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Theory of Allowed and Forbidden Transitions in Muon Capture Reactions*

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A general formula for the transition rate of the muon capture reaction, $\mu^- + (A,Z) \rightarrow \nu + (A, Z-1)$, where the final nuclear state has definite spin and parity, is given in terms of the total and orbital angular momenta of the emitted neutrino and of the spins of the initial and final nuclear states. The induced pseudoscalar interaction and the interaction due to the assumption of conserved vector current are taken into account, together with the vector and axial vector interactions. The forbiddenness of the muon capture reaction is defined in a manner analogous to the theory of the beta decay. The spin and parity changes can assume the values $(0+, 1+)$, $(0-, 1-, 2-)$, $[n(-)^n, n+1\ (-)^n]$ for the allowed, first forbidden, and nth $(n \geq 2)$ forbidden transitions, respectively. ($+$ and $-$ mean the parity change "no" and "yes".) For these transitions, the number of reduced nuclear matrix elements involved is nine, sixteen, and fourteen, respectively. The transition rate of muon capture reaction is reduced by a factor of 10^3, approximately, for a two-unit increase of the forbiddenness, if the atomic number and the energy of neutrino are constant. The contribution from the higher order transition to the lower one is less than 0.1% in the medium and light nuclei. Explicit formulas for the transition rate are given for the allowed, first forbidden and nth forbidden transitions. They are related to the corresponding formulas of beta decay. Our formalism was applied to the calculation of the partial muon capture rate by C^{12} ending in the ground state of B^{12}. The numerical analysis indicates that measurement of this capture rate can determine whether the conserved vector current interaction term exists in nature only if the coupling constant of the induced pseudoscalar interaction and the nuclear wave functions are well known. The transition rates are given in Table V and Fig. 1, for the j-j coupling shell model and harmonic oscillator wave functions. They are 9–13% smaller than those given by Fujii and Primakoff.

1. INTRODUCTION

In the past three years, remarkable progress has been achieved in the study of beta decay, both theoretically and experimentally. The basic question of the types of interaction has been settled in favor of V–A interaction, with $C_A \approx -1.2C_V$, $C_V \approx C_V'$, and $C_A \approx C_A'$.[1] It has also been shown that the V–A interaction is generally acceptable for the weak processes involving elementary particles.[2–5] Among these weak processes, muon capture by an atomic nucleus and beta decay are of particular interest, as they involve no strange particles. Nevertheless, both theory and experiment involving the muon capture reaction are not yet well established compared with those of beta decay. One of the reasons is the complexicity of the muon capture reaction, where the released energy is quite large (of the order of 100 Mev). This energy will excite many nuclear levels of the daughter nucleus. Therefore, the partial transition rate between the definite nuclear states is difficult to measure experimentally, while that of beta decay is quite easy. On the other hand, the total capture rate, which is much easier to obtain experimentally, is rather insensitive to the type of interaction, because the character of the elementary process is masked by nuclear effects. The theoretical and experimental status at present has been reviewed by Primakoff.[6]

Since, however, the partial transition rate between two definite nuclear states in the muon capture reaction depends strongly on the type of the interaction, it is worthwhile to develop a general formalism for this problem, in anticipation of future experiments. One such attempt among many is by Primakoff and one of the present authors (A.F.).[7] Assuming the lepton bare-nucleon coupling to be V–A, they derived an effective Hamiltonian. The nuclear matrix elements were estimated by using a specific nuclear model. Since they worked out only special cases of spin and parity changes ($C^{12} \rightarrow B^{12}$, $Li^6 \rightarrow He^6$, and $He^3 \rightarrow H^3$), their treatment is somewhat less general with respect to the angular momenta of the leptons and the spins of nuclear states. Furthermore, they neglected some nuclear matrix elements and all cross terms between the remaining nuclear matrix elements.

The purpose of the present paper is to present a general formalism for the partial transition rate of muon capture between definite nuclear states. This formalism is completely analogous to that of beta decay, and leads to the theory of orbital electron capture, if we replace the muon by an electron. Since our calculation is completely in the spherical representation, the relations among nuclear spins and lepton angular momenta are very clear.

In Sec. 2, the interaction Hamiltonian of the muon capture reaction is discussed. The induced pseudoscalar

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[1] For example, see the review article by C. S. Wu, Revs. Modern Phys. **31**, 783 (1959).

[2] J. J. Sakurai, Nuovo cimento **7**, 649 (1958).

[3] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[4] M. Gell-Mann, Phys. Rev. **111**, 362 (1958).

[5] E. C. G. Sudarshan and R. E. Marshak, Phys. Rev. **109**, 1860 (1958).

[6] H. Primakoff, Revs. Modern Phys. **31**, 802 (1959).

[7] A. Fujii and H. Primakoff, Nuovo cimento **12**, 327 (1959). L. Wolfenstein, Bull. Am. Phys. Soc. **4**, 81 (1959).

interaction and the interaction due to the assumption of conserved vector current are both taken into account. In Sec. 3, the nuclear part of the interaction Hamiltonian is transformed into a nonrelativistic form while the lepton part is still kept relativistic. In Sec. 4, all matrix elements of the muon capture reaction are evaluated, where the muon is absorbed by the nucleus from a given orbital state, a neutrino of given total and orbital angular momenta is emitted, and the nucleus changes from one state of given spin and parity to another state. Using the result of Sec. 4, a general formula for the transition probability is given in Sec. 5. This formula is also applicable to the orbital electron capture rate, which will be investigated in a future paper. In nature, the muon is mostly absorbed from the K orbit of the μ-mesonic atom. The transition rate for this case is given in the approximation $\alpha Z \ll 1$, in Sec. 6. In Sec. 7, we discuss the definition of forbiddenness in the muon capture reaction, which is the same as that in the theory of beta decay. We further classify all possible reduced nuclear matrix elements according to forbiddenness. This classification is somewhat different from that of the theory of beta decay. The former focuses its attention on the rank of the tensor, the latter on the orbital angular momentum of the lepton system. In this way, the contribution of the $(n+2)$th forbiddeness to the nth one is less than 0.1% in *both* cases.[8] The transition rate is reduced by a factor of 10^3 in the muon capture, assuming the level distance between the initial and final nuclear states and the atomic number to be constant, and by a factor of 10^8 in the beta decay. In Sec. 8, explicit formulas for the transition rate are given for the allowed, first forbidden, and nth forbidden transitions. In Sec. 9, the muon capture rate is related to the transition rate of beta decay. In Sec. 10, our formalism is applied to muon capture by C^{12}. The j-j coupling shell model and the harmonic oscillator functions are adopted, for the estimation of the nuclear matrix elements, as was done by Fujii and Primakoff.[7] Our theory gives a 9–13% decrease in the transition rate compared with theirs. The numerical analysis indicates that the measurement of the muon capture rate between the ground states of C^{12} and B^{12} sheds light on the assumption of the conserved vector current, provided our knowledge of the induced pseudoscalar coupling constant, C_P, and the nuclear wave functions is fairly accurate. In Appendix, the reduced nuclear matrix elements for muon capture by C^{12} are evaluated. The reader who is not too interested in mathematical detail may skip Secs. 3–6 entirely.

Although we adopt the muon wave functions (G_{-1} and F_{-1}) for a point nucleus in Secs. 6–8, the general results in Secs. 4 and 5 hold with modified G_{-1} and F_{-1} for a real nucleus. Explicit calculation for this is being made.

[8] Although the beta-ray spectrum in $B^{12}\to C^{12}$ varies about 10% in the whole energy region, the effect in its ft value is about 0.15%. This is due to the cancellation of the energy integral of $\frac{1}{3}KL_0+N_0$. In most beta decays, this effect is less than 0.01%.

2. INTERACTION HAMILTONIAN FOR MUON CAPTURE REACTION

We assume the lepton bare-nucleon coupling is via vector and axial vector interactions of the Fermi-type. Thus, the most general interaction Hamiltonian density for the reaction, $\mu^- + p \to \nu + n$, is[9]

$$H = \bar{\psi}_n \mathcal{H} \psi_p,$$

with

$$\begin{aligned}\sqrt{2}\mathcal{H} = {} & \gamma_\lambda[C_V(\bar{\psi}_\nu\gamma_\lambda\psi_\mu)+C_V'(\bar{\psi}_\nu\gamma_\lambda\gamma_5\psi_\mu)] \\ & +i\gamma_\lambda\gamma_5[C_A(\bar{\psi}_\nu i\gamma_\lambda\gamma_5\psi_\mu)+C_A'(\bar{\psi}_\nu i\gamma_\lambda\psi_\mu)] \\ & +\gamma_5[C_P(\bar{\psi}_\nu\gamma_5\psi_\mu)-C_P'(\bar{\psi}_\nu\psi_\mu)] \\ & +\sigma_{\lambda\rho}[C_M p_\rho(\bar{\psi}_\nu i\gamma_\lambda\psi_\mu)+C_M' p_\rho(\bar{\psi}_\nu i\gamma_\lambda\gamma_5\psi_\mu)], \quad (1) \\ & \bar{\psi}=\psi^\dagger\gamma_4, \quad \text{and} \quad \sigma_{\lambda\rho}=\tfrac{1}{2}[\gamma_\lambda\gamma_\rho-\gamma_\rho\gamma_\lambda].\end{aligned}$$

The dagger and asterisk mean the Hamiltonian conjugate and complex conjugate, hereafter. Here the subscript, n, p, ν, and μ stand for the neutron, proton, neutrino, and muon, respectively. In the fourth line of Eq. (1), the differential operators p_ρ, $[-i\nabla, -\partial/\partial t]$, act on the lepton covariants, but not on ψ_p. The first, the second and the third lines represent the vector, axial vector and the induced scalar interactions, respectively. The fourth line is the interaction, which is introduced by the assumption of the conserved vector current.[3,4,10] Assuming the time reversal invariance of Eq. (1), we can choose all coupling constants C_i and C_i' to be real. The relative strength of C_i and C_i' is not specified in the above, but the fact that the helicity of the neutrino is almost completely left handed concludes[1]

$$C_i = C_i'. \quad (2)$$

We assume that the relation also holds in muon capture reactions. Comparison between the "weak current" and electromagnetic current gives[4]

$$C_M = C_V(\mu_p - \mu_n)/2M, \quad \mu_p - \mu_n = 3.706. \quad (3)$$

Here μ_p and μ_n are the anomalous magnetic moments of the proton and neutron in nuclear magnetons, and M is the nucleon mass. By the dispersion-theoretical approach, Goldberger and Treiman[9] obtained the relation

$$C_P = 8C_A. \quad (4)$$

The experiments on the beta decay of the neutron and nuclear 0–0 transitions give[1]

$$C_A = -1.21C_V. \quad (5)$$

[9] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

[10] We note here that the interaction due to the assumption of conserved vector current is automatically taken into account in the theory of the elementary particles given by S. Sakata, Progr. Theoret. Phys. (Kyoto) **16**, 686 (1956), where fundamental particles are assumed to be proton, neutron, and lambda only. There, the pion is considered as a bound state of the nucleon-antinucleon system. Thus, the detection of C_M term becomes of twofold importance, namely for verification of Sakata model of elementary particles and for that of the Feynman-Gell-Mann interaction of beta decay. One of the authors (M.M.) would like to express his sincere thanks to Dr. S. Okubo, University of Naples, Italy, for a stimulating discussion.

