Sm^{152} and Sm^{154} are 27 and 74 times larger,[1] respectively, and the (n,α) cross sections on these same nuclides exhibit the still larger deviations of 99 and 165, respectively. For (n,p) reactions Brown and Muirhead[18] have calculated cross sections from a direct interaction model which assumes that these interactions, in which no intermediate nucleus is formed, occur by collision with nucleons throughout the whole nuclear volume with equal probability for all nucleons. In Table III we have listed for certain (n,p) reactions a comparison with these direct interaction calculations.

Although this direct interaction model gives better agreement with experiment (Table III),[17] it cannot be developed to include the (n,α) process, nor does it take into account that surface reactions are considerably more probable, especially in the heavier mass regions, than are direct interactions with nucleons deep in the nucleus.[19] Wilkinson[20] suggests that nucleon clusters, e.g., alpha-particles and deuterons) tend to exist in the region of low nuclear binding in the diffuse nuclear surface. Such a tendency for preformed alpha-particle clusters to exist in the surface suggests itself as a possible explanation for the large (n,α) cross sections in the heavy mass region. Such a model also might be a possible explanation for the near integral multiple decrease in (n,α) cross sections in the low Z region at 14 Mev pointed out by Levkovskii.[2] Since the number of such clusters would be much lower in the low Z region, any change in the number of clusters available to the incoming projectile (as for example with increasing A at constant Z) would be much more marked in the cross sections than would be the case at high Z.

ACKNOWLEDGMENTS

We are indebted to Mr. J. Wray for operation of the accelerator during these experiments.

[18] G. Brown and H. Muirhead, Phil. Mag. **2**, 473 (1957).
[19] G. B. Chadwick, S. A. Duranni, P. B. Jones, J. W. G. Wignall, and D. H. Wilkinson, Phil. Mag. **3**, 1193 (1958); and P. B. Jones, Phil. Mag. **3**, 33 (1958).
[20] D. H. Wilkinson, Phil. Mag. 4, 215 (1959).

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Decay of Be^{9*} (2.43-Mev State)*†

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The decay of the 2.43-Mev state of Be^9 is treated theoretically. Of the open two-body decay channels all but one involve a nuclear state, the energy of which is not well defined. The usual formalisms have been generalized to take this into account. The estimate of the decay rates is made by means of a variational internal wave function for the Be^{9*} state, based upon the alpha-particle model. It is found that the principal mode of decay is to He^5+He^4. Model—dependent arguments are given to show that decay to the ground state of Be^8 should be inhibited. Furthermore, the momentum and angular distributions of alphas emitted in the decay through several two-particle decay modes are computed. These latter calculations do not assume any specific nuclear model, but depend on the weak assumption that the state is excited by a direct reaction. Comparison with recent measurements indicates that in addition to the He^5+He^4 decay, approximately 7% of the decay occurs to the ground state of Be^8, which is consistent with our calculations.

I. INTRODUCTION

THE decay of Be^{9*} excited by various reaction mechanisms has been the source of several studies in the past. For example, early investigations[1] of the decay from the 2.43-Mev excited level have indicated that it proceeds mainly by emission of a neutron to the ground state of Be^8. However, recent coincidence measurements by Bodansky, Eccles, and Halpern[2] (hereafter referred to as BEH) have set an upper limit of 10% on decays through this channel and have concluded that emission occurs primarily by means of other processes. We shall analyze the decay of the 2.43-Mev state theoretically and compare predictions with experiment. Since this decay occurs in part through intermediate states (e.g., He^5), the energy of which is not well defined, we have had to generalize the usual emission width relations to take this into account explicitly.[3] This has been done along the lines suggested by Watson.[4]

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‡ Presently on leave at the Institute for Theoretical Physics, Copenhagen, Denmark.
[1] G. A. Dissanaike and J. O. Newton, Proc. Phys. Soc. (London) **A65**, 675 (1952). G. M. Frye and J. H. Gammel, Phys. Rev. **103**, 328 (1956).
[2] D. Bodansky, S. F. Eccles, and I. Halpern, Phys. Rev. **108**, 1019 (1957).
[3] A. C. Riviere, Nuclear Phys. **2**, 81 (1956/57); T. A. Welton, Phys. Rev. **95**, 302 (1954).
[4] K. M. Watson, Phys. Rev. **88**, 1136 (1952).

TABLE I. Angular momentum states available to the residual nucleus and the emitted nucleon.

