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Pion-Pion and Pion-Kaon Scattering*

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An easier derivation of Chew-Mandelstam's effective-range formula for pion-pion scattering is given using the conventional Feynman method with the interaction Hamiltonian $H_1=4\pi\lambda\phi^4$ in an approximation where only the chain diagrams are included. Furthermore, we have calculated a correction term to this formula due to the cross-diagram, both for S waves as well as for P waves. The method has been applied to pion-kaon scattering.

(I) INTRODUCTION

RECENTLY, Chew and Mandelstam[1] investigated the pion-pion scattering problem by using the Mandelstam representation.[2] They obtained simultaneous integral equations for the pion-pion scattering amplitudes corresponding to the various angular momentum and isotopic spin states. However, their equations turn out to be quite complicated and consequently in order to solve them they made several approximations; in particular, they neglected the cross-diagram (see the next section, Fig. 2) in the two-meson approximation. In this way, they obtained effective-range formulae for the S-wave phase-shifts corresponding to pion-pion scattering. On the other hand, the calculation of the P-wave phase-shifts seems to be rather difficult with this method.

We propose in this note to do the following: first, we shall show that we can derive exactly the same formulas in a much simpler manner by using the conventional covariant calculation of chain-diagrams (see the next section, Fig. 1). Furthermore, we shall calculate the correction due to the ladder diagram (cross-diagram) as a perturbation. We shall also derive the corresponding formulas for the P-wave phase-shifts. These calculations indicate that the cross diagrams are too important to be neglected; consequently, the previous calculations by Chew and Mandelstam may not suffice.

The same method is applied to pion-kaon scattering, although in this case there is an ambiguity in the definition of the coupling constant; calculation is given in Sec. 3.

(2) PION-PION SCATTERING

Let us consider the S-matrix element for pion-pion scattering, where two incoming pions with four momenta p and q and isospins α and β, respectively, are scattered into a final state with four momenta p' and q' and isospins α' and β', respectively.

Let us define the $\mathcal{G}$'s by

$$S(p_\alpha,q_\beta\rightarrow p_{\alpha'}',q_{\beta'}') = (2i/\pi)\delta^{(4)}(p+q-p'-q')(1/\omega^2)\mathcal{G}_{\alpha\beta,\alpha'\beta'}, \quad (1)$$

$$\mathcal{G}_{\alpha\beta,\alpha'\beta'}\equiv(\alpha\beta|P_0|\alpha'\beta')\mathcal{G}_0 + (\alpha\beta|P_1|\alpha'\beta')\mathcal{G}_1+(\alpha\beta|P_2|\alpha'\beta')\mathcal{G}_2, \quad (2)$$

where ω is the energy of either pion in the bary-centric system and

$$\begin{aligned}(\alpha\beta|P_0|\alpha'\beta')&=\tfrac{1}{3}\delta_{\alpha'\beta'}\delta_{\alpha\beta},\\ (\alpha\beta|P_1|\alpha'\beta')&=\tfrac{1}{2}(\delta_{\alpha\alpha'}\delta_{\beta\beta'}-\delta_{\alpha\beta'}\delta_{\beta\alpha'}),\\ (\alpha\beta|P_2|\alpha'\beta')&=\tfrac{1}{2}(\delta_{\alpha\alpha'}\delta_{\beta\beta'}+\delta_{\alpha\beta'}\delta_{\beta\alpha'}-\tfrac{2}{3}\delta_{\alpha\beta}\delta_{\alpha'\beta'})\end{aligned} \quad (3)$$

are the projection operators for the isotopic spin states $I=0$, 1 and 2, respectively. Then $\mathcal{G}_I$ is simply connected with the phase-shift $\delta_l^{(I)}$ by the formula

$$\mathcal{G}_I=\left(\frac{\nu+1}{\nu}\right)^{\frac{1}{2}}\sum_{l=0}^{\infty}(2l+1) \times\exp(i\delta_l^{(I)})\sin\delta_l^{(I)}P_l(\cos\theta), \quad (4)$$

where

$$\nu=\omega^2-1 \quad (5)$$

is the square of the magnitude of the pion-momentum in the bary-centric system and we have put the pion mass equal to unity.