We adopt the following representation of the Dirac matrices, which is common in the theory of beta decay.[11,12]

$$\gamma=-i\beta\alpha=\begin{pmatrix}0 & -i\sigma\\ i\sigma & 0\end{pmatrix},\quad \gamma_4=-\beta=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

$$\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4=-i\alpha_1\alpha_2\alpha_3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{6}$$

$$\alpha=\begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix},\quad \sigma=\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix},\quad 1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Here 1 and σ are the two by two unit and Pauli matrices.

Using Eqs. (2)–(6), the Hamiltonian density becomes

$$H=\psi_n^\dagger \mathbf{H}\psi_p,$$

with

$$\begin{aligned}\mathbf{H}=&C_V[1\cdot L(1)-\alpha\cdot L(\alpha)]\\ &+C_A[\sigma\cdot L(\sigma)-\gamma_5\cdot L(\gamma_5)]+C_P\beta\gamma_5\cdot L(\beta\gamma_5)\\ &+(C_V/2M)(\mu_p-\mu_n)[-i\beta\sigma\cdot\mathbf{p}\times L(\alpha)\\ &\qquad+\beta\alpha\cdot\mathbf{p}L(1)-i\beta\alpha\cdot p_4L(\alpha)].\end{aligned} \tag{7}$$

Here the lepton covariants are abbreviated as

$$L(\sigma)=\psi_\nu^\dagger[(1+\gamma_5)/\sqrt{2}]\sigma\psi_\mu,\ \text{etc.} \tag{8}$$

Therefore,

$$L(\alpha)=L(\sigma),\quad\text{and}\quad L(1)=L(\gamma_5). \tag{9}$$

In the theory of beta decay, the interaction Hamiltonian is expanded in the multipole order of the emitted electron-neutrino system, and the whole calculation is relativistic. The nuclear matrix elements (i.e., the nucleon part of this expansion) are also left in relativistic form, as phenomenological parameters which may be determined by experimental data, e.g., by the shape of the electron spectrum. On the other hand, we know only the transition rate in the muon capture reaction, but not the spectral resolution of the neutrino. Therefore, we cannot leave the nuclear matrix elements as phenomenological parameters. (The situation is the same in calculation of the ft value of beta decay.) For this purpose, it is necessary to reduce all the relativistic nuclear matrix elements (namely, the momentum-type matrix elements) into nonrelativistic forms. We thus transform, first, the nuclear part of the interaction Hamiltonian, Eq. (7), into nonrelativistic form leaving the relativistic lepton part, whose evaluation is very easy. In the entire calculation of the muon capture rate, Eq. (11) below is expanded in the multipole order of the emitted neutrino in a way analogous to that of the theory of beta decay.

[11] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

[12] M. Yamada and M. Morita, Progr. Theoret. Phys. (Kyoto) **8**, 431 (1952).

3. NONRELATIVISTIC REDUCTION OF NUCLEON OPERATORS

The interaction Hamiltonian density, Eq. (7), involves odd and even operators. We transform these operators into equivalent even operators by the Foldy-Wouthuysen transformation modified by Rose and Osborn.[13] We replace the relativistic nucleon wave function

$$\psi=\begin{pmatrix}v\\ u\end{pmatrix},\quad\text{by}\quad \psi=\begin{bmatrix}-\dfrac{1}{2M}\sigma\cdot\mathbf{p}u\\ u\end{bmatrix}, \tag{10}$$

where u and v are large and small components of the Dirac spinor, corresponding to the eigenvalue -1 and $+1$ of β, respectively.

The transformed interaction Hamiltonian is given by[14]

$$H=u_n^\dagger\mathbf{H}u_p,$$

with

$$\begin{aligned}\mathbf{H}=&C_V1\cdot L(1)+C_A\sigma\cdot L(\sigma)\\ &+(C_V/2M)[2L(\alpha)\cdot\mathbf{p}+\mathbf{p}\cdot L(\alpha)+i\sigma\cdot\mathbf{p}\times L(\alpha)]\\ &+(C_A/2M)[2L(\gamma_5)\sigma\cdot\mathbf{p}+\sigma\cdot\mathbf{p}L(\gamma_5)]\\ &-(C_P/2M)\sigma\cdot\mathbf{p}L(\beta\gamma_5)\\ &\qquad+(\mu_p-\mu_n)(C_V/2M)[i\sigma\cdot\mathbf{p}\times L(\alpha)].\end{aligned} \tag{11}$$

In Eq. (11), all terms of order $(p/2M)^2$ are omitted. The differential operators $\mathbf{p}$ standing on the left of the lepton covariants act only on them, but not on u_p. Equation (11) is equivalent to Eq. (4) of Fujii and Primakoff,[7] since the nonrelativistic muon wave function obeys the relation

$$L(\gamma_5)=L(\beta\gamma_5). \tag{12}$$

4. THE SPHERICAL COMPONENTS OF THE MATRIX ELEMENTS

Throughout this paper, we adopt the units $\hbar=c=m_e=1$. The free neutrino state is specified by its linear momentum $\mathbf{q}$ and the direction of spin $m=\pm\frac{1}{2}$. Expanding the plane wave neutrino in the normalized spherical spinor $\chi_{\kappa\mu}$,[15]

$$\psi_\nu(\mathbf{q},m;\mathbf{r})=\sum_{\kappa\mu} i^l(l\,\tfrac{1}{2}\,\mu-m\;m\,|\,j\mu)Y_l^{\mu-m}(\hat{q})\psi_{\kappa\mu}^{(\nu)}, \tag{13}$$

with

$$\psi_{\kappa\mu}^{(\nu)}=\begin{pmatrix}-if_\kappa\chi_{-\kappa\mu}\\ g_\kappa\chi_{-\kappa\mu}\end{pmatrix}, \tag{14}$$

$$g_\kappa=\pi^{-\frac{1}{2}}j_l(qr),\quad f_\kappa=\pi^{-\frac{1}{2}}S_\kappa j_{\bar{l}}(qr), \tag{15}$$

$$\chi_{\kappa\mu}=\textstyle\sum_{m'}(l\,\tfrac{1}{2}\,\mu-m'\;m'\,|\,j\mu)Y_l^{\mu-m'}(\hat{r})\psi_{\frac{1}{2}}^{m'}. \tag{16}$$

[13] M. E. Rose and R. K. Osborn, Phys. Rev. **93**, 1315 (1954).

[14] We have opposite sign for the axial vector and the pseudoscalar interactions compared with the results given in Table I of reference 13.

[15] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), Chap. IX.

Here $j_l(qr)$'s are spherical Bessel functions, $\psi_{\frac{1}{2}}^{m'}$'s are the spin wave functions of two dimension, and S_κ is the sign of κ. The total and orbital angular momentum, j and l, of the neutrino for a particular value of κ is given by

$$\begin{aligned} l&=\kappa \quad\text{and}\quad j=l-\tfrac{1}{2} \quad\text{for}\quad \kappa>0,\\ l&=-\kappa-1 \quad\text{and}\quad j=l+\tfrac{1}{2} \quad\text{for}\quad \kappa<0. \end{aligned} \tag{17}$$

For example, $\kappa=-1, 1, -2$, and 2 corresponding to $s_{\frac{1}{2}}$, $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, and $d_{\frac{3}{2}}$, respectively. Here summation over κ means the sum over all possible values of *j and l*. In the above, subscripts, l and $\bar{l}$ of the spherical Bessel functions denote the orbital angular momenta corresponding to κ and $-\kappa$, respectively. $\hat{q}$ and $\hat{r}$ are the unit vectors in the direction of the neutrino momentum and space coordinate, respectively. The normalization of ψ_ν is such that

$$\int \psi_\nu^\dagger(\mathbf{q}', m' ; \mathbf{r})\psi_\nu(\mathbf{q}, m, ; \mathbf{r})d\mathbf{r}=\delta_{mm'}\delta(\mathbf{q}'-\mathbf{q}), \tag{18}$$

where the integral extends over whole space.

The bound state wave function of the muon is expressed in the same way as that of the neutrino, but it has only a single value of corresponding to a definite orbit. Namely,

$$\psi_\mu(\kappa,\mu ; \mathbf{r})=\psi_{\kappa\mu}^{(\mu)}=\begin{pmatrix} -iF_\kappa\chi_{-\kappa\mu} \\ G_\kappa\chi_{\kappa\mu} \end{pmatrix}, \tag{19}$$

G_κ and F_κ being the radial wave functions of the bound state. Explicit forms of these wave functions for a point nucleus are given, e.g., by Bethe and Salpeter,[16] for $1s_{\frac{1}{2}}$, $2s_{\frac{1}{2}}$, $2p_{\frac{1}{2}}$, and $2p_{\frac{3}{2}}$. In the lowest state $1s_{\frac{1}{2}}$

$$\begin{aligned} &\kappa=-1, \quad \mu=\pm\tfrac{1}{2},\\ &G_{-1}=(2Z/a_0)^{\frac{3}{2}}\\ &\qquad\times[(1+\gamma)/2\Gamma(2\gamma+1)]^{\frac{1}{2}}e^{-Zr/a_0}(2Zr/a_0)^{\gamma-1},\\ &F_{-1}=-[(1-\gamma)/(1+\gamma)]^{\frac{1}{2}}G_{-1} \text{ with } \gamma=[1-(\alpha Z)^2]^{\frac{1}{2}}. \end{aligned} \tag{20}$$

Here a_0 is the Bohr radius of the μ-mesonic atom and Z is the atomic number of the nucleus. If we assume αZ (α fine structure constant) to be very small, they become

$$\begin{aligned} &G_{-1}=2(\alpha Zm_\mu')^{\frac{3}{2}}e^{-\alpha Zm_\mu' r},\\ &F_{-1}=0, \quad\text{with}\quad m_\mu'=m_\mu[1+(m_\mu/AM)]^{-1}. \end{aligned} \tag{21}$$

Here m_μ is the muon mass. The m_μ' is the muon reduced mass in the parent μ-mesonic atom. The normalization for ψ_μ is such that

$$\int \psi_\mu^\dagger(\kappa', \mu'; \mathbf{r})\psi_\mu(\kappa, \mu; \mathbf{r})d\mathbf{r}=\delta_{\kappa\kappa'}\delta_{\mu\mu'}. \tag{22}$$

Using these lepton wave functions, we evaluate the spherical component of the interaction Hamiltonian defined by

$$(\kappa\mu|\mathbf{H}|\kappa'\mu')\equiv(\psi_{\kappa\mu}^{(\nu)}, \mathbf{H}\psi_{\kappa'\mu'}^{(\mu)}). \tag{23}$$

Calculation of the spherical component of each term in Eq. (11) can be performed by a method similar to that developed by Rose and his collaborator,[13,15,17] Takebe,[18] and others. Here we give the results only.

