TABLE II. Variation of α's with a.

$a \backslash \alpha$	α_1	α_2	α_3	α_4	α_5	α_6	$-\alpha_7$	α_8	$-\alpha_9$
0.0	1.1253	1.1510	1.8602	0.8886	0.0652	2.3490	0.4485	0.9777	0.9576
0.1	1.1633	1.0959	1.7256	0.8596	0.0533	2.3616	0.4647	1.0614	0.8888
0.2	1.1938	1.0588	1.6487	0.8376	0.0480	2.4022	0.4701	1.1275	0.8389
0.3	1.2203	1.0299	1.5921	0.8194	0.0445	2.4496	0.4719	1.1848	0.7994
0.4	1.2439	1.0059	1.5469	0.8038	0.0418	2.4993	0.4719	1.2365	0.7668
0.5	1.2654	0.9855	1.5093	0.7902	0.0397	2.5497	0.4710	1.2841	0.7391
0.6	1.2852	0.9676	1.4769	0.7781	0.0379	2.6002	0.4694	1.3283	0.7149
0.7	1.3036	0.9516	1.4486	0.7671	0.0364	2.6502	0.4673	1.3699	0.6935
0.8	1.3208	0.9372	1.4233	0.7571	0.0351	2.6997	0.4650	1.4093	0.6744
0.9	1.337	0.9241	1.4005	0.7478	0.0340	2.7487	0.4625	1.4469	0.6571
1.0	1.3525	0.9121	1.3799	0.7393	0.0329	2.7969	0.4600	1.4828	0.6414
2.0	1.4759	0.8276	1.2382	0.6775	0.0264	3.2454	0.4333	1.7835	0.5343
3.0	1.5676	0.7752	1.1531	0.6379	0.0229	3.6457	0.4101	2.0222	0.4716
4.0	1.6422	0.7375	1.0930	0.6089	0.0206	4.0128	0.3905	2.2266	0.4285
5.0	1.7059	0.7083	1.0469	0.5862	0.0189	4.3555	0.3739	2.4084	0.3963
6.0	1.7619	0.6846	1.0096	0.5676	0.0176	4.6794	0.3595	2.5737	0.3710
7.0	1.8120	0.6646	0.9786	0.5518	0.0165	4.9880	0.3469	2.7263	0.3503
8.0	1.8577	0.6476	0.9521	0.5383	0.0156	5.2841	0.3358	2.8688	0.3329
9.0	1.8997	0.6326	0.9290	0.5264	0.0149	5.5696	0.3258	3.0030	0.3181
10.0	1.9387	0.6194	0.9086	0.5158	0.0142	5.8459	0.3169	3.1302	0.3052

Hence

$$\Phi_1(r,s) = -\frac{4\pi}{n}\int_0^\infty \tau^r v^s \frac{\partial f_0}{\partial v} dv$$
$$= 4l^r(2kT/m)^{(s-3)/2}(2a)^{(s-r-3)/4}\frac{\Pi((s-r-3)/4)}{\Pi(1/2)},$$

where

$$\Pi(t) = \int_0^\infty z^t e^{-z} dz = \Gamma(t+1)$$

is a tabulated function.[6]

[6] E. Jahnke and F. Emde, *Tables oƒ Functions* (Dover Publications, New York, 1945), p. 9.

NUMERICAL RESULTS

Table II illustrates the variation of α's with a, obtained numerically on the Univac 1105.

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Plasma Oscillations of a Large Number of Electron Beams*

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Longitudinal oscillations of a large number of electron beams are investigated. The normal modes for the beams are found. An orthogonality relation between the modes is obtained and is used to solve the initial value problem and the problem of forced oscillations. It is demonstrated that no signal propagates faster than the fastest beam. The problem of passing to the limit of a continuous velocity distribution is considered in detail. It is shown that in the limit the results of Landau, Van Kampen, and others are recovered. The problem of Landau damping is discussed from the point of view of the beams.

I. INTRODUCTION

IN this paper a theory for the longitudinal oscillation of a large number of electron beams is presented. The term beam is used here to denote a stream of electrons which is infinite in extent and which has a definite velocity (no random motion within a beam). This work is, of course, closely related to the many papers which have appeared on the subject of plasma oscillations.[1–6] A large portion of this paper is devoted

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[2] N. G. Van Kampen, Physica **21**, 949–63 (1955).
[3] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1851 and 1864 (1949).
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to showing that the results of Landau[1] and Van Kampen[2] can be obtained by passing to the limit of an infinite number of beams in such a way as to approach a continuous distribution function.

The method of attack used here is similar to that of Van Kampen in that we look for normal modes. Through the use of a discrete set of beams the problem of the singular integrals which he encountered through the use of a continuous distribution function is avoided (see Van Kampen[2]). This method of avoiding the trouble was proposed by Ecker.[6] We find that any number of discrete beams greater than one is unstable. However, as one passes to the continuous distribution function limit the growth rate of most of these instabilities goes to zero. In the limit we recover all the usual results obtained by using the technique of Landau and Van Kampen.

The first part of this paper is devoted to obtaining the normal modes for an arbitrary number of beams and to using them to solve the initial value problem and the problem of forced oscillations. The last sections will be devoted to the problem of passing to the continuous distribution function limit and to the recovering of the results of other authors.

EQUILIBRIUM SITUATION AND BASIC ASSUMPTIONS

The situation that we wish to investigate is the small-amplitude longitudinal oscillations of an arbitrary number of electron beams. The beams are taken to be infinite in extent, and to have well-defined velocity (no thermal motion of an individual beam). They are assumed to pass through a uniform neutralizing background of positive ions which are taken to be infinitely massive and thus immobile. It is assumed that the beams can be treated as continuous charged fluids, that collisions between individual electrons and electrons and ions can be neglected, and that the motion may be treated by the linearized equations.