Mode	Residual nucleus	Angular momentum of residual nucleus	Emitted particle	Angular momentum of emitted particle
(1a)	Be^8	0	n	3
(1b)	Be^8	2	n	1, 3, 5,
(2)	He^5	1	He^4	2, 4
		2		1, 3, 5
		3		0, 2, 4, 6

The principal two-body decay channels available to the decay of the 2.43-Mev level of Be^9 are[5]

$$\begin{aligned}
&(1a) \quad Be^{9*} \rightarrow Be^8 + n \\
&\qquad\;\; Be^8 \rightarrow He^4 + He^4, \\
&(1b) \quad Be^{9*} \rightarrow Be^{8*} + n \\
&\qquad\;\; Be^{8*} \rightarrow He^4 + He^4, \\
&(2) \quad Be^{9*} \rightarrow He^5 + He^4 \\
&\qquad\;\; He^5 \rightarrow He^4 + n, \\
&(3) \quad Be^{9*} \rightarrow Be^9 + \gamma.
\end{aligned}$$

The decay modes (1b) and (2) are allowed because of the broad widths of the 2.9-Mev state of Be^8 and of the ground state of He^5, respectively (see Fig. 1).

Because the absolute magnitude of the partial widths or decay rates through the above channels cannot be calculated without assuming a specific model to describe the internal wave functions of Be^{9*}, we shall first compute the energy and angular distributions of the alpha particles emitted in processes (1a), (1b), and (2). These processes depend primarily on a knowledge of the angular momentum of the initial state and the final state wave functions. They are, therefore, more readily believable and one might hope that the experimental measurements of BEH can differentiate between some of the processes in question. Such will indeed turn out to be the case. It is for the above reason that we postpone to the last section the computation of the absolute magnitudes of the decay rates which we base upon a variational wave function using the alpha-particle model of Be^9. There are, however, some simple qualitative arguments which demonstrate the decay through mode (1a) should be inhibited. These arguments are based upon angular momentum and parity considerations applied to current nuclear models.

II. QUALITATIVE DESCRIPTION OF DECAY

A spin and parity assignment of $\frac{5}{2}-$ for the 2.43-Mev level is strongly suggested by both pickup[6] and inelastic scattering.[7] This level assignment is also consistent

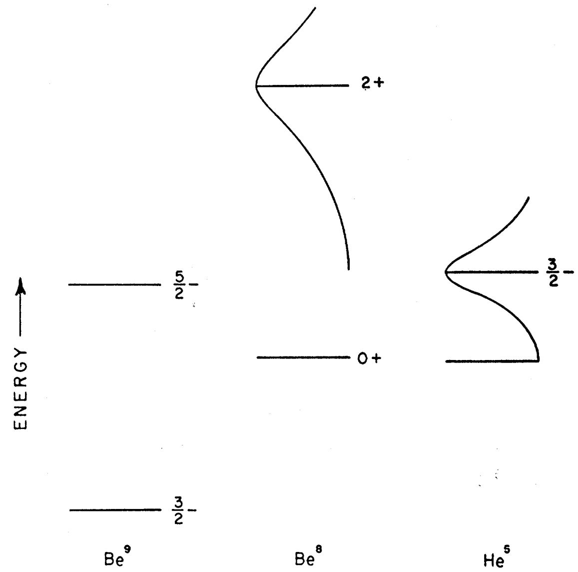

FIG. 1. Relative energies of the states of the residual nuclei of the decay modes, showing their relative widths and positions.

with the predictions of the alpha-particle model,[8] and the intermediate coupling shell model,[9] and we shall assume it to be correct. The lowest orbital angular momentum combinations are given in Table I for the two-particle decay modes considered.

In the jj, LS, or intermediate coupling shell model without configuration mixing, the Be^9 ground state is composed of four angular momentum S-state and five P-state nucleons. The low excited states of negative parity, such as the 2.43-Mev level, are than explained by recoupling the nucleons in the P shell to add to a different angular momentum. These low states cannot arise by means of the excitation of a single nucleon or a pair of nucleons to a new shell because of the large energy required. Thus, except for center-of-mass effects,[10] the wave functions of the nucleons in the 2.43-Mev state of Be^9 have no orbital angular momentum $l=3$ component, and reference to Table I shows that decay by mode (1a) is then forbidden. It furthermore follows from this table that the same argument does not restrict decay modes (1b) and (2).

In the alpha-particle model for the ground state of Be^9, the neutron is strongly coupled to the motion of the two alphas. In a coordinate system in which the axis of the alpha particles is the z axis, the neutron is in a $P_{\frac{3}{2}}$ state with a projection on the z axis of $\pm\frac{3}{2}$. In a deformed nucleus as this model describes, the total angular momentum, j, is not a good quantum number but is approximately so because of a large energy

[5] We make the usual assumption that three-body decay can be neglected when two-body decay can occur.
[6] F. L. Ribe and J. D. Seagrave, Phys. Rev. **94**, 934 (1954).
[7] G. W. Farwell and D. D. Kerlee, Bull. Am. Phys. Soc. **1**, 20 (1956); R. G. Summers-Gill, University of California Radiation Laboratory Report UCRL-3388, 1956 (unpublished); S. W. Rasmussen, Phys. Rev. **103**, 186 (1956).
[8] A review of experimental and theoretical literature is given by J. S. Blair and E. M. Henley, Phys. Rev. **112**, 2029 (1958).
[9] D. Kurath, Phys. Rev. **101**, 216 (1956).
[10] The admixture of $l=3$ due to center-of-mass effects is expected to be of the order $(M_n/M_{Be^9})^2 \approx 1\%$.

difference to states of different j. The 2.43-Mev state is given by this model as a rotational state of spin $\frac{5}{2}$- with the same neutron $P_{\frac{3}{2}}$ wave functions as was the ground state. The transformation of the neutron wave function to a space-fixed axis does not introduce any new neutron angular momentum states, and therefore, mode (1a) is again forbidden. In transforming the neutron wave function to the space-fixed frame one finds that the two alpha particles have a spin-2 component and this means that mode (1b) is allowed. The same conclusion for mode (2) is arrived at as for the shell model. The decay is not restricted.