$$1\cdot(\kappa\mu|1|\kappa'\mu')=(-)^{\mu+j'+l'}\sum_{vu}(jj' -\mu\mu'|u\,\mu'-\mu)\mathfrak{Y}_{0vu}^{\mu'-\mu}(\hat{r})[g_\kappa G_{\kappa'}S_{0vu}(\kappa,\kappa')-f_\kappa F_{\kappa'}S_{0vu}(-\kappa, -\kappa')]\delta_{vu}, \tag{24}$$

with

$$\mathfrak{Y}_{0vu}^{\mu'-\mu}(\hat{r})=(4\pi)^{-\frac{1}{2}}Y_v^{\mu'-\mu}(\hat{r}). \tag{25}$$

$$\boldsymbol{\sigma}\cdot(\kappa\mu|\boldsymbol{\sigma}|\kappa'\mu')=(-)^{\mu+j'+l'+1}\sum_{vu}(jj' -\mu\mu'|u\,\mu'-\mu)\mathfrak{Y}_{1vu}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})[g_\kappa G_{\kappa'}S_{1vu}(\kappa,\kappa')-f_\kappa F_{\kappa'}S_{1vu}(-\kappa, -\kappa')], \tag{26}$$

with

$$\mathfrak{Y}_{1vu}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})=\sum_m(1v -m\; m-\mu+\mu'|u\,\mu'-\mu)Y_v^{m+\mu'-\mu}(\hat{r})\mathfrak{Y}_1^{-m}(\boldsymbol{\sigma}). \tag{27}$$

Here Y_l^m and $\mathfrak{Y}_l^m$ are the spherical and solid harmonics defined in reference.[19] In Eqs. (24) and (26), S_{0vu} and S_{1vu} are given by

$$S_{kvu}(\kappa,\kappa')=[2(2l+1)(2l'+1)(2j+1)(2j'+1)]^{\frac{1}{2}}(ll'00|v0)\begin{bmatrix} l & l' & v\\ j & j' & u\\ \frac{1}{2} & \frac{1}{2} & k \end{bmatrix} \quad\text{for } k=0 \text{ and } 1, \tag{28}$$

with

$$\begin{bmatrix} l & l' & v\\ j & j' & u\\ \frac{1}{2} & \frac{1}{2} & k \end{bmatrix}=\sum_f(2f+1)W(l'j'\tfrac{1}{2}k,\tfrac{1}{2}f)W(j\tfrac{1}{2}vl',lf)W(jj'vk,uf). \tag{29}$$

[16] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Academic Press Inc., New York, 1959), p. 69.

[17] L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. **25**, 729 (1953).

[18] H. Takebe, Progr. Theoret. Phys. (Kyoto) **12**, 561 (1954).

[19] They are given, e.g., by A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957), p. 124.

Here k, v, and u are the resultant spin, orbital, and total angular momentum of the neutrino-muon system, respectively.

$$(\kappa\mu|\boldsymbol{\alpha}|\kappa'\mu')\cdot\mathbf{p}=(-)^{\mu+j'+l'+1}\sum_{vu}(jj'\ -\mu\ \mu'|u\ \mu'-\mu)i[f_\kappa G_{\kappa'}S_{1vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{1vu}(\kappa,-\kappa')]\mathfrak{Y}_{1vu}{}^{\mu'-\mu}(\hat{r},\mathbf{p}). \quad (30)$$

Here the definition of $\mathfrak{Y}_{1vu}{}^{\mu'-\mu}(\hat{r},\mathbf{p})$ is similar to Eq. (27).

$$\mathbf{p}\cdot(\kappa\mu|\boldsymbol{\alpha}|\kappa'\mu')=3^{\frac{1}{2}}(-)^{\mu+j'+l'+1}\sum_{vu}(jj'\ -\mu\mu'|u\ \mu'-\mu)$$
$$\times\{[(v+1)/(2v+3)]^{\frac{1}{2}}\mathfrak{Y}_{0\ v+1\ u}{}^{\mu'-\mu}(\hat{r})\delta_{v+1\ u}D_+-[v/(2v-1)]^{\frac{1}{2}}\mathfrak{Y}_{0\ v-1\ u}{}^{\mu'-\mu}(\hat{r})\delta_{v-1\ u}D_-\}$$
$$\times[f_\kappa G_{\kappa'}S_{1vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{1vu}(\kappa,-\kappa')], \quad (31)$$

with[20]

$$D_+=\frac{d}{dr}-\frac{v}{r},\quad D_-=\frac{d}{dr}+\frac{v+1}{r}. \quad (32)$$

$$i\boldsymbol{\sigma}\cdot\mathbf{p}\times(\kappa\mu|\boldsymbol{\alpha}|\kappa'\mu')=6^{\frac{1}{2}}(-)^{\mu+j'+l'+1}\sum_{vu}(jj'\ -\mu\mu'|u\ \mu'-\mu)$$
$$\times[(v+1)^{\frac{1}{2}}W(1\,1\,u\,v,1\,v+1)\mathfrak{Y}_{1\ v+1\ u}{}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})D_+-v^{\frac{1}{2}}W(1\,1\,u\,v,1\,v-1)\mathfrak{Y}_{1\ v-1\ u}{}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})D_-]$$
$$\times[f_\kappa G_{\kappa'}S_{1vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{1vu}(\kappa,-\kappa')]. \quad (33)$$

$$(\kappa\mu|\gamma_5|\kappa'\mu')\boldsymbol{\sigma}\cdot\mathbf{p}=(-)^{\mu+j'+l'}\sum_{vu}(jj'\ -\mu\mu'|u\ \mu'-\mu)$$
$$\times i\mathfrak{Y}_{0vu}{}^{\mu'-\mu}(\hat{r})[f_\kappa G_{\kappa'}S_{0vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{0vu}(\kappa,-\kappa')]\boldsymbol{\sigma}\cdot\mathbf{p}\delta_{vu}. \quad (34)$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}\left(\kappa\mu\left|\begin{matrix}\gamma_5\\ \beta\gamma_5\end{matrix}\right|\kappa'\mu'\right)=3^{-\frac{1}{2}}(-)^{\mu+j'+l'+1}\sum_{vu}(jj'\ -\mu\mu'|u\ \mu'-\mu)$$
$$\times\{[(v+1)/(2v+1)]^{\frac{1}{2}}\mathfrak{Y}_{1\ v+1\ u}{}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})D_+-[v/(2v+1)]^{\frac{1}{2}}\mathfrak{Y}_{1\ v-1\ u}{}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})D_-\}$$
$$\times[f_\kappa G_{\kappa'}S_{0vu}(-\kappa,\kappa')\pm g_\kappa F_{\kappa'}S_{0vu}(\kappa,-\kappa')]\delta_{vu}. \quad (35)$$

As is well known, the transition rate for the $C_i'^*C_j'$ term is exactly the same as that for the $C_i^*C_j$ term. Therefore, we shall not give here the spherical components for C_i' terms.

5. TRANSITION RATE

To illustrate the method of calculation, let us consider the matrix element of the reaction, $\mu^-+p\rightarrow\nu+n$, induced by the main even part of the axial vector interaction. By Eqs. (11), (13), (19), and (26),

$$\text{M.E.}=C_A\int(u_p{}^\dagger\boldsymbol{\sigma}u_n)(\psi_\nu{}^\dagger\boldsymbol{\sigma}\psi_\mu)d\mathbf{r}$$
$$=C_A\sum_{\kappa\mu}(u_p{}^\dagger\boldsymbol{\sigma}u_n)i^{-l}(l\,\tfrac{1}{2}\,\mu-m\ m|j\ \mu)Y_l{}^{\mu-m*}(\hat{q})\ (\kappa\mu|\boldsymbol{\sigma}|\kappa'\mu')$$
$$=C_A\sum_{\kappa\mu}\sum_{vu}i^{-l}(l\,\tfrac{1}{2}\,\mu-m\ m|j\ \mu)Y_l{}^{\mu-m*}(\hat{q})(-)^{\mu+j'+l'+1}(jj'\ -\mu\ \mu'|u\ \mu'-\mu)$$
$$\times\int u_p{}^\dagger\mathfrak{Y}_{1vu}{}^{\mu'-\mu}(\hat{r},\boldsymbol{\sigma})[g_\kappa G_{\kappa'}S_{1vu}(\kappa,\kappa')-f_\kappa F_{\kappa'}S_{1vu}(-\kappa,-\kappa')]u_nd\mathbf{r}. \quad (36)$$

The matrix element for the reaction, $\mu^-+(A,Z)\rightarrow\nu+(A,Z-1)$, is obtained from the above equation assuming that every dressed nucleon in a nucleus does not perturb the other. We introduce the subscript s labeling the nucleons and the operator $\tau_-{}^s$ which transform the sth nucleon from proton to neutron state. The wave functions of the neutron and proton in Eq. (36) are replaced by those of the initial and final nucleus, whose total and magnetic quantum numbers are represented by J_iM_i and J_fM_f. Thus

$$\text{M.E.}=C_A\int U_{J_f}{}^{M_f\dagger}\sum_{s=1}^{A}\boldsymbol{\sigma}_s\tau_-{}^s\psi_\nu{}^\dagger\boldsymbol{\sigma}\psi_\mu U_{J_i}{}^{M_i}d\mathbf{r}_1d\mathbf{r}_2\cdots d\mathbf{r}_A. \quad (37)$$

[20] The subscripts of D are inverted compared with those in Eq. (47) of reference 13, because our D_+ (D_-) increases (decreases) the order of the spherical Bessel function by one unit.

TABLE I. $C^{(i)}$ and $\mathfrak{M}_{vu}{}^{(i)}$ in Eq. (40). $\mathfrak{M}_{vu}{}^{(i)}$ is defined by

$$\int U_{J_f}{}^{M_f\dagger}\sum_{s=1}^{A}\Xi_s\tau_-{}^sU_{J_i}{}^{M_i}d\mathbf{r}_1d\mathbf{r}_2\cdots d\mathbf{r}_A=(J_i\,u\,M_i\,M_f-M_i|J_f\,M_f)\mathfrak{M}_{vu}{}^{(i)},$$

where Ξ_s is given in the table. In the last line, + and − signs refer to $i=7$ and 8, respectively.