BASIC EQUATION

In this paper we will look for plain longitudinal oscillations which may be taken to propagate in the x direction. Since motions in the y and z directions do not influence the x motion, they may be ignored. The linearized equations of motion for this system are given by Eqs. (1) through (3).

$$\frac{\partial v_\sigma}{\partial t}+V_\sigma\frac{\partial v_\sigma}{\partial x}=-\frac{eE}{m}, \tag{1}$$

$$\frac{\partial n_\sigma}{\partial t}+N_\sigma\frac{\partial v_\sigma}{\partial x}+V_\sigma\frac{\partial n_\sigma}{\partial x}=0, \tag{2}$$

$$\frac{\partial E}{\partial x}=-4\pi e\sum_\sigma n_\sigma, \tag{3}$$

Here n_σ and v_σ are the perturbations in the number density and velocity of the σth beam while N_σ and V_σ are the corresponding unperturbed quantities. E is the electric field and is determined from Poisson's Eq. (3).

DISPERSION RELATION

We now look for solutions of the form

$$A(x,t)=Ae^{i(\omega t-kx)},$$

where A is any one of the quantities n_σ, v_σ, or E. Substituting this form in (1) through (3) yields

$$i(\omega-kV_\sigma)v_\sigma=-eE/m, \tag{4}$$

$$(\omega-kV_\sigma)n_\sigma-kN_\sigma v_\sigma=0, \tag{5}$$

$$ikE=4\pi e\sum_\sigma n_\sigma. \tag{6}$$

Eliminating E and v_σ yields

$$(\omega-kV_\sigma)^2 n_\sigma=(4\pi e^2 N_\sigma/m)\sum_\mu n_\mu. \tag{7}$$

Since the amplitude of the waves is arbitrary, the n_σ may be normalized so that

$$\sum_\sigma n_\sigma=1, \tag{8}$$

whence n_σ, v_σ, and E are given by

$$n_\sigma(\omega,k)=\frac{4\pi e^2 N_\sigma/m}{(\omega-kV_\sigma)^2}, \tag{9}$$

$$v_\sigma(\omega,k)=\frac{4\pi e^2/m}{(\omega-kv_\sigma)}, \tag{10}$$

$$E(\omega,k)=-4\pi ei/k. \tag{11}$$

The dispersion relation which ω and k must satisfy is obtained by substituting (9) in (8) and is given by

$$\frac{4\pi e^2}{m}\sum_\sigma\frac{N_\sigma}{(\omega-kV_\sigma)^2}=1. \tag{12}$$

If the left- and right-hand sides of (12) are plotted as functions of ω for fixed k we get a diagram like that shown in Fig. 1.

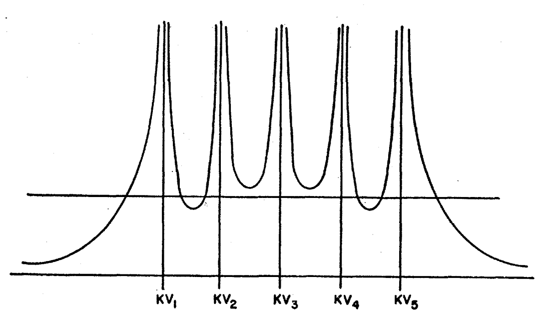

FIG. 1. A plot of the quantity $-(4\pi e^2/m)\Sigma N_\sigma/(\omega-kV_\sigma)^2$ as a function of ω.

The sum becomes infinite every time one of the denominators goes to zero. Each of the points at which the sum crosses one is a root of (12). These are all real roots. There are in general also complex roots of the dispersion relation. There are in fact twice as many modes as beams. This may be quickly seen by writing (12) in polynomial form. Each of these ω's gives a possible mode of oscillation for the beams for the given k. The system has one longitudinal degree of freedom per beam for fixed k. These may be thought of as the amplitudes of the kth Fourier components of the number density of the beams. It takes two constants $[n_\sigma(k), v_\sigma(k)]$ per degree of freedom to specify the state of the system. The arbitrary amplitudes of the modes found above supply just this number of constants so that we expect to be able to express any motion of the beams in terms of these modes.

ORTHOGONALITY RELATION

The modes obtained from (10), (11), and (12) satisfy an orthogonality relation which may be obtained as follows. Let k be fixed and let ω and ω' be two solutions of the dispersion relation (12). Let n_σ and n_σ' be the corresponding perturbations of the number densities. Multiply (7) by the normalized perturbation number density n_σ'/N_σ and (7') [(7) with primed quantities] by n_σ/N_σ. Subtracting the second of these results from the first and summing over σ gives

$$(\omega-\omega')\sum_\sigma[\omega+\omega'-2kV_\sigma]\frac{n_\sigma n_\sigma'}{N_\sigma} = \frac{4\pi e^2}{m}\sum_{\sigma\mu}(n_\sigma' n_\mu - n_\mu' n_\sigma) = 0. \quad (13)$$

The last equality follows from interchange of σ and μ. Now if

$$\omega \neq \omega',$$

then

$$\sum_\sigma(\omega+\omega'-2kV_\sigma)n_\sigma n_\sigma'/N_\sigma = 0, \quad (14)$$

while if $\omega=\omega'$ this sum is in general not zero. For notational convenience we will let

$$2\sum_\sigma(\omega-kV_\sigma)n_\sigma^2/N_\sigma = H(\omega,k). \quad (15)$$

Equations (14) and (15) demonstrate an orthogonality between modes with the same k, but different ω's. Modes with different k's are orthogonal in the usual Fourier sense.

The only ω's for which $H(\omega,k)$ is zero are those for which (12) has a double root. This is readily seen if it is noted that $H(\omega,k)$ is proportional to the derivative of (12) with respect to ω. In the case that (12) has a double root we must employ a slightly different procedure. A short discussion of this is given in Appendix 1. Here, however, it will be assumed that H is not zero.

