset S_2}[\alpha(S'')+\dim S'']. \quad (3)$$

(b1) If $D_{S_2}(f)<0$, then $f_{S_2}\epsilon A_{n-k_2}$, with

$$\alpha_{S_2}(S')=\max_{\Lambda(S_2)S''=S'}[\alpha(S'')+\dim S''-\dim S']. \quad (4)$$

(a2) If $f_{S_2}\epsilon A_{n-k_2}$, then f_I as given by (2) converges absolutely if $D_{S_1}(f_{S_2})<0$, where

$$D_{S_1}(f_{S_2})=\max_{S'\subset S_1}[\alpha_{S_2}(S')+\dim S']. \quad (5)$$

(b2) If $f_{S_2}\epsilon A_{n-k_2}$ and $D_{S_1}(f_{S_2})<0$, then f_I as given by (2) belongs to A_{n-k-1}, with

$$\alpha_I(S)=\max_{\Lambda(S_1)S'=S}[\alpha_{S_2}(S')+\dim S'-\dim S]. \quad (6)$$

The whole integral clearly converges if both $D_{S_2}(f)<0$ and $D_{S_1}(f_{S_2})<0$, i.e., if $D_I<0$ where

$$D_I=\max[D_{S_2}(f),D_{S_1}(f_{S_2})]. \quad (7)$$

Inserting (3), (4), and (5) into (7), we obtain

$$D_I=\max_{S''}{}^*[\alpha(S'')+\dim S''], \quad (8)$$

where "max*" runs over all S'' satisfying $S''\subset S_2$ *or* $\Lambda(S_2)S''\subset S_1$. According to statement (*D*) in the Appendix, this means that max* runs over all S'' satisfying the condition $S''\subset S_1+S_2$, so that (8) becomes

$$D_I=\max_{S''\subset I}[\alpha(S'')+\dim S''], \quad (9)$$

which completes the proof of part (a) of the theorem. Turning to part (b), we see that if $D_I<0$ then combining (b1), (b2), and (7) we have $f_I\epsilon A_{n-k-1}$. Combining (4) and (6), the asymptotic power of f_I is

$$\begin{aligned}\alpha_I(S)=&\max_{\Lambda(S_1)S'=S}\Big[\dim S'-\dim S\\&+\max_{\Lambda(S_2)S''=S'}(\alpha(S'')+\dim S''-\dim S')\Big] \quad (10)\\=&\max_{\substack{\Lambda(S_1)S'=S\\ \Lambda(S_2)S''=S'}}[\alpha(S'')+\dim S''-\dim S].\end{aligned}$$

By statement (E) in the Appendix the double condition $\Lambda(S_1)S'=S$ and $\Lambda(S_2)S''=S'$ can be replaced by

$$S=\Lambda(S_1+S_2)S''=\Lambda(I)S'', \quad (11)$$

so that (10) proves part (b) of the theorem.

(B): We wish to consider $I=\{L\}$, and

$$f_I(\mathbf{P})=f_{\mathbf{L}}(\mathbf{P})=\int_{-\infty}^{\infty}f(\mathbf{P}+\mathbf{L}y)dy, \quad (12)$$

where $f\epsilon A_n$. According to (III-1), we have

$$f(\mathbf{P}+\mathbf{L}y)=O\{y^{\alpha(\mathbf{L})}(\ln y)^{\beta(\mathbf{L})}\} \quad \text{as} \quad y\to\infty, \quad (13)$$

so clearly (12) converges absolutely if

$$\alpha(\mathbf{L})+1<0. \quad (14)$$

Since the only non-null subspace of I is I itself, we have according to (III-11),

$$D_I=\alpha(\mathbf{L})+1, \quad (15)$$

so that part (a) of the theorem is verified.

(C): Suppose we choose any sequence $\mathbf{L}_1\cdots\mathbf{L}_m$ of vectors ϵR_n (independent of each other and of $\mathbf{L}$) and a finite region W in R_n; our task is then to prove that if $f\epsilon A_n$, $f_{\mathbf{L}}\epsilon A_{n-1}$, or in other words,

$$f_{\mathbf{L}}(\mathbf{P})=O\{\eta_1^{\alpha_{\mathbf{L}}(\mathbf{L}_1)}(\ln\eta_1)^{\beta_{\mathbf{L}}(\mathbf{L}_1)}\cdots\eta_m^{\alpha_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_m\})}\times(\ln\eta_m)^{\beta_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_m\})}\} \quad (16)$$

for

$$\mathbf{P}=\mathbf{L}_1\eta_1\cdots\eta_m+\mathbf{L}_2\eta_2\cdots\eta_m+\cdots+\mathbf{L}_m\eta_m+\mathbf{C}, \quad (17)$$

where $\eta_1\cdots\eta_m$ tend independently to infinity with $\mathbf{C}$ confined to W. We must also verify (III-12), showing that

$$\alpha_{\mathbf{L}}(S)=\max_{\Lambda(\mathbf{L})S'=S}[\alpha(S')+\dim S'-\dim S]. \quad (18)$$

To this end we will first describe a decomposition of the interval $-\infty<y<\infty$ in (12) into a finite set of intervals J, each of which contribute a "term" to (18). Consider the sequence of $m+1$ independent vectors,

$$\mathbf{L}_1+u_1\mathbf{L},\ \mathbf{L}_2+u_2\mathbf{L},\ \cdots,\ \mathbf{L}_r+u_r\mathbf{L},\ \mathbf{L},\ \mathbf{L}_{r+1},\ \cdots,\ \mathbf{L}_m,$$

where $0\leqslant r\leqslant m$ and $u_1\cdots u_r$ are a set of r real variables. Since $f\epsilon A_n$, there must exist a set of numbers

$$b_l(u_1\cdots u_r)>1 \quad (0\leqslant l\leqslant r), \quad M(u_1\cdots u_r)>0,$$

such that

$$\begin{aligned}|f((&\mathbf{L}_1+u_1\mathbf{L})\eta_1\cdots\eta_m\eta_0+(\mathbf{L}_2+u_2\mathbf{L})\eta_2\cdots\eta_m\eta_0+\cdots\\&+(\mathbf{L}_r+u_r\mathbf{L})\eta_r\cdots\eta_m\eta_0+\mathbf{L}\eta_{r+1}\cdots\eta_m\eta_0\\&+\mathbf{L}_{r+1}\eta_{r+1}\cdots\eta_m+\cdots+\mathbf{L}_m\eta_m+\mathbf{C})|\\&\leqslant M(u_1\cdots u_r)\eta_1^{\alpha(\{\mathbf{L}_1+u_1\mathbf{L}\})}(\ln\eta_1)^{\beta(\{\mathbf{L}_1+u_1\mathbf{L}\})}\cdots\\&\times\eta_r^{\alpha(\{\mathbf{L}_1+u_1\mathbf{L},\cdots,\mathbf{L}_r+u_r\mathbf{L}\})}(\ln\eta_r)^{\beta(\{\mathbf{L}_1+u_1\mathbf{L},\cdots,\mathbf{L}_r+u_r\mathbf{L}\})}\\&\times\eta_0^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})}(\ln\eta_0)^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})}\\&\times\eta_{r+1}^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_{r+1}\mathbf{L}\})}(\ln\eta_{r+1})^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_{r+1}\mathbf{L}\})}\cdots\\&\times\eta_m^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_m\mathbf{L}\})}(\ln\eta_m)^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_m\mathbf{L}\})}, \quad (19)\end{aligned}$$

when all $\eta_l\geq b_l(u_1\cdots u_r)$ and $\mathbf{C}\epsilon W$.