Now, the pion-pion interaction Hamiltonian is given by

$$H_1=4\pi\lambda(\phi_\alpha\cdot\phi_\alpha)^2. \quad (6)$$

First, let us consider a sum only of chain-diagrams (Fig. 1). These diagrams only contribute to the S waves. We shall show shortly that this approximation gives exactly the same result obtained by Chew and Mandelstam.[1]

In our approximation, $\mathcal{G}_{\alpha\beta,\alpha'\beta'}$ defined by Eq. (1) satisfies the following algebraic equation:

$$\mathcal{G}_{\alpha\beta,\alpha'\beta'}=-\lambda\{\delta_{\alpha\beta}\delta_{\alpha'\beta'}+\delta_{\alpha\beta'}\delta_{\beta\alpha'}+\delta_{\alpha\alpha'}\delta_{\beta\beta'}\} +(\lambda/\pi)J\{\delta_{\alpha\beta}\delta_{\alpha''\beta''}+\delta_{\alpha\beta''}\delta_{\beta\alpha''}+\delta_{\alpha\alpha''}\delta_{\beta\beta''}\} \times\mathcal{G}_{\alpha''\beta'',\alpha'\beta'}, \quad (7)$$

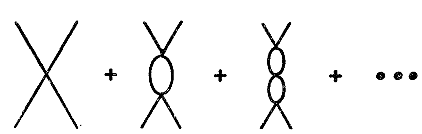

FIG. 1. Chain diagrams.

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[1] G. F. Chew and S. Mandelstam, University of California Radiation Laboratory Report UCRL-8728 (unpublished).
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

where

$$J=\left(\frac{i}{\pi^2}\right)\int\int d^4p''d^4q''\,\delta(p''+q''-p-q)\times\frac{1}{p''^2+\mu^2}\frac{1}{q''^2+\mu^2}. \quad (8)$$

Actually, J is logarithmically divergent and we must subtract the divergence according to the usual procedure. We do this subtraction according to the prescription given by Chew and Mandelstam.[1] Namely, we subtract from J the value of J at the point

$$(p+q)^2=(p-q')^2=(p-p')^2=-\tfrac{4}{3}. \quad (9)$$

It may be worthwhile to point out that this subtraction of divergences from the diagrams in Fig. (1) does not by itself give the desired form $-4\pi\delta\lambda\phi^4$. This is due to the fact that we have not included the cross-diagrams.[3] However, this does not lead to any difficulties, since according to the renormalization procedure,[4] we may consistently drop all divergences in every diagram under consideration provided we use the physical masses. If we were to consider *all* diagrams, this renormalization subtraction is equivalent to a counter term $-4\pi\delta\lambda\phi^4$. If we now consider the approximation of picking up only the chain diagrams of Fig. (1) we may simply drop the divergences in these diagrams; and this is precisely what is done here. Using Eqs. (2) and (3), we can solve Eq. (7) to obtain the formula:

$$\frac{1}{g_I}=\frac{1}{a_I}-\frac{2}{\pi}\left\{\sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}-\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}}\times\ln[\nu^{\frac{1}{2}}+(\nu+1)^{\frac{1}{2}}]\right\}-i\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}}, \quad (10)$$

where

$$a_0=-5\lambda,\quad a_2=-2\lambda. \quad (11)$$

In view of Eq. (4) and noting that we have only S waves here, we get:

$$\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}}\cot\delta_S{}^{(I)}=\frac{1}{a_I}-\frac{2}{\pi}\left\{\sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}-\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}}\ln[\nu^{\frac{1}{2}}+(\nu+1)^{\frac{1}{2}}]\right\}. \quad (12)$$

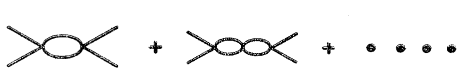

FIG. 2. Cross diagrams.

This is exactly the same result obtained by Chew and Mandelstam.[1]