INITIAL VALUE PROBLEM

Equations (14) and (15) may be used to solve the general initial value problem. Since the Fourier analysis is straight forward we will restrict ourselves to a fixed k. Assume that at $t=0$ the amplitude of the kth Fourier components of n_σ and v_σ are $\eta_\sigma(k)$ and $\nu_\sigma(k)$. The n_σ's and the v_σ's may be expanded in terms of the normal modes so that we have

$$n_\sigma(k,x,t) = \sum_\omega C(\omega,k)n_\sigma(\omega,k)e^{i(\omega t-kx)}, \quad (16)$$

$$v_\sigma(k,x,t) = \sum_\omega \frac{C(\omega,k)(\omega-kV_\sigma)}{N_\sigma k}n_\sigma(\omega,k)e^{i(\omega t-kx)}, \quad (17)$$

where the sums are over all roots of the dispersion relation for the given k and the $n_\sigma(k,\omega)$ are given by (10). If (16) for $t=0$ is multiplied by $[(\omega'-kV_\sigma)n_\sigma(\omega,k)]/N_\sigma$ and (17) for $t=0$ is multiplied by $kn_\sigma(\omega',k)$ and the results subtracted one finds, with the aid of the orthogonality conditions (14) and (15),

$$C(\omega',k) = [1/H(\omega',k)]\sum_\sigma\{[(\omega'-kV_\sigma)\eta_\sigma(k)n_\sigma(\omega',k)/N_\sigma] + kn_\sigma(\omega',k)\nu_\sigma(k)\}. \quad (18)$$

Thus the C's are determined and we have found the motion in terms of the normal modes.

The electric field may be found in terms of the C's by making use of (11). It is given by

$$E_k = (4\pi ei/k)\sum_\omega C(\omega,k)e^{i(\omega t-kx)}. \quad (19)$$

LANDAU DAMPING FOR THE BEAMS

On the basis of the previous sections we may form the following physical picture of how the motion of the beams will develop in time. In general an initial perturbation of the beams will contain all possible modes to a greater or lesser degree. The amplitude of each mode will depend on the details of the initial perturbation. These modes will start out more or less in phase. If we confine ourselves to a fixed k we see that the various modes have different frequencies. As time goes on, they will get out of phase with each other and so their coherent effects will die out. All macroscopic quantities such as the electric field which depends on the coherence of the various waves will thus die out and the initial perturbation will appear damped. This damping is the result of phase mixing of the various modes. It is not due to the damping of individual modes. This apparent damping is just *Landau damping*. Van Kampen's treatment of plasma oscillations yields a similar physical picture.

One might expect that any finite number of beams would return to their original state after a sufficiently long time. This would indeed be true, but for the fact that the beams are in general unstable. For a large number of beams, the growth rate of the instabilities is in general very small as will be shown later. Thus, the instabilities will not be sufficiently strong to over shadow the phase mixing or Landau damping, described above, but they are strong enough to prevent return to the initial state. For any finite number of beams the instabilities will get the upper hand on the Landau damping sooner or later provided some other process such as collisional damping does not prevent this. At such a time the most unstable mode will start to dominate the picture.

Trapping of electrons in the wave troughs is not allowed in this treatment since this introduced nonlinear effects. Thus, Landau damping cannot be attributed to trapped electrons. When trapping occurs, the linearized treatment breaks down. More discussion of this is given in the sections on the continuous distribution function limit and the discussion of the limits of the linearized theory.

MAXIMUM SPEED OF PROPAGATION OF A DISTURBANCE

Let us assume that at $t=0$ the beams are disturbed as follows:

$$\begin{aligned} n_\sigma(0)&=\eta_\sigma(0)e^{\epsilon x}\sin k_0x, && x\leq 0\\ v_\sigma(0)&=\nu_\sigma(0)e^{\epsilon x}\sin k_0x, && x\leq 0\\ n_\sigma(0)&\equiv 0, && x>0\\ v_\sigma(0)&\equiv 0, && x>0. \end{aligned}$$

Here ϵ is to be small and the term $e^{\epsilon x}$ is added only to give convergence at $x=-\infty$. The Fourier components of n_σ and v_σ are given by

$$\begin{aligned} n_\sigma(k)&=\eta_\sigma(0)k_0/[(k-i\epsilon)^2-k_0^2],\\ v_\sigma(k)&=\nu_\sigma(0)k_0/[(k-i\epsilon)^2-k_0^2]. \end{aligned} \tag{20}$$

The $C(\omega,k)$'s will have poles at $k=\pm k_0+i\epsilon$ and will have no other poles. We assume that $H(\omega,k)\neq 0$ for all ω which are excited; $H(\omega,k)$ is zero only when ω is a double root of the dispersion relation. These modes require the slightly different treatment outlined in Appendix 1. Now n_σ is given by

$$n_\sigma(x,t)=\int dk\sum_{\omega(k)}C(\omega,k)n_\sigma(\omega,k)e^{i(\omega t-kx)}. \tag{21}$$

A similar expression gives v_σ. If each root $\omega(k)$ of the dispersion relation is thought of as a continuous function of k then the summation and integration may be interchanged. We thus obtain a sum of integrals, one integral being obtained for each curve $\omega(k)$. Now for very large k, the roots of (12) approach $\omega(k)=kV_\sigma$ where the V_σ's are the beam velocities. Thus, for $t=0$ and $x>0$ the k contours may be closed by a large semicircle with

$$\mathrm{Im}(k)<0,$$

while for $x<0$ they may be closed by a semicircle with

$$\mathrm{Im}(k)>0.$$

For all x such that

$$x-V_{\max}t>0, \tag{22}$$

where $V_{\max}$ is the maximum beam velocity, the integrals may be closed in the same manner as for $t=0$, $x>0$. The integrals are thus zero since there are no poles inside the contour. Thus no signal travels faster than the fastest beam.