i	$C^{(i)}$	Ξ_s
1	C_V	$\mathfrak{Y}_{0vu}{}^{M_f-M_i}(\hat{r}_s)[g_\kappa G_{\kappa'}S_{0vu}(\kappa,\kappa')-f_\kappa F_{\kappa'}S_{0vu}(-\kappa,-\kappa')]\delta_{vu}$
2	$-C_A$	$\mathfrak{Y}_{1vu}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)[g_\kappa G_{\kappa'}S_{1vu}(\kappa,\kappa')-f_\kappa F_{\kappa'}S_{1vu}(-\kappa,-\kappa')]$
3	$-C_V/M$	$i[f_\kappa G_{\kappa'}S_{1vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{1vu}(\kappa,-\kappa')]\mathfrak{Y}_{1vu}{}^{M_f-M_i}(\hat{r}_s,\mathbf{p}_s)$
4	$-\sqrt{3}C_V/2M$	$\{[(v+1)/(2v+3)]^{\frac12}\mathfrak{Y}_{0\,v+1\,u}{}^{M_f-M_i}(\hat{r}_s)\delta_{v+1\,u}D_+-[v/(2v-1)]^{\frac12}\mathfrak{Y}_{0\,v-1\,u}{}^{M_f-M_i}(\hat{r}_s)\delta_{v-1\,u}D_-\}$ $\times[f_\kappa G_{\kappa'}S_{1vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{1vu}(\kappa,-\kappa')]$
5	$-(\frac{3}{2})^{\frac12}C_V(1+\mu_p-\mu_n)/M$	$[(v+1)^{\frac12}W(1\,1\,u\,v,1v+1)\mathfrak{Y}_{1\,v+1\,u}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)D_+-v^{\frac12}W(1\,1\,u\,v,1v-1)\mathfrak{Y}_{1\,v-1\,u}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)D_-]$ $\times[f_\kappa G_{\kappa'}S_{1vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{1vu}(\kappa,-\kappa')]$
6	C_A/M	$i\mathfrak{Y}_{0vu}{}^{M_f-M_i}(\hat{r}_s)[f_\kappa G_{\kappa'}S_{0vu}(-\kappa,\kappa')+g_\kappa F_{\kappa'}S_{0vu}(\kappa,-\kappa')]\boldsymbol{\sigma}_s\cdot\mathbf{p}_s$
7 8	$-C_A/2\sqrt{3}M$ $C_P/2\sqrt{3}M$	$\{[(v+1)/(2v+1)]^{\frac12}\mathfrak{Y}_{1\,v+1\,u}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)D_+-[v/(2v+1)]^{\frac12}\mathfrak{Y}_{1\,v-1\,u}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)D_-\}$ $\times[f_\kappa G_{\kappa'}S_{0vu}(-\kappa,\kappa')\pm g_\kappa F_{\kappa'}S_{0vu}(\kappa,-\kappa')]\delta_{vu}$

Furthermore, we define the reduced nuclear matrix element by using the Wigner-Eckert theory,[10]

$$\int U_{J_f}{}^{M_f\dagger}\sum_{s=1}^{A}\{\mathfrak{Y}_{1vu}{}^{\mu'-\mu}(\hat{r}_s,\boldsymbol{\sigma}_s)[g_\kappa G_{\kappa'}S_{1vu}(\kappa,\kappa')-f_\kappa F_{\kappa'}S_{1vu}(-\kappa,-\kappa')]\}\tau_-{}^sU_{J_i}{}^{M_i}d\mathbf{r}_1d\mathbf{r}_2\cdots d\mathbf{r}_A \equiv(J_iu\,M_i\,\mu'-\mu|J_f\,M_f)\mathfrak{M}_{vu}. \quad (38)$$

The reduced nuclear matrix element $\mathfrak{M}_{vu}$ does not depend on the magnetic quantum number and it is, in principle, calculable if one knows the nuclear wave functions of the initial and final states, by the defining Eq. (38).

$$\text{M.E.}=\sum_{\kappa\mu}\sum_{vu}i^{-l}(l\,\tfrac{1}{2}\,\mu-m\,m|j\,m)Y_l{}^{\mu-m*}(\hat{q})(-)^{\mu+j'+l'+1}(jj'\,-\mu\,\mu'|u\,\mu'-\mu)(J_iuM_i\,\mu'-\mu|J_fM_f)C_A\mathfrak{M}_{vu}. \quad (39)$$

Generalizing the argument to the Hamiltonian density of Eq. (11), the matrix element is obtained by replacing $C_A\mathfrak{M}_{vu}$ by

$$\sum_i C^{(i)}\mathfrak{M}_{vu}{}^{(i)}.$$

Here $C^{(i)}$ and $\mathfrak{M}_{vu}{}^{(i)}$ are given in Table I. The superscript (i) specifies the individual term in Eq. (11). We now sum over all magnetic quantum numbers, integrate over $\mathbf{q}$, and average over the initial substates in the absolute square of the matrix element. The result is

$$\langle|\text{M.E.}|^2\rangle_{\text{av}}$$

$$=[(2j'+1)(2J_i+1)]^{-1}\int d\Omega_q\sum_{m\mu'}\sum_{M_iM_f}|\text{M.E.}|^2$$

$$=[(2J_f+1)/(2j'+1)(2J_i+1)]$$

$$\times\sum_{ij}\sum_{\kappa}\sum_{u}C^{(i)*}C^{(j)}[\sum_{v}\mathfrak{M}_{vu}{}^{(i)*}][\sum_{v'}\mathfrak{M}_{v'u}{}^{(j)}]. \quad (40)$$

Since the subscript u of the nuclear matrix element indicates the rank of the tensor, we have no cross term between two $\mathfrak{M}$'s with different u. The orthonormality of the $\chi_{\kappa\mu}$ brings the sum over κ, which specifies j and l of the neutrino simultaneously, outside of the absolute square. On the other hand, there are some cross terms among $\mathfrak{M}$'s with different v. By the definition of $\mathfrak{M}_{vu}{}^{(i)}$, the parity changes of the nuclear states is $(-)^v$ for $i=1$, 2 and $(-)^{v+1}$ for $i=3, 4, \cdots, 8$. Consequently, two matrix elements interfere with each other, if $v-v'=$ even and i and i' are in the same group, or $v-v'=$ odd and i and i' are in different groups.

The transition rate of the muon capture reaction from a particular nuclear state, J_i, to another one, J_f is given by

$$\mathcal{W}=2\pi\langle|\text{M.E.}|^2\rangle_{\text{av}}q^2dq/dE_f, \quad (41)$$

with

$$dq/dE_f=[1-q(m_\mu+AM)^{-1}], \quad (42)$$

and

$$q=(m_\mu-W_0)[1-\tfrac{1}{2}m_\mu(m_\mu+AM)^{-1}]. \quad (43)$$

Here W_0 is the energy difference between the initial and final nuclear states plus one electron mass.

In this section, we did not specify any particular value of κ' for the bound muon. This means that our formula applies to the capture rate of the muon from any orbit of the μ-mesonic atom, or to the capture rate from two or more orbits, if necessary. However, the muon in the higher orbits goes down to the K orbit ($\kappa'=-1$) by electromagnetic transitions before being captured by the nucleus. Consequently, the muon capture takes place from the K orbit. The general results in Secs. 4 and 5 hold with more realistic muon wave functions than Eq. (20) or (21). We can take the finite size and the relativistic effects on the muon wave function into account, with modified G_{-1} and F_{-1} for a real nucleus.[21]

[21] These effects would be of order of αZ, which is 4.4% in C^{12}. After this work was completed, a paper by Flamend and Ford, Phys. Rev. 116, 159 (1959), has been called to our attention. They showed, in fact, these effects for muon capture in C^{12} to be −6% correction in the capture rate.

6. MUON CAPTURE RATE FROM K ORBIT

The muon quantum number is $\kappa'=-1$ for the K orbit ($1s_{\frac{1}{2}}$). We assume the nuclear charge to be not so large, $\alpha Z\ll 1$. In this approximation, the radial wave functions of the muon are given by

$$G_{-1}=2(\alpha Zm_\mu')^{\frac{3}{2}}e^{-\alpha Zm_\mu' r} \quad\text{and}\quad F_{-1}=0. \tag{44}$$

Using the results in Secs. 4 and 5, the muon capture rate is given by

$$\mathcal{W}=4P(\alpha Zm_\mu')^3[(2J_f+1)/(2J_i+1)][1-q(m_\mu+AM)^{-1}]q^2, \tag{45}$$

with

$$P=\sum_\kappa\sum_u|C_V\mathfrak{M}[0\,l\,u]S_{0u}(\kappa)\delta_{lu}-C_A\mathfrak{M}[1\,l\,u]S_{1u}(\kappa)-(C_A/M)\mathfrak{M}[1\,\bar{l}\,u\,p]S_{1u}'(-\kappa)$$
$$+3^{\frac{1}{2}}(C_Vq/2M)\{[(\bar{l}+1)/(2\bar{l}+3)]^{\frac{1}{2}}\mathfrak{M}[0\,\bar{l}+1\,u\,+]\delta_{\bar{l}+1\,u}+[\bar{l}/(2\bar{l}-1)]^{\frac{1}{2}}\mathfrak{M}[0\,\bar{l}-1\,u\,-]\delta_{\bar{l}-1\,u}\}S_{1u}'(-\kappa)$$
$$+(\tfrac{3}{2})^{\frac{1}{2}}(C_Vq/M)(1+\mu_p-\mu_n)\{(\bar{l}+1)^{\frac{1}{2}}W(1\,1\,u\,\bar{l},1\,\bar{l}+1)\mathfrak{M}[1\,\bar{l}+1\,u\,+]+\bar{l}^{\frac{1}{2}}W(1\,1\,u\,\bar{l},1\,\bar{l}-1)\mathfrak{M}[1\,\bar{l}-1\,u\,-]\}$$
$$\times S_{1u}'(-\kappa)+(C_A/M)\mathfrak{M}[0\,\bar{l}\,u\,p]S_{0u}'(-\kappa)\delta_{\bar{l}u}+(\tfrac{1}{3})^{\frac{1}{2}}(C_A-C_P)(q/2M)$$
$$\times\{[(\bar{l}+1)/(2\bar{l}+1)]^{\frac{1}{2}}\mathfrak{M}[1\,\bar{l}+1\,u\,+]+[\bar{l}/(2\bar{l}+1)]^{\frac{1}{2}}\mathfrak{M}[1\,\bar{l}-1\,u\,-]\}S_{0u}'(-\kappa)\delta_{\bar{l}u}|^2. \tag{46}$$

Here we use the following abbreviations:

$$S_{ku}(\kappa)=S_{kvu}(\kappa,-1)$$
$$=[2(2j+1)]^{\frac{1}{2}}W(\tfrac{1}{2}\,1\,j\,l,\tfrac{1}{2}u)\delta_{lv}\quad\text{for}\quad k=1 \tag{47}$$
$$=[(2j+1)/(2l+1)]^{\frac{1}{2}}\delta_{lv}\quad\text{for}\quad k=0, \tag{48}$$

and

$$S_{ku}'(-\kappa)=S_\kappa S_{ku}(-\kappa),\quad S_\kappa=\text{sign of }\kappa. \tag{49}$$

The reduced nuclear matrix elements are listed in Table II.

In the expression for P, the summation over v disappear, because the muon has zero orbital angular momentum $l'=0$, and consequently $v=l$ or $v=\bar{l}$. Summation over u for a fixed κ is necessary only for the second, the third, and the fifth terms in Eq. (46). In fact, this summation consists of only two terms because of the triangular condition among the arguments of the Racah coefficient in $S_{ku}(\kappa)$ and $S_{ku}'(-\kappa)$. Although Eq. (46) looks complicated, the selection rule on the spin and parity reduces the number of the nonvanishing reduced nuclear matrix elements into manageable size. This will be done for different forbiddenness in the next section.

TABLE II. Definition of reduced nuclear matrix elements in the muon capture reaction,

$$\int U_{J_f}{}^{M_f\dagger}\sum_{s=1}^{A}e^{-\alpha Zm_\mu' r_s}\Phi_s\tau_-{}^sU_{J_i}{}^{M_i}d\mathbf{r}_1d\mathbf{r}_2\cdots d\mathbf{r}_A$$
$$\equiv(\mathfrak{M}[k\,w\,u]\quad\text{or}\quad\mathfrak{M}[k\,w\,u\,{}^{\pm}_{p}])(J_i\,u\,M_i\,M_f-M_i|J_f\,M_f),$$

where Φ_s is given in the table.