Now, we will calculate the contribution from the cross-diagram (ladder-graph), which are pictured in Fig. (2). This graph contains a contribution to the P-wave phase-shifts. Evaluation of these diagrams is obtained most simply by changing $q\rightleftarrows -p'$ and $\beta\rightleftarrows\alpha'$ (or $q\rightleftarrows -q'$ and $\beta\rightleftarrows\beta'$) in the $g_{\alpha\beta,\alpha'\beta'}$ of Fig. 1 already given. However, the partial wave expansions are quite complicated so that we calculate them by perturbation theory. The formula (12) now becomes:

$$\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}}\cot\delta_S{}^{(I)}=\frac{1}{a_I}-\frac{2}{\pi}\left\{\sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}-\left(\frac{\nu}{\nu+1}\right)^{\frac{1}{2}}\ln[\sqrt{\nu}+(\nu+1)^{\frac{1}{2}}]\right\}+\frac{1}{\pi}b_I\left[K(\nu)-\sqrt{2}\tan^{-1}\frac{1}{\sqrt{2}}\right], \quad (13)$$

where

$$b_0=12/5,\quad b_2=9, \quad (14)$$

and $K(\nu)$ is given by

$$K(\nu)=\int_0^1 dx\left\{\frac{1}{(1-x^2)}-\frac{1}{\nu}\frac{x^2}{(1-x^2)^2}\ln[1+\nu(1-x^2)]\right\}=\frac{1}{2\nu}\left\{\ln[\sqrt{\nu}+(\nu+1)^{\frac{1}{2}}]+[\nu(\nu+1)]^{\frac{1}{2}}\right\}^2-\tfrac{1}{2}(\nu+2). \quad (15)$$

For small values ν, $K(\nu)$ behaves as

$$K(\nu)\simeq 1+\tfrac{1}{6}\nu+O(\nu^2). \quad (16)$$

As we can see from Eqs. (13) and (14), the correction from the cross-diagram is opposite in sign and quite large for the $I=2$ state (about 50% correction) but not so large for the $I=0$ state (about 20% correction). Thus, the argument concerning the unliklihood of a resonance[1] in the S state will be maintained. However, because of the rather large corrections, especially for the $I=2$ state, it is desirable to investigate the cross-diagrams more carefully, not by perturbation theory as we have done.

In a similar approximation, we obtain the corresponding formula for the P wave:

$$\left(\frac{\nu^3}{\nu+1}\right)^{\frac{1}{2}}\cot\delta_P=\frac{1}{a_1}, \quad (13')$$

where

$$a_1=(10/9\pi)\lambda^2M(\nu),$$

[3] The author owes this remark to Professor C. J. Goebel.

[4] A. Salam, Phys. Rev. 82, 217 (1951).

and $M(\nu)$ is given by

$$M(\nu)=-\frac{18}{\nu^2}\int_0^1 dx\,\frac{x^2}{(1-x^2)^2}\times\left\{2-\frac{2+\nu(1-x^2)}{\nu(1-x^2)}\ln[1+\nu(1-x^2)]\right\}$$
$$=9\left\{\frac{\nu-1}{2\nu^2}-\frac{2\nu+1}{2\nu^3}\ln^2[\sqrt{\nu}+(\nu+1)^{\frac{1}{2}}]+\frac{[\nu(\nu+1)]^{\frac{1}{2}}}{\nu^3}\ln[\nu^{\frac{1}{2}}+(\nu+1)^{\frac{1}{2}}]\right\}. \quad (14')$$

For small ν, $M(\nu)$ behaves as

$$M(\nu)\simeq 1-\tfrac{2}{5}\nu+O(\nu^2).$$

Unfortunately, this value of a_1 at zero-kinetic energy $\nu=0$ is too small by a factor 50, compared to the value used by Rodberg[5] to give a good fit of the double pion production data. However, we must bear in mind the possibility of a P-wave resonance[1] and the resulting enhancement of the P-wave amplitude. To find a resonance it would be necessary to calculate the cross-diagrams nonperturbatively so that we can make no statement. On the other hand, a P-wave resonance, if any, may not be so favorable as far as the explanation of the second resonance of pion-nucleon scattering is concerned,[6] since it will not give a dominant $I=\frac{1}{2}$ state compared to the $I=\frac{3}{2}$ state. On the basis of Peierls' model,[7] Schnitzer and Goebel[8] investigated the possible explanation of this second resonance. If we take S-wave pion-pion scattering to be dominant, the $I=\frac{1}{2}$ state for two pions and one nucleon system is automatically favored on kinematical grounds. However, it is rather difficult to explain a sharp rise of the double pion production cross section by means of this model. One possible way out of the difficulty is to assume an S-wave resonance[6] in the two-pion state with $I=0$, although this seems somewhat unlikely.[1]

(3) PION-KAON SCATTERING

The same technique given in the previous section can be applied equally well for pion-kaon scattering. However, the main ambiguity here is that we do not have a convenient unique subtraction prescription such as Eq. (9), due to the lack of exchange symmetry between pion and kaon.