For positions x which do not satisfy (22) some of the integrals must be closed on a large semicircle with

$$\mathrm{Im}(k)>0.$$

These are the integrals for which

$$\omega(k)\to kV_\sigma \quad\text{as}\quad k\to\infty$$

with

$$V_\sigma>x/t.$$

Since we may associate the curve $\omega(k)$ which goes as kV_σ for large k with the σ'th beam we see that in a very real sense the disturbance is carried on the beams.

FORCED OSCILLATIONS

The problem of forced oscillations may be solved in a manner similar to that used for the initial value problem. We must add a forcing term F_σ to the right-hand side of (1) so that it reads

$$m\left(\frac{\partial v_\sigma}{\partial t}+V_\sigma\frac{\partial v_\sigma}{\partial x}\right)=-eE+F_\sigma. \tag{24}$$

F_σ is the external force per particle applied to the σth beam. Such a force might be supplied by a grid inserted in the beams. It would then arise from an electric field which is not self-consistent with the beam motion and which thus has its sources in charges not belonging to the beams. Here, however, it will be thought of simply as an external mechanical force.

The F_σ may be Fourier analyzed in space and time so that we need only solve the problem for a single driving frequency Ω and for a single k. The motion of the beams may be expanded in terms of the normal modes so that $n_\sigma(x,t)$ and $v_\sigma(x,t)$ may be written in the form

$$n_\sigma(k,x,\Omega,t)=\sum_\omega C(\Omega,\omega,k)n_\sigma(\omega,k)e^{i(\Omega t-kx)}, \tag{25}$$

$$v_\sigma(k,x,\Omega,t)=\sum_\omega\frac{C(\Omega,\omega,k)(\omega-kV_\sigma)}{N_\sigma k}n_\sigma(\omega,k)e^{i(\Omega t-kx)}. \tag{26}$$

Here as in the case of the free oscillations the sums are over all the roots of the dispersion relation (12) for the given k, and the $n_\sigma(\omega,k)$ are the corresponding number density perturbations given by (9). Substituting in (24) and making use of the orthogonal relations (14) and (15) yields

$$C(\Omega,\omega,k)=-\frac{ik}{4\pi e}\sum_\sigma\frac{F_\sigma(k,\Omega)n_\sigma(\omega,k)}{(\Omega-\omega)H(k,\omega)}. \tag{27}$$

Solutions to the free equations may, of course, be added to (25) and (26) so as to satisfy arbitrary initial conditions. Resonant solutions may be treated as limits of the nonresonant cases.

CONTINUOUS DISTRIBUTION FUNCTION AS THE LIMIT OF AN INFINITE NUMBER OF BEAMS

Consider now the problem of letting the number of beams go to infinity in such a way as to approach a continuous distribution function. One might expect that the dispersion relation (12) would go over into

$$\frac{4\pi e^2}{m}\int\frac{N(v)dV}{(\omega-kV)^2}=1. \tag{28}$$

However, the integrand in (28) is singular and one does not know how to treat it *a priori*. One may get around the difficulty by treating the problem by means of Laplace transform theory as Landau does or by allowing delta functions in the perturbed distribution function as Van Kampen does. However, it should be possible to handle the trouble by taking the limit of (12) in the proper way. By this means we will be led to results which closely resemble those of Van Kampen and from which the results of previous authors can be obtained.

In order to carry out the limiting process we will take the electrons to be distributed among a large number of beams which are equally spaced in velocity. This spacing will be taken to be δ. The number density will be a function of the beam velocity. Equation (12) now becomes

$$\lim_{\delta\to 0}\frac{4\pi e^2}{m}\sum_{\sigma=-\infty}^{\sigma=\infty}\frac{N(\sigma\delta)\delta}{(\omega-k\sigma\delta)^2}=1. \tag{29}$$

Here $\sigma\delta$ and $N(\sigma\delta)\delta$ are the velocity and number density of the σth beam. $N(v)$ is the velocity distribution function and is assumed to be a continuous function of v.

If the quantity

$$\frac{4\pi e^2}{m}\left(\frac{\pi^2 N(\omega/k)}{k^2\delta\sin^2(\pi\omega/k\delta)}-\frac{\pi N'(\omega/k)}{k^2}\cot\frac{\pi\omega}{k\delta}\right), \tag{30}$$

$$N'(\omega/k)=dN(\omega/k)/d(\omega/k),$$

is added to and subtracted from the left-hand side of (29) we find

$$\frac{4\pi e^2}{m}\left[\frac{\pi^2 N(\omega/k)}{k^2\delta\sin^2(\pi\omega/k\delta)}-\frac{\pi N'(\omega/k)}{k^2}\cot\frac{\pi\omega}{k\delta}-\frac{\pi N'(\omega/k)\delta}{\omega}\right.$$
$$\left.+\sum_{-\infty}^{\infty}\left(\frac{[N(\sigma\delta)-N(\omega/k)]\delta}{(\omega-\sigma k\delta)^2}+\frac{2N'(\omega/k)\omega\delta/k}{(\omega^2-\sigma^2k^2\delta^2)}\right)\right]=1, \tag{31}$$

where use has been made of the relations (see Knopp[7])

$$\frac{\pi^2}{\sin^2\pi x}=\sum_{-\infty}^{\infty}\frac{1}{(x-\sigma)^2}, \tag{32}$$

$$\pi\cot\pi x=-\frac{1}{x}+\sum_{-\infty}^{\infty}\frac{2x}{(x^2-\sigma^2)}. \tag{33}$$

The term inside the summation in (31) has no poles so that the sum passes smoothly to an integral as δ goes to zero.

Equation (31) may therefore be closely approximated by

$$\frac{4\pi e^2}{m}\left[\frac{\pi^2 N(\omega/k)}{k^2\delta\sin^2(\pi\omega/k\delta)}-\frac{\pi N'(\omega/k)}{k^2}\cot\frac{\pi\omega}{k\delta}\right.$$
$$\left.-\int_{-\infty}^{\infty}dV\left(\frac{N'(V)}{k(\omega-kV)}+\frac{2N'(\omega/k)\omega/k}{(\omega^2-k^2V^2)}\right)\right]=1, \tag{34}$$

for small δ. This form closely resembles Van Kampen's dispersion relation. The $\sin^{-2}$ and cot terms give us the freedom which he obtained by allowing his perturbed distribution functions to contain delta functions.