$\mathfrak{M}[\ ]$	Φ_s
$\mathfrak{M}[0\,w\,u]$	$j_w(qr_s)\mathfrak{Y}_{0wu}{}^{M_f-M_i}(\hat{r}_s)\delta_{wu}$
$\mathfrak{M}[1\,w\,u]$	$j_w(qr_s)\mathfrak{Y}_{1wu}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)$
$\mathfrak{M}[0\,w\,u\,\pm]$	$[j_w(qr_s)\pm\alpha Z(m_\mu'/p_\nu)j_{w\mp1}(qr_s)]\mathfrak{Y}_{0wu}{}^{M_f-M_i}(\hat{r}_s)\delta_{wu}$
$\mathfrak{M}[1\,w\,u\,\pm]$	$[j_w(qr_s)\pm\alpha Z(m_\mu'/p_\nu)j_{w\mp1}(qr_s)]\mathfrak{Y}_{1wu}{}^{M_f-M_i}(\hat{r}_s,\boldsymbol{\sigma}_s)$
$\mathfrak{M}[0\,w\,u\,p]$	$ij_w(qr_s)\mathfrak{Y}_{0wu}(\hat{r}_s)\boldsymbol{\sigma}_s\cdot\mathbf{p}_s\delta_{wu}$
$\mathfrak{M}[1\,w\,u\,p]$	$ij_w(qr_s)\mathfrak{Y}_{1wu}(\hat{r}_s,\mathbf{p}_s)$

7. ALLOWED AND FORBIDDEN TRANSITIONS IN MUON CAPTURE REACTION

In analogy to the theory of beta decay,[11,12,22] we define the forbiddenness of muon capture reaction in the following and discuss the connection with the forbiddenness of beta decay.

The reduced nuclear matrix elements are specified by three numbers, $k\,w\,u$, and an additional symbol, $+$ or $-$ or p. Their explicit forms are given in Table II. We list all possible reduced nuclear matrix elements according to forbiddenness of the muon capture reaction in Table III. The definition of the forbiddenness is exactly the same as that in the theory of beta decay. It is given in the second and third rows of Table III as parity and spin changes between the initial and final nuclear levels. On the other hand, the classification of the reduced nuclear matrix elements with respect to forbiddenness is somewhat different from that in the theory of beta decay,[11,12,22] and the number of the nuclear matrix elements belonging to a particular forbiddenness is considerably large. It is 9, 16, and 14 for the allowed, first forbidden and other transitions, respectively. The selection rule for the nuclear spin of the matrix element is $J_i+J_f\geq u\geq|J_i-J_f|$ because the third number u represents the rank of the tensor. The parity change is given by the second number w, namely $(-)^w$ and $(-)^{w+1}$ for $\mathfrak{M}[k\,w\,u]$ (or $\mathfrak{M}[k\,w\,u\,\pm]$) and $\mathfrak{M}[k\,w\,u\,p]$, respectively, because the w represents the rank of the spherical harmonics with the argument $\hat{r}$ and the additional minus parity is required by the operator $\mathbf{p}$. w is equal to v for $\mathfrak{M}[k\,w\,u]$ or $\mathfrak{M}[k\,w\,u\,p]$ and $v\mp1$ for $\mathfrak{M}[k\,w\,u\,\pm]$. Here v is the orbital angular momentum of the neutrino. k represents the resultant spin of the neutrino-muon system.

The above classification of the reduced nuclear matrix elements is based on the tensorial rank of the nuclear matrix elements and the property of the neutrino wave

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[22] M. Morita, Phys. Rev. **113**, 1584 (1959); Nuclear Phys. **14**, 106 (1959).

TABLE III. Reduced nuclear matrix elements in the allowed and forbidden transitions of the muon capture reaction. They are classified with three numbers, k, w, u, and additional symbols $\pm$ or p in each forbiddenness. Here u is the rank of the tensor. For a capture from K orbit $\mathfrak{M}[1\ u+1\ u\ -]$ and $\mathfrak{M}[1\ u-1\ u\ +]$ will not appear, because of the selection rule. In the second row, the plus and minus indicate no parity change and parity change, respectively.

Forbiddenness	Allowed	First	Second	Third	Fourth	nth
Parity change	+	−	+	−	+	$(-)^n$
Spin change $\lvert J_f-J_i\rvert$	0, 1	0, 1, 2	2, 3	3, 4	4, 5	n, $(n+1)$
Matrix elements						
$\mathfrak{M}[0\ u\ u]$, $\mathfrak{M}[0\ u\ u\ \pm]$	0 0 0[a]	0 1 1	0 2 2	0 3 3	0 4 4	$0\ n\ n$
$\mathfrak{M}[1\ u+1\ u]$, $\mathfrak{M}[1\ u+1\ u\ +]$	1 2 1	1 1 0[a] 1 3 2	1 4 3	1 5 4	1 6 5	$1(n+2)(n+1)$
$\mathfrak{M}[1\ u\ u]$, $\mathfrak{M}[1\ u\ u\ \pm]$		1 1 1	1 2 2	1 3 3	1 4 4	$1\ n\ n$
$\mathfrak{M}[1\ u-1\ u]$, $\mathfrak{M}[1\ u-1\ u\ -]$	1 0 1	1 1 2	1 2 3	1 3 4	1 4 5	$1\ n(n+1)$
$\mathfrak{M}[0\ u\ u\ p]$	0 1 1	0 0 0 0 2 2	0 3 3	0 4 4	0 5 5	$0(n+1)(n+1)$
$\mathfrak{M}[1\ u+1\ u\ p]$	1 1 0	1 2 1	1 3 2	1 4 3	1 5 4	$1(n+1)\ n$
$\mathfrak{M}[1\ u\ u\ p]$	1 1 1	1 2 2	1 3 3	1 4 4	1 5 5	$1(n+1)(n+1)$
$\mathfrak{M}[1\ u-1\ u\ p]$		1 0 1	1 1 2	1 2 3	1 3 4	$1(n-1)\ n$
No. of matrix elements	9	16	14	14	14	14

[a] There are no $\mathfrak{M}[0\ 0\ 0\ +]$ and $\mathfrak{M}[1\ 1\ 0\ +]$.

function involved as an integrand, namely the spherical Bessel function of the order w. The argument of the spherical Bessel function $j_w(qr)$ is roughly 1 or 2 at the nuclear radius for light or medium nuclei. Consulting the values of $j_w(qr)$'s in the numerical tables of spherical Bessel functions[23] in the range of $qr=0-2$, we conclude that

$$j_{w+2}(qr)\lesssim(10^{-1}\sim10^{-2})\times j_w(qr). \quad (50)$$

This leads to

$$\mathfrak{M}[k\ w+2\ u']\lesssim3\times10^{-2}\mathfrak{M}[k\ w\ u],$$
$$\mathfrak{M}[k\ w+2\ u'\ \pm]\lesssim3\times10^{-2}\mathfrak{M}[k\ w\ u\ \pm],$$
$$\mathfrak{M}[k\ w+2\ u'\ p]\lesssim3\times10^{-2}\mathfrak{M}[k\ w\ u\ p]. \quad (51)$$

Here u' may or may not be equal to u. The above approximation may not hold in heavy nuclei because of the large nuclear radius. Also the atomic number of the nucleus becomes large, and consequently, the neglect of F_{-1} is not justified. Thus, the theory will be more complicated, as seen in Sec. 5. For the nuclear matrix elements involving p, $\mathfrak{M}[k\ w\ u]$, which represents the relativistic correction, we have

$$(1/M)\mathfrak{M}[k\ w\ u\ p]\sim(p/M)\mathfrak{M}[k\ w\ u], \quad (52)$$

with $p/M\sim v/c\sim10^{-1}$, where the p and v are the nucleon momentum and its velocity in the nucleus. Taking these relative magnitudes of the nuclear matrix elements into account, we obtain the classification shown in Table III. Another relation for the nuclear matrix elements is

$$\mathfrak{M}[k\ w\ u-]\approx\mathfrak{M}[k\ w\ u], \quad (53)$$

because the second term in the integrand is very small compared with the first (see Table II). But $\mathfrak{M}[k\ w\ u+]$ is different from $\mathfrak{M}[k\ w\ u]$ because the two terms in the integrand (see also Table II) have almost the same magnitude.

[23] E.g., see National Bureau of Standards. *Tables of Spherical Bessel Functions* (Columbia University Press, New York, 1947).

Since we have no cross term between two nuclear matrix elements with different u's, the contribution of the $(n+2)$th forbiddenness to the nth forbiddenness is very small and of the order of 0.1% or less. This means that the daughter nucleus of the muon capture reaction goes mostly to states which have $|J_i-J_f|=0$, 1 without parity change, or $|J_i-J_f|=0$, 1, 2 with parity change, in the low Z region.

So far we have assumed K orbit capture and neglected the small component of the muon wave function F_{-1}. When more exact muon wave function is used, the above classification into forbiddenness still holds. In the theory of orbital electron capture or beta decay, the quantum number κ' for the initial lepton state is not restricted to -1 only. In the conventional theories of beta decay[11,12] and orbital capture,[24,25] all quantities are expanded in powers of $(\mathbf{q}+\mathbf{p}_e)\cdot\mathbf{r}$, which is of the order of 0.01. Therefore, the lowest term in this expansion is always dominant and the higher terms are negligible. However, if $(\mathbf{q}+\mathbf{p}_e)\cdot\mathbf{r}$ becomes of the order of 0.1, e.g., in the case of beta decays of B^{12} and N^{12}, we must take higher order terms, too.[8] This has been done by one of the authors.[22] The present formalism of muon capture has a close analogy to the theory of beta decay with higher order corrections, but the difference in the classification of the reduced nuclear matrix elements is shown in Table IV. As is seen there, the classification of nuclear matrix elements in muon capture is made by the third number u (the tensorial rank), in beta decay theories by the second number w.

8. EXPLICIT FORMULAS FOR THE MUON CAPTURE RATE

According to the definition discussed in the last section, the muon capture rate will be given for each

[24] S. R. de Groot and H. A. Tolhoek, Physica **16**, 456 (1950).
[25] H. Brysk and M. E. Rose, Oak Ridge National Laboratory Report ORNL-1830, 1955 (unpublished).

TABLE IV. Comparison of the theory of muon capture and that of beta decay or orbital electron capture. The reduced nuclear matrix elements are classified according to forbiddenness in each theory. In the table, "the nth" means the nth forbidden transition, etc. † means that the relevant nuclear matrix element is neglected, being regarded as a small quantity. Two exceptions are mentioned at the bottom of the table.