Now let us consider the closed interval $(-b_0, b_0)$. [When we refer to an interval as (a,b) we mean the set of all u with $(a\leqslant u\leqslant b)$; by "$b_0$" we mean the b_l function with $l=r=0$.] Every point u on this line is in the interior of a closed interval, $[u-b_1^{-1}(u),\ u+b_1^{-1}(u)]$. Therefore, by the Heine-Borel theorem[6] we can find a finite set of points U_i with $|U_i|\leqslant b_0$, each U_i contained in an interval $(U_i-\lambda_i,\ U_i+\lambda_i)$, such that the intervals $(U_i-\lambda_i,\ U_1+\lambda_i)$ cover the entire closed interval $(-b_0, b_0)$, and such that $0<\lambda_i\leqslant b_1^{-1}(U_i)$. Now consider any particular i, and the closed interval $[-b_0(U_i),\ b_0(U_i)]$. Again we may use the Heine-Borel theorem, and obtain a finite set of points U_{ij} with $|U_{ij}|\leqslant b_0(U_i)$, and a set of closed intervals $(U_{ij}-\lambda_{ij},\ U_{ij}+\lambda_{ij})$, which cover the finite line $[-b_0(U_i),\ b_0(U_i)]$, and such that $\lambda_{ij}\leqslant b_2^{-1}(U_i,U_{ij})$. Continuing in this fashion, we find m finite sets of points U_{i_1}, $U_{i_1i_2}$, $\cdots$, $U_{i_1\cdots i_m}$ and numbers λ_{i_1}, $\lambda_{i_1i_2}$, $\cdots$, $\lambda_{i_1\cdots i_m}$, such that for any $r\leqslant m$, the

[6] See, e.g., E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1927), fourth edition, p. 53.

$(U_{i_1\cdots i_r}-\lambda_{i_1\cdots i_r},\ U_{i_1\cdots i_r}+\lambda_{i_1\cdots i_r})$ cover the line

$$(-b_0(i_1\cdots i_{r-1}),\quad b_0(i_1\cdots i_{r-1})), \tag{20}$$

$$0<\lambda_{i_1\cdots i_r}\leqslant 1/b_r(i_1\cdots i_r), \tag{21}$$

$$|U_{i_1\cdots i_r}|\leqslant b_0(i_1\cdots i_{r-1}), \tag{22}$$

where we introduce the notation

$$b_l(i_1\cdots i_r)\equiv b_l(U_{i_1},U_{i_1i_2},\cdots,U_{i_1i_2\cdots i_r}). \tag{23}$$

We shall use (20), (21), (22) to split up the integration in (12). Take any set of $\eta_1\cdots\eta_m>1$. We will define a set of intervals $J_{i_1\cdots i_r}{}^{\pm}(\eta)$ $(r\leqslant m)$ to consist of all y that may be written

$$y=U_{i_1}\eta_1\cdots\eta_m+U_{i_1i_2}\eta_2\cdots\eta_m+\cdots U_{i_1\cdots i_r}\eta_r\cdots\eta_m+z\eta_{r+1}\cdots\eta_m, \tag{24}$$

where

$$\eta_r\lambda_{i_1\cdots i_r}\geq|z|=\pm z\geq b_0(i_1\cdots i_r). \tag{25}$$

For the case $r=0$ the intervals $J^{\pm}(\eta)$ are defined to consist of all y with

$$\pm y=|y|\geq b_0\eta_1\cdots\eta_m, \tag{26}$$

and the intervals $J_{i_1\cdots i_m}{}^0$ are defined to consist of all y that may be written

$$y=U_{i_1}\eta_1\cdots\eta_m+U_{i_1i_2}\eta_2\cdots\eta_m+\cdots+U_{i_1\cdots i_m}\eta_m+z, \tag{27}$$

with

$$|z|\leqslant b_0(i_1\cdots i_m). \tag{28}$$

It is easy to see that every real y belongs to at least one of these intervals. For if y is not in $J^{\pm}(\eta)$, then $|y|\leqslant b_0\eta_1\cdots\eta_m$. However, the finite line $(-b_0\eta_1\cdots\eta_m,\ +b_0\eta_1\cdots\eta_m)$ is covered by the intervals $(U_i\eta_1\cdots\eta_m-\lambda_i\eta_1\cdots\eta_m,\ U_i\eta_1\cdots\eta_m+\lambda_i\eta_1\cdots\eta_m)$. Therefore, we may set

$$y=U_{i_1}\eta_1\cdots\eta_m+y',\quad |y'|\leqslant\eta_1\cdots\eta_m\lambda_{i_1},$$

for some i_1. This implies that if $\pm y'\geq\eta_2\cdots\eta_m b_0(i_1)$ then $y\epsilon J_{i_1}{}^{\pm}(\eta)$ according to (24), (25). On the other hand, if $|y'|\leqslant\eta_2\cdots\eta_m b_0(i_1)$, we can again place it in a covering interval, so that

$$y=U_{i_1}\eta_1\cdots\eta_m+U_{i_1i_2}\eta_2\cdots\eta_m+y'',\quad |y''|\leqslant\eta_2\cdots\eta_m\lambda_{i_1i_2}.$$

Thus if $\pm y''\geq\eta_2\cdots\eta_m b_0(i_1i_2)$, $y\epsilon J_{i_1i_2}{}^{\pm}(\eta)$. This process can clearly be continued until the final alternative, which is $y\epsilon J_{i_1\cdots i_m}{}^0(\eta)$. It therefore follows that

$$|f_{\mathbf{L}}(\mathbf{P})|\leqslant\sum_{\pm}\sum_{r=0}^{m}\sum_{i_1\cdots i_r}\int_{J_{i_1\cdots i_r}{}^{\pm}(\eta)}dy|f(\mathbf{P}+\mathbf{L}y)|+\sum_{i_1\cdots i_m}\int_{J_{i_1\cdots i_m}{}^0(\eta)}dy|f(\mathbf{P}+\mathbf{L}y)|. \tag{29}$$

(D): Now we shall examine the asymptotic behavior of each term in (29) with $\mathbf{P}$ given by (17).