Consider now the roots of (34). First suppose that there exists an ω with finite imaginary part for which

$$1+\frac{4\pi e^2}{m}\int_{-\infty}^{\infty}\frac{N'(V)}{k(\omega-kV)}dV=0. \tag{35}$$

Then $\sin^{-2}(\pi\omega/k\delta)$ is exponentially small in $1/\delta$ for this ω and so is the expression

$$\frac{N'(\omega/k)}{k}\left(\pi\cot\frac{\pi\omega}{k\delta}+\int_{-\infty}^{\infty}\frac{2N'(\omega/k)\omega dV}{(\omega^2-k^2V^2)}\right). \tag{36}$$

Thus this ω is a solution of (34) in the limit of zero δ and hence the roots of (35) give all modes with finite imaginary part. This is not surprising since for such modes no trouble arises from the integrand in (28) and we may obtain (35) by a straight forward limiting process plus an integration by parts.

The complex roots of (35) cannot be all the roots of (34) since it was shown previously that there are two modes per beam for a given k and (35) can at best yield a small fraction of this number. The large number of modes which we have missed must have imaginary parts which go to zero with δ. We will therefore write ω in the form

$$\omega=\alpha+i\beta, \tag{37}$$

where β must go to zero with δ. Now

$$1+\frac{4\pi e^2}{m}\int_{-\infty}^{\infty}dV\left(\frac{N'(v)}{k(\omega-kV)}+\frac{2N'(\omega/k)\omega/k}{(\omega^2-k^2V^2)}\right)$$

[7] Konrad Knopp, *Theory of Functions* (Dover Book Publishers, Inc., New York, 1957).

is in general finite so that $[\delta \sin^2(\pi\omega/k\delta)]^{-1}$ and $\cot(\pi\omega/k\delta)$ must be finite. Therefore, $\sin^2(\pi\omega/k\delta)$ must be of the order of $1/\delta$ and hence β must go as

$$\pm\delta \ln\delta.$$

With this behavior for β, $\cot(\pi\omega/k\delta)$ will approach $\pm i$ as δ goes to zero. Here $\pm$ has the opposite sign from β.

This qualitative knowledge of the behavior of β tells us that

$$\lim_{\delta\to 0}\beta/k\delta \to \pm\infty.$$

Making use of this fact and writing α as $nk\delta+\alpha_1$ (n an integer) with α_1 of the order of δ so that we obtain the roots in the vicinity of α_0 (where $\alpha_0=nk\delta$), (34) becomes

$$-\frac{4\pi e^2}{m}\left(\frac{-\pi^2N(\alpha_0/k)\cos[2\pi(\alpha_0+\alpha_1)/k\delta]\pm i\sin[2\pi(\alpha_0+\alpha_1)/k\delta]}{k^2\cos[2\pi\beta/k\delta]}\pm\frac{i\pi N'(\alpha_0/k)}{k^2}\right)=1+\frac{4\pi e^2}{m}\mathrm{P}\int_{-\infty}^{\infty}\frac{N'(V)dV}{k(\alpha_0-kV)}. \quad (38)$$

Here use has been made of the fact that $\sinh\theta\simeq\pm\cosh\theta$ for $|\theta|$ large. As before $\pm$ has the opposite sign from β. Solving (38) for α_1, and β gives

$$\tan\frac{2\pi\alpha_1}{k\delta}=-\left(\frac{4\pi e^2}{km}\right)\pi\frac{N'}{k}\left(\frac{\alpha_0}{k}\right)\bigg/\left(1+\frac{4\pi e^2}{m}\mathrm{P}\int_{-\infty}^{\infty}\frac{N'(v)dV}{k(\alpha_0-kV)}\right), \quad (39)$$

$$\beta=\pm\frac{k\delta}{2\pi}\left\{\ln\left(\frac{k\delta}{\pi N(\alpha_0/k)}\right)+\ln\left[\left(1+\frac{4\pi e^2}{mk}\mathrm{P}\int_{\infty}^{\infty}\frac{N'(v)dV}{(\alpha_0-kv)}\right)^2\left(\frac{mk}{4\pi^2e^2}\right)^2+\frac{N'(\alpha_0/k)}{k^2}\right]\right\}. \quad (40)$$

Equation (39) yields roots

$$2\pi\alpha/k\delta=\theta\pm n\pi, \quad (41)$$

or

$$\alpha_1=(k\delta/2\pi)\theta\pm(nk\delta/2), \quad (42)$$

where θ is the principal solution of (39) for $2\pi\alpha_1/k\delta$ and lies between 0 and π. However, only half these roots satisfy (38) since $\cos(2\pi\alpha_1/k\delta)$ must have the opposite sign to

$$1+\frac{4\pi e^2}{m}P\int_{-\infty}^{\infty}\frac{N'(V)dV}{k(\alpha_0-kV)}.$$

Thus the roots are spaced $k\delta$ apart in α_1. The natural frequencies of the beams

$$\sigma k\delta=\alpha \quad (43)$$

also have this spacing. Thus the α's we obtain have real parts between the α's associated with the beams. There are two roots for each α since β can be either positive or negative. Thus we obtain two modes for each beam as required.

INITIAL VALUE PROBLEM FOR AN INFINITE NUMBER OF BEAMS

We may now solve the initial value problem for an infinite set of beams. We must find the limit of (18) as δ goes to zero. The only ω's which give trouble are those whose imaginary part goes to zero with δ. These ω's will be considered in detail. The $C(\omega,k)$'s for complex ω may be obtained by taking a straight forward limit of (18).