Matrix elements	Theory of muon capture	Theory of beta decay or orbital electron capture: With higher order correction[b]	Theory of beta decay or orbital electron capture: Without higher order correction
$\mathfrak{M}[0\ n\ n]$, $\mathfrak{M}[0\ n\ n\ \pm]$	the nth	the nth	the nth
$\mathfrak{M}[1\ n\ n-1]$, $\mathfrak{M}[1\ n\ n-1\ +]$	the $(n-2)$th[a]	the nth	†[c]
$\mathfrak{M}[1\ n\ n]$, $\mathfrak{M}[1\ n\ n\ \pm]$	the nth	the nth	the nth
$\mathfrak{M}[1\ n\ n+1]$, $\mathfrak{M}[1\ n\ n+1\ -]$	the nth	the nth	the nth
$\mathfrak{M}[0\ n\ n\ p]$,	the $(n-1)$th[a]	the $(n+1)$th	†[c]
$\mathfrak{M}[1\ n\ n-1\ p]$	the $(n-1)$th	the $(n+1)$th	†
$\mathfrak{M}[1\ n\ n\ p]$	the $(n-1)$th	the $(n+1)$th	†
$\mathfrak{M}[1\ n\ n+1\ p]$	the $(n+1)$th	the $(n+1)$th	the $(n+1)$th

[a] $\mathfrak{M}[1\ 1\ 0]$ and $\mathfrak{M}[0\ 0\ 0\ p]$ are in the first forbidden transition.
[b] See reference 23.
[c] $\mathfrak{M}[1\ 1\ 0]$ and $\mathfrak{M}[0\ 0\ 0\ p]$ are in the first forbidden transition and included.

forbiddenness. In the calculation of Eq. (46), we neglect all terms of order $(p/2M)^2$ except for C_P^2, which would be large compared with the other coupling constants. For $J_i \neq 0$ the result depends on the assumption that the hyperfine states of the μ-mesonic atom are statistically populated.[26] This effect of the hyperfine splitting is omitted. We also adopt the muon wave function, Eq. (44), for the point nucleus in the approximation $\alpha Z \ll 1$, which neglects the small component F_{-1}.[21] We abbreviate $\mathfrak{M}[k\ w\ u]$ by $[k\ w\ u]$ etc., to save space. This should lead to no confusion. These nuclear matrix elements can be evaluated from their definitions in Table III. Examples are shown for muon capture by C^{12} in the Appendix.

The muon capture rate is given by

$$\mathcal{W} = 8P_0(\alpha Z m_\mu')^3[(2J_f+1)/(2J_i+1)] \times[1-q(m_\mu+AM)^{-1}]q^2, \quad (54)$$

with

$$2P_0 = P, \quad \text{and} \quad \hbar = c = m_e = 1. \quad (55)$$

Here m_μ and m_μ' are the muon mass and its reduced mass in the parent μ-mesonic atom, respectively. The neutrino momentum q is given by Eq. (43). For a given order of forbiddenness, only those reduced nuclear matrix elements for which $J_i+J_f \geq u \geq |J_i-J_f|$ actually should be included. The P_0 for each forbiddenness is:

I. Allowed Transition

$$\begin{aligned} P_0 = {} & C_V^2[0\,0\,0]^2+\tfrac{1}{3}C_A^2([1\,0\,1]^2+[1\,2\,1]^2)+C_V^2q\,M^{-1}[0\,0\,0][0\,0\,0\ -]-\tfrac{1}{9}C_AC_VqM^{-1}(1+\mu_p-\mu_n) \\ & \times(2^{\frac{1}{2}}[1\,0\,1]-[1\,2\,1])(2^{\frac{1}{2}}[1\,0\,1\ -]-[1\,2\,1\ +])+\tfrac{1}{9}C_A(C_A-C_P)qM^{-1}([1\,0\,1]+2^{\frac{1}{2}}[1\,2\,1]) \\ & \times([1\,0\,1\ -]+2^{\frac{1}{2}}[1\,2\,1\ +])-(\tfrac{1}{3})^{\frac{1}{2}}2C_V^2M^{-1}[0\,0\,0][1\,1\,0\ p]-(\tfrac{1}{3})^{\frac{1}{2}}\tfrac{2}{3}C_AC_VM^{-1}(2^{\frac{1}{2}}[1\,0\,1]-[1\,2\,1])[1\,1\,1\ p] \\ & +\tfrac{2}{3}C_A^2M^{-1}([1\,0\,1]+2^{\frac{1}{2}}[1\,2\,1])[0\,1\,1\ p]+(C_Pq/6M)^2([1\,0\,1\ -]+2^{\frac{1}{2}}[1\,2\,1\ +])^2. \end{aligned} \quad (56)$$

II. First Forbidden Transition

$$\begin{aligned} P_0 = {} & C_V^2[0\,1\,1]^2+\tfrac{1}{3}C_A^2([1\,1\,0]^2+[1\,1\,1]^2+[1\,1\,2]^2+[1\,3\,2]^2)+\tfrac{1}{3}C_V^2qM^{-1}[0\,1\,1](2[0\,1\,1\ -]+[0\,1\,1\ +]) \\ & +\tfrac{1}{3}(\tfrac{2}{3})^{\frac{1}{2}}C_V^2qM^{-1}(1+\mu_p-\mu_n)[0\,1\,1](-[1\,1\,1\ -]+[1\,1\,1\ +])+\tfrac{1}{3}(\tfrac{2}{3})^{\frac{1}{2}}C_AC_VqM^{-1}[1\,1\,1] \\ & \times([0\,1\,1\ -]-[0\,1\,1\ +])-\tfrac{1}{3}C_AC_VqM^{-1}(1+\mu_p-\mu_n)\{\tfrac{1}{3}[1\,1\,1]([1\,1\,1\ -]+2[1\,1\,1\ +]) \\ & +\tfrac{1}{5}(3^{\frac{1}{2}}[1\,1\,2]-2^{\frac{1}{2}}[1\,3\,2])(3^{\frac{1}{2}}[1\,1\,2\ -]-2^{\frac{1}{2}}[1\,3\,2\ +])\}+(1/15)C_A(C_A-C_P)qM^{-1}\{(2^{\frac{1}{2}}[1\,1\,2]+3^{\frac{1}{2}}[1\,3\,2]) \\ & \times(2^{\frac{1}{2}}[1\,1\,2\ -]+3^{\frac{1}{2}}[1\,3\,2\ +])+5[1\,1\,0][1\,1\,0\ +]\}-\tfrac{2}{3}C_V^2M^{-1}[0\,1\,1]([1\,0\,1\ p]+2^{\frac{1}{2}}[1\,2\,1\ p]) \\ & +\tfrac{2}{3}C_AC_VM^{-1}\{(\tfrac{1}{3})^{\frac{1}{2}}[1\,1\,1](2^{\frac{1}{2}}[1\,0\,1\ p]-[1\,2\,1\ p])+(\tfrac{1}{5})^{\frac{1}{2}}(-3^{\frac{1}{2}}[1\,1\,2]+2^{\frac{1}{2}}[1\,3\,2])[1\,2\,2\ p]\} \\ & +(\tfrac{1}{3})^{\frac{1}{2}}2C_A^2M^{-1}\{[1\,1\,0][0\,0\,0\ p]+(\tfrac{1}{5})^{\frac{1}{2}}(2^{\frac{1}{2}}[1\,1\,2]+3^{\frac{1}{2}}[1\,3\,2])[0\,2\,2\ p]\} \\ & +(1/60)(C_Pq/M)^2\{5[1\,1\,0\ +]^2+(2^{\frac{1}{2}}[1\,1\,2\ -]+3^{\frac{1}{2}}[1\,3\,2\ +])^2\}. \end{aligned} \quad (57)$$

III. The nth Forbidden Transition

$$\begin{aligned} P_0 = {} & C_V^2[0\ n\ n]^2+\tfrac{1}{3}C_A^2([1\ n\ n]^2+[1\ n\ n+1]^2+[1\ n+2\ n+1]^2)+(2n+1)^{-1}C_V^2qM^{-1}[0\ n\ n] \\ & \times\{n[0\ n\ n\ +]+(n+1)[0\ n\ n\ -]\}+[n(n+1)/3]^{\frac{1}{2}}(2n+1)^{-1}C_V^2qM^{-1}(1+\mu_p-\mu_n)[0\ n\ n] \\ & \times(-[1\ n\ n\ -]+[1\ n\ n\ +])+[n(n+1)/3]^{\frac{1}{2}}(2n+1)^{-1}C_AC_VqM^{-1}[1\ n\ n]([0\ n\ n\ -]-[0\ n\ n\ +]) \\ & -\tfrac{1}{3}C_AC_VqM^{-1}(1+\mu_p-\mu_n)((2n+1)^{-1}[1\ n\ n]\{n[1\ n\ n\ -]+(n+1)[1\ n\ n\ +]\}+(2n+3)^{-1} \\ & \times\{(n+2)^{\frac{1}{2}}[1\ n\ n+1]-(n+1)^{\frac{1}{2}}[1\ n+2\ n+1]\}\{(n+2)^{\frac{1}{2}}[1\ n\ n+1\ -]-(n+1)^{\frac{1}{2}}[1\ n+2\ n+1\ +]\}) \\ & +\tfrac{1}{3}(2n+3)^{-1}C_A(C_A-C_P)qM^{-1}\{(n+1)^{\frac{1}{2}}[1\ n\ n+1]+(n+2)^{\frac{1}{2}}[1\ n+2\ n+1]\} \\ & \times\{(n+1)^{\frac{1}{2}}[1\ n\ n+1\ -]+(n+2)^{\frac{1}{2}}[1\ n+2\ n+1\ +]\}-2[3(2n+1)]^{-\frac{1}{2}}C_V^2M^{-1}[0\ n\ n] \\ & \times\{n^{\frac{1}{2}}[1\ n-1\ n\ p]+(n+1)^{\frac{1}{2}}[1\ n+1\ n\ p]\}+\tfrac{2}{3}C_AC_VM^{-1}((2n+3)^{-\frac{1}{2}}\{-(n+2)^{\frac{1}{2}}[1\ n\ n+1] \\ & +(n+1)^{\frac{1}{2}}[1\ n+2\ n+1]\}[1\ n+1\ n+1\ p]+(2n+1)^{-\frac{1}{2}}[1\ n\ n]\{(n+1)^{\frac{1}{2}}[1\ n-1\ n\ p]-n[1\ n+1\ n\ p]\}) \\ & +2[3(2n+3)]^{-\frac{1}{2}}C_A^2M^{-1}\{(n+1)^{\frac{1}{2}}[1\ n\ n+1]+(n+2)^{\frac{1}{2}}[1\ n+2\ n+1]\}[0\ n+1\ n+1\ p] \\ & +\tfrac{1}{12}(2n+3)^{-1}(C_Pq/M)^2\{(n+1)^{\frac{1}{2}}[1\ n\ n+1\ -]+(n+2)^{\frac{1}{2}}[1\ n+2\ n+1\ +]\}^2. \end{aligned} \quad (58)$$

[26] E. g., see, J. Bernstein, T. D. Lee, C. N. Yang, and H. Primakoff, Phys. Rev. 111, 313 (1958).

9. BETA DECAY AS THE INVERSE PROCESS OF MUON CAPTURE

The transition rate of beta decay from the state J_f to the state J_i, considered as the inverse nuclear process of the muon capture, can be expressed by

$$\mathcal{W}_\beta=(8/\pi)\int_1^{W_0} P_{0\beta}F(Z,W) \times(W_0-W)^2Wp_edW[(2J_i+1)/(2J_f+1)]. \quad (59)$$

Here p_e, W, and W_0 are the electron momentum, its energy and its maximum energy, respectively. $F(Z,W)$ is the Fermi function. $P_{0\beta}$ is given by a formula familiar to P_0 given in the last section. It is called the correction factor of the beta-ray spectrum, and can be expressed by the already published formulas as follows:

$$16\pi^2P_{0\beta}=\sum_L a_{LL}^{(0)} \quad \text{of references}^{27,28} \quad (60)$$

$$=\sum_L(-)^L(2L+1)^{-\frac{1}{2}}b_{LL}^{(0)} \quad \text{of references}^{29,30} \quad (61)$$

$$=C[(2J_f+1)/(2J_i+1)] \quad \text{of references}^{31,32}. \quad (62)$$

Here, it should be noticed that the reduced nuclear matrix element is defined in the transition $J_f\rightarrow J_i$ for beta decay, while $J_i\rightarrow J_f$ for muon capture.