First, suppose $y\epsilon J_{i_1\cdots i_r}{}^{\pm}(\eta)$. Combining (24) and (17), we have

$$\mathbf{P}+\mathbf{L}y=(\mathbf{L}_1+U_{i_1}\mathbf{L})\eta_1\cdots\eta_m+(\mathbf{L}_2+U_{i_1i_2}\mathbf{L})\eta_2\cdots\eta_m+\cdots+(\mathbf{L}_r+U_{i_1\cdots i_r}\mathbf{L})\eta_r\cdots\eta_m+z\mathbf{L}\eta_{r+1}\cdots\eta_m+\mathbf{L}_{r+1}\eta_{r+1}\cdots\eta_m+\cdots+\mathbf{L}_m\eta_m+\mathbf{C},$$

or by a slight re-writing,

$$\begin{aligned}\mathbf{P}+\mathbf{L}y=&(\mathbf{L}_1+U_{i_1}\mathbf{L})\eta_1\cdots\eta_{r-1}(\eta_r/|z|)|z|\eta_{r+1}\cdots\eta_m+\cdots+(\mathbf{L}_r+U_{i_1\cdots i_r}\mathbf{L})(\eta_r/|z|)\\&\times|z|\eta_{r+1}\cdots\eta_m\pm\mathbf{L}|z|\eta_{r+1}\cdots\eta_m+\mathbf{L}_{r+1}\eta_{r+1}\cdots\eta_m+\cdots+\mathbf{L}_m\eta_m+\mathbf{C}.\end{aligned} \tag{30}$$

Let us define

$$\alpha(i_1\cdots i_l)=\alpha(\{L_1+U_{i_1}L,\ L_2+U_{i_1i_2}L,\ \cdots,\ L_l+U_{i_1\cdots i_l}\}). \tag{31}$$

Applying (19) with $\eta_0=|z|$, we see that

$$\begin{aligned}|f(P+\mathbf{L}y)|\leqslant&M(i_1\cdots i_r)\eta_1{}^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\eta_2{}^{\alpha(i_1i_2)}(\ln\eta_2)^{\beta(i_1i_2)}\cdots\\&\times(\eta_r/|z|)^{\alpha(i_1\cdots i_r)}(\ln(\eta_r/|z|))^{\beta(i_1\cdots i_r)}|z|^{\alpha(\{L_1\cdots L_rL\})}\\&\times(\ln|z|)^{\beta(\{L_1\cdots L_rL\})}\eta_{r+1}{}^{\alpha(\{L_1\cdots L_{r+1}L\})}\\&\times(\ln\eta_{r+1})^{\beta(\{L_1\cdots L_{r+1}L\})}\cdots\\&\times\eta_m{}^{\alpha(\{L_1\cdots L_mL\})}(\ln\eta_m)^{\beta(\{L_1\cdots L_mL\})},\end{aligned} \tag{32}$$

providing that

$$\eta_l\geq b_l(i_1\cdots i_r)\quad(l\neq r), \tag{33}$$

$$\eta_r/|z|\geq b_r(i_1\cdots i_r), \tag{34}$$

$$|z|\geq b_0(i_1\cdots i_r). \tag{35}$$

However, since $y\epsilon J_{i_1\cdots i_r}{}^{\pm}(\eta)$, conditions (34) and (35) are automatic. [See (25), (21).] Thus, if (33) is satisfied, we have, making a change of variable in (29),

$$\begin{aligned}\int_{J_{i_1\cdots i_r}{}^{\pm}(\eta)}dy|f(\mathbf{P}+\mathbf{L}y)|\leqslant&M(i_1\cdots i_r)\eta_1{}^{\alpha(i_1)}\\&\times(\ln\eta_1)^{\beta(i_1)}\cdots\eta_{r-1}{}^{\alpha(i_1\cdots i_{r-1})}(\ln\eta_{r-1})^{\beta(i_1\cdots i_{r-1})}\\&\times\eta_{r+1}{}^{\alpha(\{L_1\cdots L_{r+1}L\})}(\ln\eta_{r+1})^{\beta(\{L_1\cdots L_{r+1}L\})}\\&\times\cdots\eta_m{}^{\alpha(\{L_1\cdots L_mL\})}(\ln\eta_m)^{\beta(\{L_1\cdots L_mL\})}\eta_{r+1}\cdots\eta_m\\&\times\int_{b_0(i_1\cdots i_r)}^{\eta_r\lambda_{i_1\cdots i_r}}d|z|(\eta_r/|z|)^{\alpha(i_1\cdots i_r)}(\ln\eta_r/|z|)^{\beta(i_1\cdots i_r)}\\&\times|z|^{\alpha(\{L_1\cdots L_rL\})}(\ln|z|)^{\beta(\{L_1\cdots L_rL\})}.\end{aligned} \tag{36}$$

It is now generally true that for any numbers $0<\lambda<1$, $b>1$, α, α', and non-negative integers β, β', as $\eta\to\infty$

$$\int_b^{\lambda\eta}(\eta/z)^{\alpha}(\ln(\eta/z))^{\beta}z^{\alpha'}(\ln z)^{\beta'}dz=O\begin{cases}\eta^{\alpha}(\ln\eta)^{\beta+\beta'+1}, & \alpha'+1=\alpha\\ \eta^{\alpha}(\ln\eta)^{\beta}, & \alpha'+1<\alpha\\ \eta^{\alpha'+1}(\ln\eta)^{\beta'}, & \alpha'+1>\alpha.\end{cases} \tag{37}$$

We can always take the original $\beta(S)$ to be non-negative integers, for since the asymptotic coefficients only set an upper bound on the behaviour of $f(\mathbf{P})$, we are free to increase them as needed. [Of course, our arbitrarily increasing the $\beta(S)$ to be non-negative integers causes the final formula for $\beta_{\mathbf{L}}(S)$ to lose its significance.] Therefore we can find numbers $c(i_1\cdots i_r)>1$, $N(i_1\cdots i_r)>0$ such that

$$\int_{J_{i_1\cdots i_r}^{\pm}(\eta)} dy|f(\mathbf{P}+\mathbf{L}y)| \leqslant M(i_1\cdots i_r)N(i_1\cdots i_r)$$
$$\times \eta_1^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\cdots\eta_{r-1}^{\alpha(i_1\cdots i_{r-1})}(\ln\eta_{r-1})^{\beta(i_1\cdots i_{r-1})}$$
$$\times \eta_{r+1}^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1}(\ln\eta_{r+1})^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})}\cdots$$
$$\times \eta_m^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_m\mathbf{L}\})+1}(\ln\eta_m)^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_m\mathbf{L}\})}$$
$$\times\begin{cases}\eta_r^{\alpha(i_1\cdots i_r)}(\ln\eta_r)^{\beta(i_1\cdots i_r)+\beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1}, & \text{if } \alpha(i_1\cdots i_r)=\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1\\ \eta_r^{\alpha(i_1\cdots i_r)}(\ln\eta_r)^{\beta(i_1\cdots i_r)}, & \text{if } \alpha(i_1\cdots i_r)>\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1\\ \eta_r^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1}(\ln\eta_r)^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})}, & \text{if } \alpha(i_1\cdots i_r)<\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1,\end{cases} \tag{38}$$

whenever

$$\eta_l\geq b_l(i_1\cdots i_r)\quad (l\neq r),\quad \eta_r\geq c(i_1\cdots i_r). \tag{39}$$

[Note that the interval $J_{i_1\cdots i_r}^{\pm}(\eta)$ does not contribute unless $\eta_r\geq b_r(i_1\cdots i_r)b_0(i_1\cdots i_r)$.]