First consider the limit of $H(\omega,k)$. This is given by

$$\lim_{\delta\to 0}\sum_{\sigma}\left(\frac{4\pi e^2}{m}\right)^2\frac{N(\sigma\delta)\delta}{(\omega-\sigma k\delta)^3}=\lim_{\delta\to 0}H(\omega,k). \quad (44)$$

Here as was the case with the dispersion relation the function inside the sum is singular and we handled the limit in a manner similar to that employed there. To the left-hand side of (44) we add and subtract

$$2\left(\frac{4\pi e^2}{m}\right)^2\left(-\frac{N(\omega/k)\pi^3\cos(\pi\omega/k\delta)}{k^3\delta^2\sin^3(\pi\omega/k\delta)}-\frac{N'(\omega/k)\pi^2}{k^3\delta\sin^2(\pi\omega/k\delta)}+\frac{N''(\omega/k)\pi\cot(\pi\omega/k\delta)}{2k^3}\right). \quad (45)$$

These terms may also be written in the form of sums by making use of (32), (33), and (46)

$$\pi^3\frac{\cos\pi x}{\sin^3\pi x}=\sum_{-\infty}^{\infty}\frac{1}{(x-\sigma)^3}. \quad (46)$$

In this way we again obtain an expression inside the summation which has no poles and which, hence, passes over into a well-defined integral in the limit. If we now make use of the solutions for ω (α and β) which were found in the previous section we find that the first term in (45) is dominant and gives the limiting value of $H(\omega,k)$. The limit of this term is

$$\lim_{\delta\to 0}H(\omega,k)=\pm\frac{2\pi i}{k\delta}\frac{4\pi e^2}{m}\left(1+\frac{4\pi e^2}{mk}\mp i\pi N'(\alpha/k)+P\int_{-\infty}^{\infty}\frac{N'(v)dV}{(\alpha-kV)}\right). \quad (47)$$

$\omega=\alpha\pm i\beta$, $\pm$ has the sign of β. We shall let $H(\omega,k)=h_{\mp}(\alpha,k)/k\delta$ (subscript has the opposite sign to β). Equation (47) may also be written in the form

$$h_{\mp}(\alpha,k)=\pm 2\pi i\left(\frac{4\pi e^2}{m}\int_{-\infty}^{\infty}\frac{N'(v)dV}{(\alpha_{\mp}-kV)}+1\right), \quad (48)$$

where the $\mp$ sign on α means that α is to be given an infinitesimal negative or positive imaginary part so that for the $-$ sign the integration contour for the velocity (the real axis) passes above the pole of the integrand and for the $+$ sign it passes below the pole.

Returning to the numerator of (18) we must find

$$\lim_{\delta\to 0}\frac{4\pi e^2}{m}\sum_{\sigma}\left(\frac{\eta(\sigma\delta,k)\delta}{(\omega-\sigma k\delta)}+\frac{k\delta N(\sigma\delta)\nu(\sigma\delta,k)}{(\omega-\sigma k\delta)^2}\right). \qquad (49)$$

As before, we have a singular function inside the summation and we handle it as we did $H(\alpha,k)$. Adding and subtracting the appropriate terms so as to eliminate the poles inside the sum and making use of the solutions for ω found in the previous section, (49) reduces to

$$\frac{4\pi e^2}{m}\Bigg[\int_{-\infty}^{\infty}\frac{N(v,k)dV}{(\alpha_{\mp}-kV)} -\int_{-\infty}^{\infty}\frac{dV[N'(V)\nu(v,k)+\nu'(v,k)N(v)]}{(\alpha_{\mp}-kV)} +k\nu(\alpha/k,k)\left(1+\frac{4\pi e^2}{m}\int_{-\infty}^{\infty}\frac{N'(v)dV}{(\alpha_{\mp}-kV)}\right)\Bigg]=g_{\mp}(\alpha,k), \qquad (50)$$

$\pm$ has sign of β, $\mp$ sign on α has same meaning as given above.

The development of the electric field in time is given by (19) for fixed k.

$$E_k(x,t)=-(4\pi ie/k)\sum_{\omega} C(\omega,k)e^{i(\omega t-kx)}.$$

The limiting form of this equation is

$$\int_{\substack{-\infty\\ \omega(\text{real}-i\epsilon)}}^{\infty}\frac{g_-(\omega,k)e^{i(\omega t-kx)}}{h_-(\omega,k)}d\omega+\int_{\substack{-\infty\\ \omega(\text{real}+i\epsilon)}}^{\infty}\frac{g_+(\omega,k)e^{i(\omega t-kx)}}{h_+(\omega,k)}d\omega +\sum\nolimits_{\text{cr}}\frac{g(\omega,k)e^{i(\omega t-kx)}}{h(\omega,k)}. \qquad (51)$$

where the integrals are taken over ω for ω real $-$ an infinitesimal imaginary part and for ω real $+$ an infinitesimal imaginary part. In the summation, the subscript cr indicates the sum is over all complex roots of Eq. 12. Here the sums over the roots $\alpha\pm i\beta$ (real ω in limit) have been replaced by the integrals along the real axis by making the substitution

$$(g/H)\Delta\omega=(g/H)k\delta=(g/h),$$

since there is one root with $\beta>0$ and one with $\beta<0$ in each interval $\Delta\omega=k\delta$. The quantities $h(\omega,k)$ and $g(\omega,k)$ for the complex ω may be obtained by taking a straight forward limit of Eqs. (15) and (18). Performing one integration by parts on these limits leads to (52) and (53).

$$h(\omega,k)=\frac{1}{k}\left(\frac{4\pi e^2}{m}\right)^2\int_{-\infty}^{\infty}\frac{N'(V)dV}{(\omega-kV)^2}, \qquad (52)$$

$$g(\omega,k)=\frac{4\pi e^2}{m}\left(\int_{-\infty}^{\infty}\frac{\eta(V,k)N(V)dV}{(\omega-kV)} -\int_{-\infty}^{\infty}\frac{dk[N'(V)\nu(v,k)+N(v)\nu(v,k)]}{(\omega-kV)}\right). \qquad (53)$$

Examination of Eqs. (52), (53), and (48), (50) shows that the modes with complex ω's may be included in the integral expressions in the following way. If we let ω take on any complex value in the upper half plane then the expressions for g_+ and h_+ yield analytic functions of ω in the upper half plane. This follows from the fact that as we allow ω to move around in the upper half plane the singularity in the integrands in (48) and (50) does not cross the contour of integration. Likewise if ω is allowed to take on any complex value in the lower half plane the expressions for g_- and h_- yield analytic functions in the lower half plane. If we wish to give analytic definitions to h_+ and g_+ in the lower half plane and to h_- and g_- in the upper half plane we must analytically continue these functions.