The transition rates of muon capture and beta decay the relation,

$$\frac{\mathcal{W}}{\mathcal{W}_\beta}=\frac{\pi(\alpha Zm_\mu')^3P_0q^2[1-q(m_\mu+AM)^{-1}]}{\int_1^{W_0} P_{0\beta}F(Z,W)(W_0-W)^2Wp_edW}\left(\frac{2J_f+1}{2J_i+1}\right)^2. \quad (63)$$

For the allowed transition, we have

$$\mathcal{W}_\beta=(2\pi^3)^{-1}f(Z,W_0)[C_V^2\mathfrak{M}^2(1)+C_A^2\mathfrak{M}(\sigma)][(2J_i+1)/(2J_f+1)] \quad (64)$$

$$=(2\pi^3)^{-1}f(Z,W_0)\left[C_V^2\left(\int 1\right)^2+C_A^2\left(\int \sigma\right)^2\right] \quad (65)$$

$$=(8/\pi)f(Z,W_0)(C_V^2[0\,0\,0]_\beta^2+\tfrac{1}{3}C_A^2[1\,0\,1]_\beta^2)[(2J_i+1)/(2J_f+1)], \quad (66)$$

in the notation of references 27 to 30, of references 31 and 32, and of the present work, respectively. Therefore, the muon capture rate is given in terms of the beta decay transition rate as[33]

$$\mathcal{W}=\mathcal{W}_\beta\frac{\pi(\alpha Zm_\mu')^3P_0q^2[1-q(m_\mu+AM)^{-1}]}{f(Z,W_0)(C_V^2[0\,0\,0]_\beta^2+\tfrac{1}{3}C_A^2[1\,0\,1]_\beta^2)}\left(\frac{2J_f+1}{2J_i+1}\right)^2, \quad (67)$$

with

$$P_0=[\text{Eq. (56)}],$$

and

$$[k\,0\,k]_\beta^2[(2J_i+1)/(2J_f+1)]\equiv([k\,0\,k] \quad \text{with} \quad q=m_\mu'=0)$$

for both $k=0$ and 1. Here $f(Z,W_0)$ is the integrated Fermi function.

10. EXAMPLE: MUON CAPTURE RATE BY C^{12}

We calculate the muon capture rate in the transition from the ground state of C^{12} to that of B^{12}. Since the spin and parity changes are $0_+\rightarrow 1_+$, this is an allowed transition. The relevant nuclear matrix elements are [1 0 1], [1 0 1 −], [1 2 1], [1 2 1 +], [0 1 1 p], and [1 1 1 p]. The transition rate $\mathcal{W}$ is given in terms of the well known transition rate of the beta decay $B^{12}\rightarrow C^{12}$. The nuclear matrix elements are evaluated for the j-j coupling shell model with the harmonic oscillator wave functions. The detailed calculation of the reduced matrix elements is given in the appendix. The results are

$$[1\,0\,1]=-(\tfrac{1}{3})^{\frac{1}{2}}\pi^{-1}[(1-\tfrac{1}{6}\lambda)e^{-\lambda/4} -\zeta(1-\tfrac{1}{2}\lambda+\tfrac{1}{10}\lambda^2)]=-0.138, \quad (68)$$

$$[1\,0\,1\,-]=-(\tfrac{1}{3})^{\frac{1}{2}}\pi^{-1}[(1-\tfrac{1}{6}\lambda)e^{-\lambda/4} -\zeta\{\tfrac{4}{3}-\tfrac{3}{5}\lambda+(4/35)\lambda^2\}]=-0.135, \quad (69)$$

[27] Square terms of the interactions, see M. Yamada and M. Morita, Progr. Theoret. Phys. (Kyoto) **8**, 431 (1952).

[28] Cross terms among interactions, M. Morita, Progr. Theoret. Phys. (Kyoto) **10**, 363 (1953), and Y. Kato and M. Morita, Progr. Theoret. Phys. (Kyoto) **13**, 276 (1955) with errata, ibid. **14**, 174 (1955).

[29] M. Morita, Progr. Theoret. Phys. (Kyoto) **14**, 27 (1955) and **15**, 445 (1956).

[30] M. Morita and R. S. Morita, Phys. Rev. **109**, 2048 (1958).

[31] Square terms of the interactions, see E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941). There is a misprint, in C_{2V}, the second line of the page 315. $\frac{1}{3}K^3(2L_1+M_0)$ should be read as $\frac{1}{3}K^2(2L_1+M_0)$.

[32] Cross terms among interactions, see A. M. Smith, Phys. Rev. **82**, 955 (1951).

[33] This is apparently different from Eq. (10) of reference 7, in the power of the statistical weight of spins. This comes, however, from the difference of the definition in the reduced nuclear matrix elements. Taking this into account, our Eq. (63) for the allowed transition is equal to Eq. (10) of reference 7.

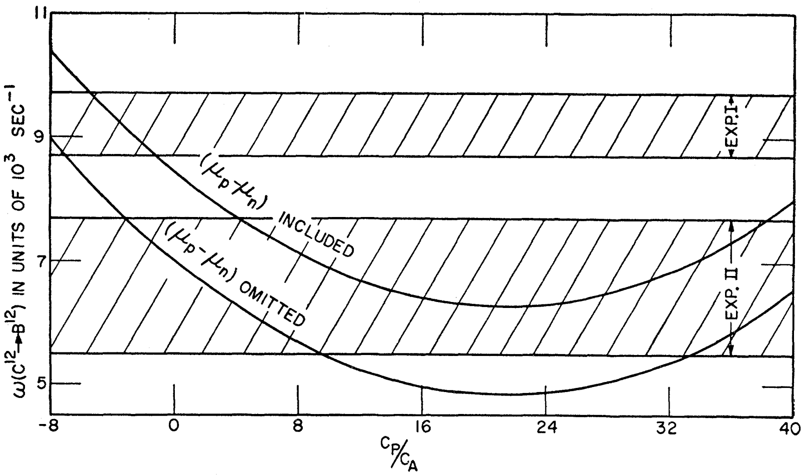

FIG. 1. Calculated transition rate for the muon capture reaction between the ground states of C^{12} and B^{12} versus C_P/C_A. C_P and C_A are the coupling constants of the induced pseudoscalar and axial vector interactions. In the figure, Exp. I and II refer to the experimental data b and c given in Table V.

$$[1\,2\,1]=(\tfrac{1}{6})^{\frac{1}{2}}(2\pi)^{-1}[\tfrac{1}{6}\lambda e^{-\lambda/4}-\zeta\{\tfrac{1}{5}\lambda-(2/35)\lambda^2\}]=0.00482, \quad (70)$$

$$[1\,2\,1\,+]=(\tfrac{1}{6})^{\frac{1}{2}}(2\pi)^{-1}[\tfrac{1}{6}\lambda e^{-\lambda/4}+\zeta\{\tfrac{1}{3}-\tfrac{3}{10}\lambda+(1/14)\lambda^2\}]=0.00584, \quad (71)$$

$$[0\,1\,1\,p]=(\tfrac{1}{3})^{\frac{1}{2}}(q/2\pi)[\tfrac{5}{6}(1-\tfrac{1}{10}\lambda)e^{-\lambda/4}-\zeta\{1-\tfrac{2}{5}\lambda+(1/14)\lambda^2\}]=0.00575M, \quad (72)$$

$$[1\,1\,1\,p]=(\tfrac{1}{2})^{\frac{1}{2}}(q/2\pi)[\tfrac{1}{3}e^{-\lambda/4}-\zeta\{\tfrac{1}{3}-\tfrac{1}{10}\lambda+(1/70)\lambda^2\}]=0.00301M, \quad (73)$$

with

$$\lambda=\tfrac{2}{5}\rho^2q^2=q^2/a,$$

and

$$\zeta=(8/3)\alpha Z m_\mu'(\pi a)^{-\frac{1}{2}}=(8/3)\alpha Z(m_\mu'/q)(\lambda/\pi)^{\frac{1}{2}}. \quad (74)$$

Here ρ is the root mean square value of the nuclear radius. We use the following experimental values[7]:

$$\begin{aligned} \rho&=2.52\times10^{-13}\text{ cm},\\ q&=91.4\text{ Mev}/c,\\ \mathcal{W}_\beta&=33.15\text{ sec}^{-1}\quad\text{for}\quad B^{12}\to C^{12},\\ f(Z,W_0)&=5.625\times10^5\quad\text{for}\quad B^{12}\to C^{12}. \end{aligned} \quad (75)$$

For the coupling constants, we assume[7]

$$\begin{aligned} C_A{}^\beta&=-1.21C_V{}^\beta,\\ C_V&=0.972C_V{}^\beta,\\ C_A&=0.999C_A{}^\beta. \end{aligned} \quad (76)$$

Here the coupling constants with the superscript β refer to beta decay.

Using Eqs. (56), (67), and (68)–(76), the muon capture rate by C^{12} is calculated for different assumptions regarding conserved vector current (see also Fig. 1):

I. $(\mu_p-\mu_n)$ term included.

$$\mathcal{W}=6.28+0.00481[(C_P/C_A)-21.2]^2. \quad (77)$$

II. $(\mu_p-\mu_n)$ term omitted.

$$\mathcal{W}=4.84+0.00481[(C_P/C_A)-21.2]^2. \quad (78)$$

Here $\mathcal{W}$ are given in units of 10^3 sec^{-1}.

Our calculation should coincide with that of Primakoff[6] and Fujii and Primakoff,[7] if we replace in Eq. (56):

$$\begin{aligned} &[1\,0\,1\,-]\quad\text{by}\quad[1\,0\,1],\\ &[1\,2\,1\,+]\quad\text{by}\quad[1\,2\,1],\\ &[0\,1\,1\,p]\text{ and }[1\,1\,1\,p]\quad\text{by zero},\\ &([1\,0\,1]\pm2^{\pm\frac{1}{2}}[1\,2\,1])^2\quad\text{by}\quad([1\,0\,1]^2+[1\,2\,1]^2). \end{aligned} \quad (79)$$

TABLE V. Theoretical and experimental transition rates for the muon capture reaction between the ground states of C^{12} and B^{12}. The last column gives the ratio of our value to that of Fujii and Primakoff.[a]

		$w(C^{12}\to B^{12})$ in 10^3 sec^{-1}		
C_P/C_A	$(\mu_p-\mu_n)$ terms	Present work	Fujii and Primakoff	Ratio
8	included	7.12	7.86	0.91
8	omitted	5.68	6.34	0.90
−8	included	10.39	11.80	0.88
−8	omitted	8.95	10.25	0.87
Experiments			9.05±0.95[b] 9.18±0.5[c] 6.6 ±1.1[d] 6.8 ±1.5[e] 5.9 ±1.5[f]	

[a] See references 6 and 7.
[b] F. B. Harrison, H. V. Argo, H. B. Kruse, and A. D. McGuire, Gatlinburg Conference on Weak Interactions, Gatlinburg, Tennessee, October, 1958 (unpublished), paper S4.
[c] J. O. Burgman, J. Fischer, E. Leontic, A. Lundby, R. Meunier, J. P. Stroot, and J. D. Teja, Phys. Rev. Letters 1, 469 (1958).
[d] J. G. Fetkovich, T. H. Fields, and R. L. McIlwain, Bull. Am. Phys. Soc. 4, 81 (1959).
[e] W. A. Love, S. Marder, I. Nadelhaft, R. T. Siegel, and A. E. Taylor, Bull. Am. Phys. Soc. 4, 81 (1959).
[f] T. N. K. Godfrey, Ph.D. thesis, Princeton University, 1954 (unpublished) and Phys. Rev. 92, 512 (1953).