Now let us consider the two infinite intervals

$$\int_{J^{\pm}(\eta)} dy|f(\mathbf{P}+\mathbf{L}y)| = \int_{b_0\eta_1\cdots\eta_m}^{\infty} |f(\mathbf{P}\pm\mathbf{L}y)|dy. \tag{40}$$

After making a change of variable,

$$\int_{J^{\pm}(\eta)} dy|f(\mathbf{P}+\mathbf{L}y)| = \eta_1\cdots\eta_m\int_{b_0}^{\infty}|f(\pm\mathbf{L}z\eta_1\cdots\eta_m + \mathbf{L}_1\eta_1\cdots\eta_m+\cdots+\mathbf{L}_m\eta_m+\mathbf{C})|dz. \tag{41}$$

Therefore, applying (19) for $r=0$, $\eta_0=z$ we have

$$\int_{J^{\pm}(\eta)} dy|f(\mathbf{P}+\mathbf{L}y)| \leqslant MN\eta_1^{\alpha(\{\mathbf{L}_1\mathbf{L}\})+1}(\ln\eta_1)^{\beta(\{\mathbf{L}_1\mathbf{L}\})} \times\cdots\eta_m^{\alpha(\{\mathbf{L}_1\cdots\mathbf{L}_m\mathbf{L}\})+1}(\ln\eta_m)^{\beta(\{\mathbf{L}_1\cdots\mathbf{L}_m\mathbf{L}\})}, \tag{42}$$

whenever

$$\eta_l\geq b_l\quad (1\leqslant l\leqslant m). \tag{43}$$

Here N is the finite positive number

$$N=\int_{b_0}^{\infty} z^{\alpha(\mathbf{L})}(\ln z)^{\beta(\mathbf{L})}dz, \tag{44}$$

where we recall that $\alpha(\mathbf{L})+1<0$ by hypothesis.

Finally, we consider the case $y\epsilon J_{i_1\cdots i_m}{}^0(\eta)$. Combining (27) and (17), we have

$$\mathbf{P}+\mathbf{L}y=(\mathbf{L}_1+U_{i_1}\mathbf{L})\eta_1\cdots\eta_m+(\mathbf{L}_2+U_{i_1i_2}\mathbf{L})\eta_2\cdots\eta_m +\cdots+(\mathbf{L}_m+U_{i_1\cdots i_m}\mathbf{L})\eta_m+\mathbf{L}z+\ , \tag{45}$$

where $|z|\leqslant b_0(i_1\cdots i_m)$. Suppose we now define a new finite region R', consisting of all vectors

$$\mathbf{C}'=\mathbf{L}z+\mathbf{C},$$

where $\mathbf{C}\epsilon R$ and $|z|\leqslant b_0(i_1\cdots i_m)$. Since $f\epsilon A_n$, we can find numbers $M'(i_1\cdots i_m)>0$, $b_l'(i_1\cdots i_m)>1$ such that

$$|f((\mathbf{L}_1+U_{i_1}\mathbf{L})\eta_1\cdots\eta_m+\cdots+(\mathbf{L}_m+U_{i_1\cdots i_m}\mathbf{L})\eta_m+\mathbf{C}')| \leqslant M'(i_1\cdots i_m)\eta_1^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\cdots \times\eta_m^{\alpha(i_1\cdots i_m)}(\ln\eta_m)^{\beta(i_1\cdots i_m)}, \tag{46}$$

wherever $\mathbf{C}'\epsilon R'$ and $\eta_l\geq b_l'(\eta_1\cdots\eta_m)$. Therefore,

$$\int_{J_{i_1\cdots i_m}{}^0(\eta)} dy|f(\mathbf{P}+\mathbf{L}y)| \leqslant 2b_0(i_1\cdots i_m)M'(i_1\cdots i_m) \times\eta_1^{\alpha(i_1)}(\ln\eta_1)^{\beta(i_1)}\cdots\eta_m^{\alpha(i_1\cdots i_m)}(\ln\eta_m)^{\beta(i_1\cdots i_m)}, \tag{47}$$

provided that

$$\eta_l\geq b_l'(i_1\cdots i_m)\quad (1\leqslant l\leqslant m). \tag{48}$$

All we need do to finish the proof is to inspect (38), (42), and (47), together with the corresponding conditions on η, (39), (43), and (48), and use (29). We see that $f_{\mathbf{L}}\epsilon A_{n-1}$, with

$$\alpha_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_r\}) =\max[\alpha(i_1\cdots i_r),\ \alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1],$$

$$\beta_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_r\}) =\max\begin{cases}\beta(i_1\cdots i_r), & \text{for } \alpha(i_1\cdots i_r)=\alpha_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_r\})\\ \beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\}), & \text{for } \alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1=\alpha_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_r\})\\ \beta(i_1\cdots i_r)+\beta(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1, & \text{for } \alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1 =\alpha(i_1\cdots i_r)=\alpha_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_r\}).\end{cases} \tag{49}$$

The formula for $\beta_{\mathbf{L}}$, while giving a correct upper bound on the number of logarithms, is an overestimate, and will not be further discussed. (It may be noted, though, that $\beta_{\mathbf{L}}$ is still a non-negative integer.) The formula for $\alpha_{\mathbf{L}}$ may be rewritten:

$$\alpha_{\mathbf{L}}(\{\mathbf{L}_1\cdots\mathbf{L}_r\}) =\max[\alpha(\mathbf{L}_1+u_1\mathbf{L},\ \mathbf{L}_2+u_2\mathbf{L},\ \cdots,\ \mathbf{L}_r+u_r\mathbf{L}),\ \alpha(\{\mathbf{L}_1\cdots\mathbf{L}_r\mathbf{L}\})+1], \tag{51}$$

where the $u_1\cdots u_r$ take only the finite set of values $U_{i1}, \cdots, U_{i1\cdots i_r}$. According to statement C in the Appendix, this formula is equivalent to (18), which was to be proven.

The most interesting possible extension of this theorem would be to introduce some sort of "positivity" conditions on $f(\mathbf{P})$ which would enable us to set a lower bound, as well as an upper bound, on the asymptotic behavior of the integrals of $f(\mathbf{P})$. Our method of proof is well suited to such a program, since we display explicitly in (29) the part of the domain of integration giving each particular contribution to the asymptotic behavior. (It is easy to verify in the example of Sec. II that the covering intervals J can be arranged so that they don't overlap.) It might then be possible to show that certain theories are rigorously nonrenormalizable in the Heisenberg representation.