Now consider the scattering of a pion with momentum p and isospin α by a kaon with momentum q into a corresponding final state with momenta p' and q' and with pion-isospin β. Defining the $\mathcal{G}$'s by

$$S(p_\alpha+q\rightarrow p_\beta'+q')=\left(\frac{i}{2\pi}\right)\delta(p+q-p'-q')\frac{1}{E\omega}\mathcal{G}_{\alpha,\beta}, \quad (15')$$

$$\mathcal{G}_{\alpha,\beta}=\langle\alpha|P_1|\beta\rangle\mathcal{G}_1+\langle\alpha|P_3|\beta\rangle\mathcal{G}_3, \quad (16)$$

where ω and E are the energies of pion and kaon in the bary-centric system, respectively, and

$$\begin{aligned}\langle\alpha|P_1|\beta\rangle&=\tfrac{1}{3}\tau_\alpha\tau_\beta,\\ \langle\alpha|P_3|\beta\rangle&=\delta_{\alpha\beta}-\tfrac{1}{3}\tau_\alpha\tau_\beta,\end{aligned} \quad (17)$$

are the projection operators into the total isospic spin $I=\frac{1}{2}$ and $I=\frac{3}{2}$ at the pion-kaon system, respectively. Then, $\mathcal{G}_I$ is connected with the phase-shifts $\delta_l^{(I)}$ by

$$\mathcal{G}_I=\frac{(\nu+1)^{\frac{1}{2}}+(\nu+m^2)^{\frac{1}{2}}}{\sqrt{\nu}}\sum_{l=0}^{\infty}(2l+1)\exp(i\delta_l^{(I)})\times\sin\delta_l^{(I)}\cdot P_l(\cos\theta), \quad (18)$$

where

$$\omega=(\nu+1)^{\frac{1}{2}},\quad E=(\nu+m^2)^{\frac{1}{2}}, \quad (19)$$

and m is the rest mass of the kaon. Now, the pion-kaon interaction is given by

$$H_1=4\pi\lambda'(\phi_\alpha\phi_\alpha)(\phi_k^*\phi_k). \quad (20)$$

The procedure is almost the same as in the previous section and so we do not give any details. In the chain-approximation, in which we sum up the diagrams given in Fig. 3, we obtain the following formula for the S-wave phase-shift.

$$\frac{\sqrt{\nu}}{(\nu+1)^{\frac{1}{2}}+(\nu+m^2)^{\frac{1}{2}}}\cot\delta_S^{(I)}$$
$$=-\frac{1}{\lambda'}+\frac{1}{\pi}\left\{\ln m\left(\frac{(\nu+m^2)^{\frac{1}{2}}}{(\nu+1)^{\frac{1}{2}}+(\nu+m^2)^{\frac{1}{2}}}-\frac{\alpha^2+m^2-1}{2\alpha^2}\right)+\frac{\sqrt{\nu}}{(\nu+1)^{\frac{1}{2}}+(\nu+m^2)^{\frac{1}{2}}}\ln\left[[\sqrt{\nu}+(\nu+1)^{\frac{1}{2}}]\cdot\left(\frac{(\nu+m^2)^{\frac{1}{2}}+\sqrt{\nu}}{m}\right)\right]-\frac{\beta}{2\alpha^2}\left[\tan^{-1}\left(\frac{\alpha^2+m^2-1}{\beta}\right)+\tan^{-1}\left(\frac{\alpha^2+1-m^2}{\beta}\right)\right]\right\}, \quad (21)$$

where α^2 is the subtraction parameter; we subtract the

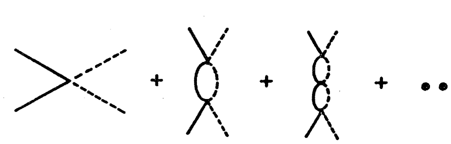

FIG. 3. Chain diagrams for pion-kaon scattering.

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[5] L. S. Rodberg, Phys. Rev. Letters **3**, 58 (1959).
[6] The author owes this remark to Mr. Schnitzer and Dr. Sakita.
[7] R. F. Peierls, Phys. Rev. **111**, 1373 (1958).
[8] H. Schnitzer and C. J. Goebel (private communication).