Now g_+/h_+ has poles at the complex roots of (35) which lie in the upper half plane and g_-/h_- has poles at the roots of (35) which lie in the lower half plane when g_+/h_+ and g_-/h_- are given as described above. Further g/h for the complex modes is $-2\pi i$ times the residue of g_+/h_+ at the roots in the upper half plane and $2\pi i$ times the residue of g_-/h_- at the roots in the lower half plane. It therefore follows that E_k is given by

$$E_k(x,t)=\int_{\substack{-\infty\\ \text{contour 1}}}^{\infty}\frac{g_-(\omega,k)e^{i(\omega t-kx)}}{h_-(\omega,k)}d\omega +\int_{\substack{-\infty\\ \text{contour 2}}}^{\infty}\frac{g_+(\omega,k)e^{i(\omega t-kx)}}{h_+(\omega,k)}d\omega, \qquad (54)$$

where the contours 1 and 2 are those shown in Fig. 2. Contour 1 passes below all the poles of h_- and contour 2 passes above all poles of h_+.

Now for $t<0$ both integrals may be closed on a large semicircle with

$$|\text{Im}(\omega)|<0, \qquad (55)$$

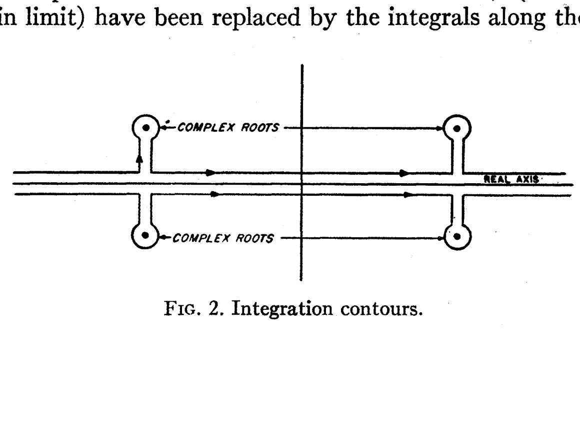

FIG. 2. Integration contours.

and for $t>0$ both may be closed on a semicircle with

$$|\mathrm{Im}(\omega)|>0, \tag{56}$$

if the definitions of g_+ and h_+ are extended by analytic continuation into the lower half plane, and those of g_- and h_- are similarly extended into the upper half plane. In both cases the integral along the semicircle vanishes. For $t<0$ the first integral vanishes while for $t>0$ the second integral vanishes.

The integral along contour 1 gives an expression for E equivalent to the expression obtained by Landau. It may be treated in the manner that Landau[1] uses and for the case of no complex roots of the dispersion relation (unstable modes) it yields the usual Landau damping. The integral along contour 2 is similar to Landau's expression except that it gives the electric field for negative time. It can also be handled in a manner similar to that used by Landau and it yields damping in the negative t direction if there are no complex modes. Thus, the wave dies out in both the positive and negative t directions so that we have symmetry between the future and past.

Expression (54) is very similar to the expressions obtained by Landau and Van Kampen. There is, however, a slight difference which is due to the difference in the method of attack. Both Landau and Van Kampen solve the problem by means of the Boltzmann equation and do not attempt to follow the motion of a single stream of particles in detail. Here, however, the motion of each stream is followed and so more information is contained in the solutions given here than in those given by them. This is why the g functions which appear in this paper are somewhat more complex than the corresponding expressions which appear in the works of these authors.

LIMITATIONS TO THE LINEARIZED THEORY

The theory that has been presented here is a linearized theory like that of most other treatments of plasma oscillations. Because of this, it breaks down if the amplitude of the oscillations becomes too large. For unstable situations this break down will always occur sooner or later. If we confine ourselves to the limiting situations for which the distribution is stable (all $\beta\rightarrow 0$) the beam instabilities will give trouble only after times of the order of logarithm of the beam spacing. Nevertheless, an examination of density perturbations given by (9), plus the fact that β goes to zero as $\delta\ln\delta$, shows that the larger the number of beams the smaller the amplitude of the wave must be so that the theory does not break down for those beams for which $(\omega-\sigma k\delta)$ is smallest. This restricts the amplitude to be of the order of $\delta\ln\delta$. However, this restriction is what is required so that the theory will hold for times of the order of the beam instability growth times $(\delta|\ln\delta|)^{-1}$. If, in fact, the beams were not unstable, choosing the amplitude to be of this order would insure that the solutions would be good for all times. If we are satisfied with solutions which are good for finite lengths of time then we need not put this restriction on the amplitude. The solutions that we obtain are then good so long as nonlinear terms are not important to the motion of any of the beams. The length of this time will, of course, depend on the amplitude. We can estimate this time for the case of a continuous distribution function to be of the order of the period of oscillation of a particle trapped in the trough of the wave, for this is roughly the time in which nonlinear terms become important for the trapped electrons. This time is of the order of

$$\tau=(m/eE_{\max}\kappa)^{\frac{1}{2}}, \tag{59}$$

where $E_{\max}$ is the maximum electric field produced by the wave. If now the Landau damping time is short compared to this time then we may expect the waves to damp out in accordance with the linearized theory. If, on the other hand, the Landau damping time is long compared to this time, we expect the linearized theory will not give an accurate picture of the long-time behavior of the wave.