It would also be very useful to refine the theorem so that we are not forced to overestimate the powers $\beta_{\mathrm{L}}(S)$ of $\ln\eta$. In order to include the possibility of negative $\beta(S)$ it would be necessary to introduce powers of $\ln\ln\eta$ into the definition (III-1).

Finally, it might be interesting to extend the theorem to the case where the α depend on the individual vectors $\mathbf{L}_1\cdots\mathbf{L}_r$ and not just the manifold $\{\mathbf{L}_1\cdots\mathbf{L}_r\}$. This is very easy to do in the case where the subspace of integration I is one-dimensional, but has no obvious physical application.

V. APPLICATION TO PERTURBATION THEORY

We shall now apply the general theorem proven above to the determination of the convergence and asymptotic properties of Green's functions in covariant perturbation theory. Our treatment will follow closely that of the simple example discussed in Sec. II and at the end of Sec. III.

Let us consider any particular Feynman diagram $\mathcal{G}$ in any local field theory. According to the usual rules there is associated with each internal and external particle line j of $\mathcal{G}$ a bare propagator $\Delta_j(p_j,\sigma)$, where p_j is the momentum four-vector carried by line j, and σ is a single label representing all discrete variables such as spins, polarizations, etc. The integrand F corresponding to diagram $\mathcal{G}$ is given as a simple product

$$F=\gamma(\sigma)\prod_{j=1}^{M}\Delta_j(p_j,\sigma), \tag{1}$$

where $\gamma(\sigma)$ is the product of all vertex factors, such as Dirac matrices, coupling constants, etc., and plays no important role here. (Since all discrete indices are subsumed under σ, the Δ_j and γ are ordinary complex functions, and can be multiplied without regard to their order.) In theories with derivative coupling, we must include in the Δ_j any factors of p_j arising from derivative coupling vertices to which line j may be attached. The Green's function G corresponding to $\mathcal{G}$ is formed by summing and integrating F over all internal σ and p_j; it is then a function of the external variables.

Now, we are supposing in (1) that all energy-momentum conservation δ functions have already been eliminated, leaving N independent momentum four-vectors, where of course $N<M$. Let us unite all components of these independent four-vectors into a single vector $\mathbf{P}$ in the $4N$-dimensional Euclidean space R_{4N}, so that each component $p_{j\mu}(\mu=0, 1, 2, 3)$ of each of the internal and external momenta can be written as a linear combination of the components of $\mathbf{P}$. To use vector notation, we introduce for each j, μ a vector $V_{j\mu}$ in R_{4N}, such that $p_{j\mu}$ is given by the scalar product

$$p_{j\mu}=\mathbf{P}\cdot\mathbf{V}_{j\mu}, \tag{2}$$

and therefore

$$F(\mathbf{P},\sigma)=\gamma(\sigma)\prod_{j=1}^{M}\Delta_j(\mathbf{P}\cdot\mathbf{V}_j,\sigma). \tag{3}$$

[For an example, see Eq. (II-5).] If a vector $\mathbf{P}$ is orthogonal to all $\mathbf{V}_{j\mu}$ for which j is an internal (external) line, then we shall say that $\mathbf{P}$ lies in the external (internal) subspaces $E(I)$; its components then involve only external (internal) momenta. The subspaces E and I are disjoint, and

$$E+I=R_{4N}. \tag{4}$$

The Green's function corresponding to graph $\mathcal{G}$ is now given by the improper integral over the internal momenta,

$$G(\mathbf{P},\sigma_{\mathrm{ext}})=\sum_{\sigma_{\mathrm{int}}}\int_{\mathbf{P}'\epsilon I}F(\mathbf{P}+\mathbf{P}',\sigma)d\mathbf{P}', \tag{5}$$

where $\mathbf{P}\epsilon E$. We will first study the asymptotic behavior of F, and then apply the general asymptotic theorem to learn what we want to know about G.

Unfortunately, as it stands the function $F(\mathbf{P},\sigma)$ does not belong to the class A_{4N}, because of the special circumstance that the propagators $\Delta_j(p_j,\sigma)$ depend on the scalar product

$$p_j^2=p_{j1}^2+p_{j2}^2+p_{j3}^2-p_{j0}^2,$$

which can vanish for nonzero p_j. In order to apply our theorem, it is necessary to rotate the contour of integration for each energy integration in (5) in a well-known manner from the real up to the imaginary axis.[2,7] A general discussion of the difficulties encountered in this step would be interesting, but beyond the scope of this work; we shall simply assume henceforth that all energy contours have been so deformed. Likewise, if the integral G is to be used as an insertion in a larger diagram, we shall be interested in its behavior for imaginary values of its energy arguments. We shall, therefore, restrict ourselves throughout to consider all four-vectors p_j with imaginary fourth component $p_{j0}=ip_{j4}$,

[7] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

and hence with positive-definite square

$$p_j^2 = p_{j1}^2 + p_{j2}^2 + p_{j3}^2 + p_{j4}^2 \geq 0. \qquad (6)$$

In this manner we circumvent, but do not solve, the special problems associated with the hyperbolic metric of space-time. [It will also be necessary in electrodynamics to introduce a small photon mass in order to avoid infrared divergences in (5).]

With the above qualifications it is easy to see that the propagators $\Delta_j(p_j,\sigma)$ have a very simple asymptotic behavior, given by

$$\Delta_j(p_j,\sigma) = O\{[(p_j^2)]^{\frac{1}{2}\alpha_j}\}, \qquad (7)$$

for each fixed σ, and for p_j tending to infinity along any direction. For example, if line j represents a spinless internal particle, $\alpha_j = -2$, since

$$\Delta_j(p_j,\sigma) = 1/(p_j^2 + \mu_j^2),$$

while if it represents an internal particle of spin $\frac{1}{2}$, $\alpha_j = -1$, because

$$\Delta_j(p_j,\sigma) = S_c(p_j) = \frac{ip_{j\mu}\gamma^\mu + m_j}{p_j^2 + m_j^2}$$
$$\leqslant [\mathrm{Tr}\{S_c(p_j)S_c^\dagger(p_j)\}]^{\frac{1}{2}} = 2/(p_j^2 + m_j^2)^{\frac{1}{2}}.$$

Using (3) and (7), we may now show that $F(\mathbf{P},\sigma)$ belongs to the class A_{4N} (defined in Sec. III) with asymptotic coefficients, for any subspace $S \subset R_{4N}$, given by

$$\alpha(S) = \sum\nolimits_j^{(S)} \alpha_j, \qquad (8)$$

$$\beta(S) = 0, \qquad (9)$$

the sum in (8) running over all j for which $\mathbf{V}_j$ is not orthogonal to the subspace S.