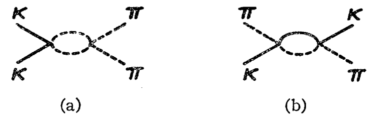

FIG. 4(a) Cross diagram contributing to Eq. (24); (b) Cross diagram neglected in Eq. (24).

divergence at the point

$$(p+q)^2=(p-q')^2=-\alpha^2,$$
$$(p-p')^2=+\gamma^2=2(\alpha^2-1-m^2). \tag{22}$$

When $m=1$ the natural choice would be $\alpha^2=-\gamma^2=+\frac{4}{3}$ as in the pion-pion scattering case [see Eq. (9)]. However, in general, there is no unique reasonable choice for α^2. β in Eq. (21) is defined by

$$\beta^2=[(m+1)^2-\alpha^2][\alpha^2-(m-1)^2], \tag{23}$$

and we chose $\beta^2>0$.

It should be noted that the formula Eq. (21) is the same for both isotopic spin $I=\frac{1}{2}$ and $I=\frac{3}{2}$. This equivalence of phase-shifts for the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ states is always true independent of any approximation, as long as we use only the interaction Eq. (20). We may also add the pion-pion interaction term Eq. (6) without changing this conclusion. This follows because both interactions Eq. (20) and Eq. (6) do not contain the isotopic spin matrices τ_α.

Now let us proceed to the cross-diagrams. In this case, we have two types of diagrams: Figs. 4(a) and Fig. 4(b). Actually, the calculation of Fig. 4(b) is rather complicated. However, since the kaon is heavy compared to the pion, Fig. 4(b) is unfavorable, compared to Fig. 4(a). Moreover, we may hope that the pion-pion interaction is much stronger than the pion-kaon interaction, and accordingly, we neglect the contribution from Fig. 4(b). Then, the correction to the formula Eq. (21) from Fig. 4(a) is obtained by adding to the right-hand side of Eq. (21) the term

$$(10/\pi)(\lambda/\lambda')[K(\nu)-J_0], \tag{24}$$

where λ is the renormalized pion-pion interaction constant defined in the previous section [Eq. (6)], $K(\nu)$ is given by Eq. (16) of the previous section, and J_0 is defined by

$$J_0=\theta\ln[(2\theta+1)/(2\theta-1)] \quad \text{if} \quad \theta^2=\tfrac{1}{4}-1/\gamma^2>0,$$
$$J_0=2\theta\tan^{-1}(1/2\theta) \quad \text{if} \quad \theta^2=(1/\gamma^2)-\tfrac{1}{4}>0. \tag{25}$$

In the above, γ^2 is the subtraction parameter defined by Eq. (22).

Similarly, for the P wave, we find

$$\frac{(\nu^3)^{\frac{1}{2}}}{(\nu+1)^{\frac{1}{2}}+(\nu+m^2)^{\frac{1}{2}}}\cot\delta_P{}^{(I)}=\frac{9}{5}\cdot\frac{\pi}{\lambda\lambda'}\cdot\frac{1}{M(\nu)}, \tag{26}$$

where $M(\nu)$ is given by Eq. (14) and in this case, also, the phase-shifts are the same for both the $I=\frac{1}{2}$ and $I=\frac{3}{2}$ states.

(4) CONCLUDING REMARKS

As we have shown in the previous sections, the result of Chew and Mandelstam for pion-pion scattering can quite easily be obtained by means of Feynman's graphical method. We see also that the approximation employed by these authors is nothing but the chain approximation. Furthermore, with our method, we can evaluate corrections from the cross-diagrams more easily.

Of course, in principle, the method used by Chew and Mandelstam is much more general than that given in the previous sections, since their assumptions are less restrictive than ours, e.g., they do not assume the consistency at the Hamiltonian formalism which we have used. In fact, the use of any Hamiltonian is completely unnecessary in their formalism. In addition, the question whether the pion is an elementary or a bound particle is completely irrelevant in their formalism, whereas this is not the case in our method. However, from the point of view of calculation, there seems to be little difference between their method and ours. Indeed, it appears that our rather old-fashioned method (in the sense that we do not use dispersion theory) can more easily yield numerical answers.

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[9] *Note added in proof.*—After this work had been done, it was called to the author's attention that a similar analysis was done by G. Bonnevay: Thése, Ecole Normale Superieure Paris, 1958 (unpublished). The author would like to thank Dr. Bonnevay for calling his attention to it.