Another place where one can expect the theory to break down is through the representation of all streams of particles as continuous fluids. When one goes to very high velocities there will be very few particles per stream. If this number is only a few particles per wavelength then the continuous fluid picture again breaks down. If the Landau damping is due to particles in such a region then again one expects the long-time behavior to be modified.

It should be emphasized that the restrictions presented above are not confined to the beam calculation, but apply also to treatments by the Boltzmann equation.[8] Similar arguments to some of those presented in this section are given in Bohm and Gross.[3]

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APPENDIX 1

Double Roots of the Dispersion Relation

If Eq. (12) has a double root, ω, we must proceed in a slightly different way from that used in the text. Instead of looking for solutions of the form

$$Ae^{i(\omega t-kx)}, \tag{60}$$

we look for one solution of this form and one solution of the form

$$(A+Bt)e^{i(\omega t-kx)}, \tag{61}$$

where, as before, these are the wave forms for any of

[8] J. R. Pierce and J. A. Morrison, Proc. Inst. Radio Engrs. ED-6, 231 (1959).

the quantities v_σ, n_σ, or E. Eliminating v_σ and E between (4), (5), and (6) we find

$$n_\sigma=(n_\sigma+\bar{n}_\sigma t)e^{i(\omega t-kx)}, \tag{62}$$

$$\left(\frac{\partial^2 n_\sigma}{\partial t^2}-2ikV_\sigma\frac{\partial n_\sigma}{\partial t}-k^2V^2n_\sigma\right)=-\frac{4\pi e^2}{m}N_\sigma\sum_\mu n_\mu, \tag{63}$$

$$\bar{n}_\sigma=\frac{4\pi e^2N_\sigma}{m(\omega-kV_\sigma)^2}\sum_\mu \bar{n}_\mu, \tag{64}$$

$$n_\sigma=\frac{2i\bar{n}_\sigma}{(\omega-kV_\sigma)}+\frac{4\pi e^2N_\sigma}{m(\omega-kV_\sigma)^2}\sum_\mu n_\mu. \tag{65}$$

We may normalize in such a way that

$$\sum_\mu \bar{n}_\mu=1, \tag{66}$$

which leads to

$$\bar{n}_\sigma=\frac{4\pi e^2N_\sigma}{m(\omega-kV_\sigma)^2}. \tag{67}$$

Equations (66) and (67) are consistent with the dispersion relation (12). Substitution of (67) in (65) and summing the result over σ leads to

$$\sum_\sigma n_\sigma=2i\sum_\sigma\frac{4\pi e^2N_\sigma}{m(\omega-kV_\sigma)^3}+\left(\frac{4\pi e^2}{m}\sum_\sigma\frac{N_\sigma}{(\omega-kV_\sigma)^2}\right)\left(\sum_\mu n_\mu\right). \tag{68}$$

Now the first term on the right-hand side of (68) is zero because ω is a double root of the dispersion relation and the last term on the right-hand side of (68) is simply

$$\sum_\mu n_\mu.$$

Thus (68) is satisfied automatically and so we may choose

$$\sum_\mu n_\mu=0. \tag{69}$$

This leads to

$$n_\sigma=\frac{2i4\pi e^2N_\sigma}{m(\omega-kV_\sigma)^3}. \tag{70}$$

The arbitrariness which arises in the choice of

$$\sum_\mu n_\mu,$$

comes simply from the fact that we may add an arbitrary amount of (60) to (61) and still have a solution.

We may find v from (2), (62), (67), and (70). These give

$$v_\sigma=\left(\frac{(\omega-kV_\sigma)}{kN_\sigma}(n_\sigma+\bar{n}_\sigma t)-\frac{in_\sigma}{kN_\sigma}\right)e^{(i\omega t-kx)}. \tag{71}$$

Let us also take $\bar{n}_\sigma$ and $\bar{v}_\sigma$ to be the density and velocity perturbation for the solution with form (60) for the given ω.

We may now solve the initial value problem. We need only consider the density and velocity perturbations due to the two modes of frequency ω since our previously derived orthogonality relation holds between all other modes and these two. Let η_σ and ν_σ be the density and velocity perturbations at $t=0$ due to these two modes.

We have

$$n_\sigma(t,x)=[a\bar{n}_\sigma+b(n_\sigma+\bar{n}_\sigma t)]e^{i(\omega t-kx)}, \tag{72}$$

$$v_\sigma(t,x)=\left[\frac{a(\omega-kV_\sigma)}{kN_\sigma}\bar{n}_\sigma+b\left(\frac{(\omega-kV_\sigma)}{kN_\sigma}(n_\sigma+\bar{n}_\sigma t)-\frac{i\bar{n}_\sigma}{kN_\sigma}\right)\right]e^{i(\omega t-kx)}, \tag{73}$$

which at $t=0$ must equal η_σ and ν_σ. Placing $t=0$ and summing over σ (72) yields

$$a=\sum_\sigma \eta_\sigma,$$

since $\bar{n}_\sigma$ satisfies (8). Multiplying (73) by

$$N_\sigma/(\omega-kV_\sigma)^2,$$

yields

$$\sum_\sigma\frac{N_\sigma\nu_\sigma}{(\omega-kV_\sigma)^2}=\frac{b}{k}\sum_\sigma\frac{4\pi e^2N_\sigma(1-i)}{n(\omega-kV_\sigma)^4}, \tag{74}$$

or

$$b=k\frac{\displaystyle\sum_\sigma\frac{N_\sigma\nu_\sigma}{(\omega-kV_\sigma)^2}}{\displaystyle\sum_\sigma\frac{4\pi e^2N_\sigma(1-i)}{m(\omega-kV_\sigma)^4}}. \tag{75}$$

The denominator of this expression can be zero only if ω is a triple root of the dispersion relation and this can never happen for a discrete set of beams.