Proof. Let us set $\mathbf{P}$ in (3) equal to

$$\mathbf{P} = \mathbf{L}_1\eta_1 \cdots \eta_m + \mathbf{L}_2\eta_2 \cdots \eta_m + \cdots + \mathbf{L}_m\eta_m + \mathbf{C}, \quad (10)$$

where $\mathbf{L}_1 \cdots \mathbf{L}_m$ are independent vectors in R_{4N} (so $m \leqslant 4N$), $\mathbf{C}$ is a vector confined to a finite region W, and $\eta_1 \cdots \eta_m$ tend independently to infinity. Then from (7) we have for fixed σ,

$$\Delta_j(\mathbf{V}_j \cdot \mathbf{P},\sigma) = O\{(\eta_l\eta_{l+1} \cdots \eta_m)^{\alpha_j}\}, \qquad (11)$$

where l is determined by the condition that

$$\mathbf{V}_j \cdot \mathbf{L}_1 = \mathbf{V}_j \cdot \mathbf{L}_2 = \cdots = \mathbf{V}_j \cdot \mathbf{L}_{l-1} = 0 \quad \text{but} \quad \mathbf{V}_j \cdot \mathbf{L}_l \neq 0. \quad (12)$$

Therefore, from (3),

$$F(\mathbf{P},\sigma) = O\{\prod_{l=1}^{M} \prod\nolimits_j^{(l)} (\eta_l\eta_{l+1} \cdots \eta_m)^{\alpha_j}\}, \qquad (13)$$

where the product $\prod_j^{(l)}$ includes only those j satisfying (12). Collecting powers of each η, we have

$$F(\mathbf{P},\sigma) = O\{\eta_1^{\alpha(1)}\eta_2^{\alpha(2)} \cdots \eta_m^{\alpha(m)}\}, \qquad (14)$$

$$\alpha_{(r)} = \sum\nolimits_j^{(r)} \alpha_j, \qquad (15)$$

where the sum $\sum_j^{(r)}$ contains only those j for which (12) is satisfied for some $l \leqslant r$, and hence just those j for which

$$\mathbf{L}_1 \cdot \mathbf{V}_j \neq 0 \quad \text{or} \quad \mathbf{L}_2 \cdot \mathbf{V}_j \neq 0 \quad \text{or} \quad \cdots \quad \text{or} \quad \mathbf{L}_r \cdot \mathbf{V}_j \neq 0,$$

or in other words, over all j for which $\mathbf{V}_j$ is not orthogonal to the subspace $\{\mathbf{L}_1 \cdots \mathbf{L}_r\}$ spanned by $\mathbf{L}_1 \cdots \mathbf{L}_r$. We therefore are entitled to write $\alpha_{(r)}$ as a function only of the subspace $\{\mathbf{L}_1 \cdots \mathbf{L}_r\}$,

$$\alpha_{(r)} = \alpha(\{\mathbf{L}_1 \cdots \mathbf{L}_r\}), \qquad (16)$$

and obtain (8) from (15).

We may now apply our general theorem. The first part tells us that the integral (5) converges if it converges superficially [i.e., $\alpha(I) + \dim I < 0$] and if all subintegrations converge superficially [i.e., $\alpha(I') + \dim I' < 0$ for $I' \subset I$]. According to the renormalization procedure suggested by Dyson[2] and perfected by Salam,[3] we must subtract from each F a series of counterterms, which have the effect of lowering *all* these superficial divergences below zero. [For example, the last subtraction term, corresponding to the subspace I itself, is a polynomial of order $\alpha(I) + \dim I$ in the external momenta.] These subtractions have been proven equivalent to a renormalization of coupling constants, masses, and fields.[4] Our theorem then verifies the conjecture used without proof by Dyson and Salam, that such subtractions actually do render all integrals convergent, to any finite order in perturbation theory.[8]

In order to apply the second part of our theorem to learn about the asymptotic behavior of G, and also to further understand its convergence properties, we shall now introduce a new concept, that of a subgraph $\mathcal{G}'$ of the graph $\mathcal{G}$.

Definition. A set $\mathcal{G}'$ of internal and external lines j form a subgraph of $\mathcal{G}$ provided that there is no vertex in $\mathcal{G}$ to which is attached just one line of those in $\mathcal{G}'$. Clearly, a subgraph $\mathcal{G}'$ may be thought of as composed of a number of paths which begin and end in external lines or each other, but which never end abruptly within $\mathcal{G}$. Some examples are presented in Fig. 2.

We shall associate with each subspace $S' \subset R_{4N}$ a subgraph $\mathcal{G}'$, consisting of all lines j such that $\mathbf{V}_j$ is not orthogonal to S'. It is easy to see that this actually does define a subgraph obeying the above definition. [Proof: Suppose we have a vertex joining lines $j_1, j_2, \cdots, j_r$. Then by momentum conservation we must have

$$\pm \mathbf{V}_{j1} \pm \mathbf{V}_{j2} \pm \cdots \pm \mathbf{V}_{jr} = 0.$$

Thus if $j_1 \cdots j_{r-1}$ are not in $\mathcal{G}'$, so that $\mathbf{V}_{j1} \cdots \mathbf{V}_{jr-1}$ are orthogonal to S', we must have V_{jr} orthogonal to S' also, and hence j_r is not in $\mathcal{G}'$.] This correspondence allows a simple interpretation of the asymptotic powers $\alpha(S')$.

[8] A detailed discussion of this point is given by N. N. Bogoliubov and D. W. Shirkov, Fortsch. Phys. 4, 438 (1956); they show that with proper use of "regulators" all integrations are rendered convergent. I wish to thank Professor M. Goldberger for bringing this reference to my attention.

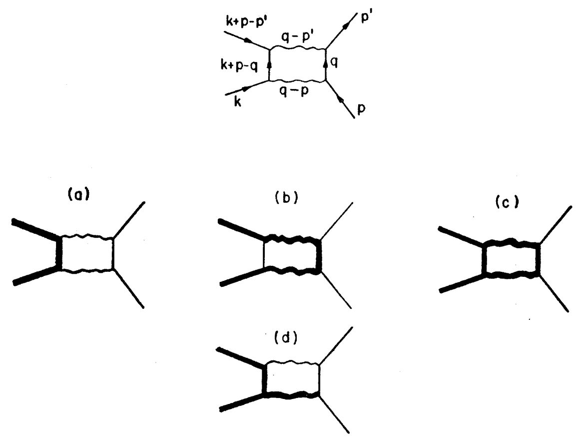

FIG. 2. An example of the use of subgraphs in determining an asymptotic behavior. Here (a), (b), (c) show subgraphs (darkened lines) of the graph pictured above, while (d) is shown as an example of a set of lines that is not a subgraph. If we label coordinate axes in a four dimensional space by k, q, p, p', the subgraphs (a), (b), (c) correspond to subspaces S' given by: (a) the line (1100); (b) the line (1000); (c) the plane spanned by (1000) and (0100). These subgraphs have dimensionality -1, -5, and $-6+4=-2$, respectively, as given by (19). Since (a), (b), (c) are obviously the only subgraphs containing external lines labelled k, $k+p-p'$ but not p or p', the maximum $[-1$, corresponding to (a)$]$ of these subgraph dimensionalities is the asymptotic power of the integral for $k\rightarrow\infty$; the integral goes as k^{-1}, times some power of $\log k$. We have purposely chosen the variable of integration q in an eccentric manner. Virtual nucleons and pions are represented here by straight and wavy lines, respectively.