This has also been checked in the numerical calculation.[34] Our values for the muon capture rate are 9–13% less than those of references 6 and 7 (see Table V).

As is seen in Fig. 1, the same transition rate often arises from more than one set of parameters. The inaccuracy involved in the nuclear wave functions brings a further complication.[35]

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APPENDIX. EVALUATION OF REDUCED NUCLEAR MATRIX ELEMENTS IN MUON CAPTURE BY C^{12}

By definition, Eq. (38), the reduced nuclear matrix elements, [1 l 1], etc., in the muon capture by C^{12} are

$$([1\,l\,1],\text{ etc.})(0\,1\,0\,M|1\,M)=\int U^\dagger(B^{12})\times\sum_{s=1}^{8}e^{-\alpha Zm_\mu' r_s}\Phi_s\tau_-{}^s U(C^{12})d\mathbf{r}_1d\mathbf{r}_2\cdots d\mathbf{r}_8. \quad (A1)$$

Here Φ_s is given in Table II. We assume the j–j coupling shell model and the harmonic oscillator wave functions. The four nucleons in the 1s shell are assumed to be inert during the ground state to ground state transition. As is given in reference 7, the nuclear wave function of C^{12} in the standard notation is

$$U_{0,0}{}^{0,0}(C^{12})=(\tfrac{1}{2})^{\frac{1}{2}}\sum_m\sum_{\tau_3=\pm\frac{1}{2}}(\tfrac{3}{2}\,\tfrac{3}{2}\,m\,-m|0\,0)\times 2\tau_3W_{\frac{3}{2},\frac{1}{2}}{}^{m,\tau_3}(2,3,\cdots,8)X_{\frac{3}{2}}{}^{-m}(1)\varphi_{\frac{1}{2}}{}^{-\tau_3}(1). \quad (A2)$$

The nuclear wave function of B^{12} is

$$U_{1,1}{}^{M,-1}(B^{12})=(\tfrac{1}{8})^{\frac{1}{2}}\sum_P\sum_m\epsilon_P(\tfrac{3}{2}\,\tfrac{1}{2}\,m\,M-m|1\,M)\times W_{\frac{3}{2},\frac{1}{2}}{}^{m,-\frac{1}{2}}(2,3,\cdots,8)X_{\frac{1}{2}}{}^{M-m}(1)\varphi_{\frac{1}{2}}{}^{-\frac{1}{2}}(1). \quad (A3)$$

Here P means the replacement of a member in $(2,3,\cdots,8)$ by 1. ϵ_P is the sign due to this permutation.

Inserting (A2) and (A3) into (A1) and integrating over the $3\times(8-1)$ dimensions, the result is

$$([1\,l\,1],\text{ etc.})=\sum_m(-)^{\frac{3}{2}-m}(\tfrac{3}{2}\,\tfrac{1}{2}\,m\,M-m|1\,M)\times\int X_{\frac{1}{2}}{}^{M-m\dagger}e^{-\alpha Zm_\mu' r}\Phi X_{\frac{3}{2}}{}^{-m}d\mathbf{r}. \quad (A4)$$

Here

$$X_j{}^m=\phi\sum_\mu(1\,\tfrac{1}{2}\,m-\mu\,\mu|j\,m)\psi_{\frac{1}{2}}{}^\mu Y_1{}^{m-\mu}(\hat{r}). \quad (A5)$$

The ϕ is the oscillator potential wave function in the 1p state,

$$\phi=Nr\exp(-ar^2/2)\quad\text{with}\quad N^2=(8/3)\pi^{-\frac{1}{2}}a^{\frac{5}{2}}. \quad (A6)$$

Inserting (A5) into (A4), we have, e.g., for [1 1 1 p]:

$$[1\,1\,1\,p]=\sum_m\sum_\mu\sum_{\mu'}\sum_{M'}(-)^{\frac{3}{2}-m}(\tfrac{3}{2}\,\tfrac{1}{2}\,m\,M-m|1\,M)\times(1\,1\,M'\,M-M'|1\,M)(1\,\tfrac{1}{2}\,M-m-\mu\,\mu|\tfrac{1}{2}\,M-m)\times(1\,\tfrac{1}{2}\,-m-\mu'\,\mu'|\tfrac{3}{2}\,-m)\times\int[\psi_{\frac{1}{2}}{}^\mu Y_1{}^{M-m-\mu}(\hat{r})\phi]^\dagger j_1(qr)e^{-\alpha Zm_\mu' r}Y_1{}^{M-M'}(\hat{r})\times[\mathfrak{Y}_1{}^{M'}(\nabla)\psi_{\frac{1}{2}}{}^{\mu'}Y_1{}^{-m-\mu'}(\hat{r})\,\phi]r^2drd\Omega. \quad (A7)$$

Summing over all magnetic quantum numbers except for M, (A7) becomes

$$[1\,1\,1\,p]=-\sum_l\pi^{-1}\cdot 2^{-\frac{1}{2}}\cdot 3(2l+1)(l\,1\,0\,0|1\,0)\times W(1\,1\,1\,1,l\,1)W(1\,\tfrac{3}{2}\,1\,\tfrac{1}{2},\tfrac{1}{2}\,1)\times\int\phi e^{-\alpha Zm_\mu' r}j_1(qr)(D_l\phi)r^2dr, \quad (A8)$$

---

[34] M. E. Rose and R. H. Good, Jr., Ann. Phys. **9**, 211 (1960). They gave a formula for the angular distribution of recoils from muon capture in $0\rightarrow J$ transitions. Equation (20) in their paper is consistent with our P_0's, if we adopt the assumption Eq. (79), except for its fourth line. Without this assumption, a formula has been given by D. Greenberg and one of the present authors (M.M.), the second part of the present paper.

[35] Using our nuclear wave functions of C^{12} and B^{12}, Eqs. (A2) and (A3), the transition rate, Eq. (65), of beta decay $B^{12}\rightarrow C^{12}$ is calculated, $\mathcal{W}_{\beta\text{ theory}}=159\text{ sec}^{-1}$. This is 4.8 times larger than its experimental value, Eq. (75). This inaccuracy of wave functions is almost cancelled in the ratio of $\mathcal{W}/\mathcal{W}_\beta$, as is seen in Fig. 1. It is, however, difficult to estimate the limit of error due to the crudeness of nuclear wave functions. L. Wolfenstein, Nuovo cimento **13**, 319 (1959) did a similar calculation of the muon capture rate in C^{12} which has been called to our attention after completion of our work. The result agrees with ours. There, the errors involved in the calculated transition rates due to the inaccuracy of the nuclear wave functions are estimated as roughly 20–30%.

with

$$D_2=(\tfrac{2}{5})^{\frac{1}{2}}\left(\frac{d}{dr}-\frac{1}{r}\right),\quad D_1=0,\quad D_0=-\left(\frac{d}{dr}+\frac{2}{r}\right).\tag{A9}$$

Similarly,

$$[0\,1\,1\,p]=\sum_l \pi^{-1}\cdot 3^{\frac{1}{2}}(2l+1)(l\,1\,0\,0|1\,0)\times[W(\tfrac{3}{2}\,1\,\tfrac{1}{2}\,1,\tfrac{1}{2}\,l)]^2\times\int \phi e^{-\alpha Z m_\mu' r} j_1(qr)(D_l\phi)r^2dr.\tag{A10}$$

$$[1\,l\,1]=-\pi^{-1}\cdot 3[3(2l+1)]^{\frac{1}{2}}\times(l\,1\,0\,0|1\,0)\begin{bmatrix}1 & l & 1\\ \frac{1}{2} & 1 & \frac{3}{2}\\ \frac{1}{2} & 1 & \frac{1}{2}\end{bmatrix}\times\int\phi^2 e^{-\alpha Z m_\mu' r}j_l(qr)r^2dr.\tag{A11}$$

Integrating over radial coordinate, we have the results given in (68)–(73) of Sec. 10.

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Application of Dispersion Relations to K_{e3} and $K_{\mu3}$ Decays*†

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The decay modes, K_{e3} and $K_{\mu3}$, are studied by means of a dispersion relation. It is assumed that the fundamental couplings involved are the strong pion and K-meson couplings to the baryons and a weak four-field coupling connecting nucleon, hyperon, and the lepton pair. The baryon-antibaryon pair contribution to the absorptive part of the decay amplitude is expressed in terms of the imaginary part of the pion propagator in the same approximation. The decay rate is determined in terms of the various coupling constants and the quantity Z which renormalizes the pion propagator. Comparison with experiment is made for the case $g_\pi^2/g_K^2=15$. The results are consistent with a hyperon leptonic decay coupling constant an order of magnitude less than the beta-decay strength.

1. INTRODUCTION

IT is not known whether four-Fermi forms are appropriate to represent the fundamental weak interactions. However, many authors[1] have suggested various weak couplings of the form $\mathfrak{L}_{\text{weak}}=\lambda J_\mu J_\mu$ where J_μ consists of vector and axial-vector combinations of various baryon and lepton pairs. So far evidence for a universal coupling constant is found only in the strangeness conserving decays; present data on hyperon decays into a nucleon plus leptons indicate that if nucleon-hyperon terms are present in J_μ, they are present in a reduced amount.[2]

Aside from the leptonic hyperon decays the most direct evidence bearing on the strangeness nonconserving "current" comes from the $K^+\to\mu^++\nu$, $K^+\to\mu^++\nu+\pi^0$ and $K^+\to e^++\nu+\pi^0$ decay modes, and the similar K^- decay modes. The partial lifetimes for these processes are fairly well known, and if it were possible to connect them to the strength of a strangeness violating Fermi interaction one would have an indication of the consistency of such a form.

In a perturbation calculation the divergence problem renders this connection impossible. Either one must choose a numerical value for a cutoff or one must introduce counter-terms in the form of fundamental K decay couplings for each process. Nevertheless there is some hope that another calculation procedure could avoid this problem; that in this case the divergence is really a consequence of the perturbation expansion.

Recently Goldberger and Treiman[3] have used a dispersion relation approach to the similar problem of $\pi\to\mu+\nu$ decay. An answer for the decay rate was obtained in terms of the pi-nucleon and mu capture coupling constants. The same method is applicable to the $K\to\mu+\nu$ mode,[4] except that here the coupling constants involved are unknown.

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