According to (8), $\alpha(S')$ is just the sum of all momentum powers α_j for all lines j in the subgraph $\mathcal{G}'$ corresponding to S'.

Of the lines in a subgraph $\mathcal{G}'$, corresponding to a subspace S', the external lines are those that are not orthogonal to the projection $\Lambda(I)S'$ of S' along I. The number of independent internal momenta in $\mathcal{G}'$ is therefore

$$\dim S'-\dim\Lambda(I)S'=\dim\Lambda(E)S'. \tag{17}$$

It follows then that the coefficient

$$\mathfrak{D}_I(\mathcal{G}')\equiv\alpha(S')+\dim S'-\dim\Lambda(I)S', \tag{18}$$

is just the net number of momentum factors of the subgraph $\mathcal{G}'$, counting α_j for each line j and 4 for each integration. In renormalizable field theories it may be shown[2] that

$$\mathfrak{D}_I(\mathcal{G}')=4-\tfrac{3}{2}F(\mathcal{G}')-B(\mathcal{G}'), \tag{19}$$

where $F(\mathcal{G}')$, $B(\mathcal{G}')$ are the numbers of spin $\frac{1}{2}$ and spin 0 (or photon) lines attached to $\mathcal{G}'$, including external lines of $\mathcal{G}$ belonging to $\mathcal{G}'$. (We are supposing from now on that no bare propagators are associated with external lines of $\mathcal{G}$.) The quantity $\mathfrak{D}_I(\mathcal{G}')$ will be called the *dimensionality* of the subgraph $\mathcal{G}'$.

In the special case where $S'\subset I$, we have $\Lambda(I)S'=0$, and $\mathcal{G}'$ is thus a subgraph including no external lines. In this case $\mathfrak{D}_I(\mathcal{G}')$ is the superficial divergence $\alpha(S')+\dim S'$ associated with the integral of the subgraph. The renormalization subtractions are designed to lower $\mathfrak{D}_I(\mathcal{G}')$ below zero for every purely internal subgraph $\mathcal{G}'$ (including $\mathcal{G}$ itself without its external lines, if $\mathcal{G}$ is a "proper" diagram.) It is important that the number of differentiations needed to perform the subtraction associated with a subgraph $\mathcal{G}'$ is not greater than $\mathfrak{D}_I(\mathcal{G}')$; this ensures that the renormalization counterterms introduced in the Lagrangian to account for these subtractions are themselves renormalizable interactions.

Now we are in a position to apply part (b) of our theorem. Equation (III-12) may be rewritten, for any $S\subset E$,

$$\alpha_I(S)=\max_{\mathcal{G}'\ni S}\mathfrak{D}_I(\mathcal{G}'). \tag{20}$$

The "max" here is over all subgraphs $\mathcal{G}'$ containing just that set $\mathcal{E}_\infty$ of external lines j for which $\mathbf{V}_j$ is not orthogonal to S. According to the interpretation of $\alpha_I(S)$ in Sec. III, and of $\mathfrak{D}_I(\mathcal{G}')$ above, we can express (20) in the rule:

If a set $\mathcal{E}_\infty$ of external lines of a graph $\mathcal{G}$ have moment) going to infinity (i.e., for line j in $\mathcal{E}_\infty$, $p_j=e_j\eta$ where $\eta\rightarrow\infty$, e_j "almost any" set of fixed nonzero four-vectors, then the integrated Green's function corresponding to $\mathcal{G}$ will behave as $O\{\eta^{\alpha_I(\mathcal{E}_\infty)}(\log\eta)^{\beta_I(\mathcal{E}_\infty)}\}$, where $\alpha_I(\mathcal{E}_\infty)$ is the maximum of the dimensionality [given by (19)] of all subgraphs $\mathcal{G}'$ of $\mathcal{G}$ including the external lines $\mathcal{E}_\infty$ and no other external lines. (A detailed example of the application of this rule is presented in Fig. 2.)

Strictly speaking, we should prove that the renormalization subtractions needed to lower all $\alpha(I')+\dim I'(I'\subset I)$ below zero do not alter the value of (20). The proof is tedious, and will be omitted. It is important to note, however, that we should restrict the subgraphs $\mathcal{G}'$ above to those that do not contain any parts entirely disconnected from $\mathcal{E}_\infty$, for since such parts are themselves purely internal subgraphs of $\mathcal{G}$, the renormalization procedure invariably lowers their dimensionality below zero.

If we consider not an individual Feynman diagram, but the whole sum of Feynman diagrams for a particular set of external lines up to some sufficiently large but finite order, we find a remarkable simplification. *When a set $\mathcal{E}_\infty$ of external lines have momenta tending to infinity, then the total Green's function has as its asymptotic power a quantity $\alpha(\mathcal{E}_\infty)$ which depends only on the numbers of lines in $\mathcal{E}_\infty$,* and is given by

$$\alpha(\mathcal{E}_\infty)=4-\tfrac{3}{2}f(\mathcal{E}_\infty)-b(\mathcal{E}_\infty)-\min_{\mathcal{E}'}[\tfrac{3}{2}f(\mathcal{E}')+b(\mathcal{E}')]. \tag{21}$$

Here $b(\mathcal{E})$, $f(\mathcal{E})$ are the number of spin 0 (or photon) or spin $\frac{1}{2}$ lines in the set $\mathcal{E}$. The minimum in (21) is taken over all sets $\mathcal{E}'$ of lines such that the virtual transition $\mathcal{E}_\infty\leftrightarrow\mathcal{E}'$ is not forbidden by selection rules. (If we are concerned with connected or proper Green's functions, we may also stipulate that $\mathcal{E}'$ must contain at least one or at least two lines.) For example, if $\mathcal{E}_\infty$ consists of a pair of incoming nucleon and antinucleon lines, the "min" in (21) is reached for $\mathcal{E}'$ a pair of pions, so that $\alpha(\mathcal{E}_\infty)$ is given by $4-3-2=-1$. This is the maximum asymptotic power of any connected diagram or sum of diagrams for which a nucleon and an antinucleon external momenta tend to infinity, with the other external momenta fixed. (The diagram of Fig. 2 shows the realization of this maximum in this case.)

In order to verify this rule we need only note that every subgraph $\mathcal{G}'$ included in (20) has attached to it *all* lines in $\mathcal{E}_\infty$, together with a set $\mathcal{E}'$ of "bridges," consisting of internal and external lines belonging to $\mathcal{G}$ but not to $\mathcal{G}'$. We can therefore write in (19)

$$F(\mathcal{G}')=f(\mathcal{E}_\infty)+f(\mathcal{E}'),\quad B(\mathcal{G}')=b(\mathcal{E}_\infty)+b(\mathcal{E}'), \tag{22}$$

and inserting (22) and (19) into (20) we obtain (21). Any possible set $\mathcal{E}'$ of bridges will occur if we go to high enough order, so that the maximum is always attained.

Our result cannot easily be extended to the logarithmic powers $\beta_I(S)$; it is known that these depend strongly on the structure and order of the graphs considered. Thus, although our proof shows that any Green's function, calculated to any finite order, belongs to a class A_n with asymptotic powers $\alpha(\mathcal{E}_\infty)$ given by (21), it is entirely possible that the logarithmic powers in the infinite sum add up in such a manner that the total Green's function does not have asymptotic power $\alpha(\mathcal{E}_\infty)$, or perhaps does not even belong to a class A_n. However, in the present state of field theory we may hope that results based on perturbation theory may serve as a useful guide.

APPENDIX: PROJECTIONS OF SUBSPACES

In Sec. III we introduce the operation of projecting one subspace along another. As our use of this operation may perhaps be unfamiliar, we shall define it more precisely, and prove some simple statements used in the proof and interpretation of our theorem.

Let I be a subspace of a vector space R_n. It is always possible to choose (not uniquely) another subspace $E\subset R_n$ such I and E are disjoint (and therefore independent) and such that $R_n=I+E$. With such a choice of E the operator $\Lambda(I)$, the usual projection along I on E, becomes well defined: For any vector $\mathbf{L}'\epsilon R_n$ we write $\mathbf{L}'=\mathbf{L}+\mathbf{L}''$, $\mathbf{L}\epsilon E$, $\mathbf{L}''\epsilon I$, and set $\Lambda(I)\mathbf{L}'=\mathbf{L}$. If a set of vectors $\mathbf{L}_i'$ span a subspace $S'\subset R_n$ then $\Lambda(I)S'\equiv S$, where S is the subspace spanned by the corresponding $\mathbf{L}_i$.

The last equation, $\Lambda(I)S'=S$, is usually taken in this paper as a condition on S', with I and S fixed disjoint subspaces of R_n. As such a condition it is actually independent of the choice of E, as shown below by statement (A); we should, properly speaking, refer to $\Lambda(I)$ as the projection onto the space R_n/I.

(A) If I and S are disjoint subspaces, and we define $\Lambda(I)$ by choosing E to be *any* subspace such that I and E are disjoint, $R_n=I+E$, and $S\subset E$, then $\Lambda(I)S'=S$ if and only if for *every* set of vectors $\mathbf{L}_1\cdots\mathbf{L}_r$ spanning S there exists a set of vectors $\mathbf{L}_1'\cdots\mathbf{L}_r'\epsilon I$ such that

$$S'=\{\mathbf{L}_1+\mathbf{L}_1',\ \cdots,\ \mathbf{L}_r+\mathbf{L}_r'\},S \tag{1}$$

Equivalently, $\Lambda(I)S'=S$ if and only if there exists *some* set of vectors $\mathbf{L}_1\cdots\mathbf{L}_r$ spanning S and $\mathbf{L}_1'\cdots\mathbf{L}_r'\epsilon I$ and satisfying (1).

[Proof: If S' is given by (1), with $\mathbf{L}_i'\epsilon I$ and $\mathbf{L}_i\epsilon S$ and thus $\mathbf{L}_i\epsilon E$, then by definition $\Lambda(I)S'=\{\mathbf{L}_1\cdots\mathbf{L}_r\}$. On the other hand, we can always write any subspace S' as in (1) where $\mathbf{L}_i'\epsilon I$, $\mathbf{L}_i\epsilon E$, and if $\Lambda(I)S'=S$ we have $S=\{\mathbf{L}_1\cdots\mathbf{L}_r\}$. Clearly we can take any new set of $\mathbf{L}_i$ spanning S and preserve (1), with a new set of $\mathbf{L}'$.]

(B) If $\Lambda(I)S'=S$, with I, S disjoint, then

$$S'\subset I+S,$$
$$\dim S\leqslant\dim S'\leqslant\dim I+\dim S.$$

(C) If $\mathbf{L}$ is a vector not in S, then $\Lambda(\{L\})S'=S$ if and only if either $S'=S+\{L\}$ or there exist numbers $u_1\cdots u_p$ with

$$S'=\{\mathbf{L}_1+u_1\mathbf{L},\ \mathbf{L}_2+u_2\mathbf{L},\ \cdots,\ \mathbf{L}_r+u_r\mathbf{L}\},$$

where $S=\{L_1\cdots L_r\}$.

(These two alternatives represent the possibilities $\dim S'=\dim S+1$ and $\dim S'=\dim S$, respectively.)

(D) If I and S are disjoint subspaces then $S'\subset S+I$ if and only if there is some subspace $S''\subset S$ such that $\Lambda(I)S'=S''$.

[Proof: The subspace S'' is just $\{\mathbf{L}_1'\cdots\mathbf{L}_r'\}$.]

(E) If S, S_1, S_2 are disjoint subspaces, and $\Lambda(S_1)S'=S$, then $\Lambda(S_2)S''=S'$ if and only if $\Lambda(S_1+S_2)S''=S$.

[Proof: We can write $S'=\{\mathbf{L}_1+\mathbf{L}_1',\ \cdots,\ \mathbf{L}_r+\mathbf{L}_r'\}$ where $S=\{\mathbf{L}_1\cdots\mathbf{L}_r\}$, $\mathbf{L}_1'\cdots\mathbf{L}_r'\epsilon S_1$. Then $\Lambda(S_2)S''=S'$ means that $S''=\{\mathbf{L}_1+\mathbf{L}_1'+\mathbf{L}_1'',\ \cdots,\ \mathbf{L}_r+\mathbf{L}_r'+\mathbf{L}_r''\}$, where $\mathbf{L}_1''\cdots\mathbf{L}_r''\epsilon S_2$. Also, $\Lambda(S_1+S_2)S''=S$ means that $S''=\mathbf{L}_1+\mathbf{L}_1''',\ \cdots,\ \mathbf{L}_r+\mathbf{L}_r'''$ where $\mathbf{L}_1'''\cdots\mathbf{L}_r'''\epsilon S_1+S_2$. The most general $\mathbf{L}_i'''\epsilon S_1+S_2$ may be written $\mathbf{L}_i'''=\mathbf{L}_i'+\mathbf{L}_i''$, so these statements are equivalent.]

ACKNOWLEDGMENTS

It is a pleasure to thank Professor N. M. Kroll, Professor T. D. Lee, Professor S. B. Treiman, and Professor A. S. Wightman for discussions on this work and related matters. I particularly wish to thank Professor Wightman for his many valuable suggestions, and for enabling this paper to satisfy Salam's criterion.

Note to be added in proof.—For the sake of mathematical rigor, the definition in Equation (III-1) of the class A_n requires a slight modification. The coefficients β should be taken as functions of the individual vectors $\mathbf{L}_1, \mathbf{L}_2, \cdots$ and not only of the subspaces $\{\mathbf{L}_1,\mathbf{L}_2,\cdots\}$. The proof in Sec. IV that if $f\epsilon A_n$ then $f_I\epsilon A_{n-k}$ is then correct, with no changes (except minor notational ones) required.