It is easy to show that similar statements and conclusions can be made for the Nilsson[11] wave functions based on the Bohr-Mottelson model. In conclusion if one does not allow configuration mixing of neutron $l=3$ waves in the description of the 2.43-Mev state of Be^9, then decay to the ground state of Be^8 is inhibited by angular momentum and parity conservation alone. The other decay modes are not hindered. In order to obtain an approximate amount of decay through the Be^8 ground state, we shall in our later calculations introduce some $l=3$ configuration by means of the alpha-particle model.

III. DECAY MODEL

A. Method of Calculation

The decay of the 2.43 level of Be^{9*} finally results in three particles, two alpha particles and a neutron. In the description of the decay we make the assumption of a two-step decay in which a first particle is emitted and the residual nucleus subsequently breaks up. We furthermore assume that the decay into each mode can be described independently of the others. The first assumption is realized if the transit time in Be^9 of the first emitted nucleon is short compared to the lifetime of the residual nucleus as is the case for the decay to the Be^8 ground state. If this condition is not met but the particles forming the residual nucleus have a strong final state interaction[4] compared with the intitial Be^{9*} decay interaction, then one effectively still has a two-step decay process. The second assumption implies that the states which result from the primary decay are orthogonal to one another. This assumption is valid for mode (1a) but not directly justifiable for mode (1b) relative to mode (2). These modes become identical if the recoil effect of the neutron motion is completely neglected. When an alpha-particle model is used to describe Be^9, the neutron wave function does not remove the lack of orthogonality referred to above because of the large energy widths of the states. Unless this second assumption is made, however, it becomes increasingly difficult to discuss the breakup into channels (1b) and (2). Since the decay via the excited state of Be^8 occurs at energies which are at least three times the half-width from resonance we expect the probability of decay through this channel to be small. In this sense the second assumption is unimportant and the calculations will bear this out.

Inspection of Table I indicates which angular momentum combinations must be considered in the decay of the $\frac{5}{2}$-state of Be^{9*} at 2.43 Mev. Because of the small energy released in any of the primary decays, the angular momentum barrier effectively limits the open channels to the lowest l value listed for each channel. Emission through higher relative angular momentum channels, when allowed, can be shown to be less probable by at least two orders of magnitude. One further factor to consider is the strength of the final state interaction of the residual nucleus. In the He^5 states of $L\geq 2$ one has weak interactions in which the first assumption is violated. These cases are to be thought of as three-body decays. Because of the weak final state interactions they will have a small probability and we shall neglect them.

B. Expansion of the Wave Function

The wave function of the decay nucleus, Ψ_λ, is an eigenfunction of the total Hamiltonian, H,

$$H\Psi_\lambda=E_\lambda\Psi_\lambda, \tag{1}$$

and can be expanded into a complete orthonormal set,[12] $\psi_c(\mathbf{r})\phi_p(\mathbf{R})$

$$\Psi_\lambda=\sum_{cp} C_{\lambda;cp}\psi_c(\mathbf{r})\phi_p(\mathbf{R}). \tag{2}$$

Here $\psi_c(\mathbf{r})$ describes the stationary states of the residual nucleus and $\phi_p(\mathbf{R})$ is the wave function of the emitted particle. It has the form $\phi_p(R)=v_p(KR)Y_p(\Omega_p)$, where $v_p(KR)$ satisfies the boundary condition of an outgoing wave of wave number K. The set $\psi_c(\mathbf{r})\phi_p(\mathbf{R})$ is a solution of the wave equation

$$H_0\psi_c(\mathbf{r})\phi_p(\mathbf{R})=E_{cp}\psi_c(\mathbf{r})\phi_p(\mathbf{R}), \tag{3}$$

where H_0 is the Hamiltonian of the residual nucleus plus that of the emitted particle moving in an average potential of this nucleus. Since the sets Ψ_λ and $\psi_c\phi_p$ are both orthonormal and complete sets, the expansion coefficients satisfy the usual relations,

$$\sum_{cp} C_{\lambda;cp}C_{\lambda';cp}=\delta_{\lambda\lambda'}, \tag{4}$$

$$\sum_\lambda C_{\lambda;cp}C_{\lambda;c'p'}=\delta_{cc'}\delta_{pp'}, \tag{5}$$

$$C_{\lambda;cp}=\int\psi_c\phi_p\Psi_\lambda d\tau_r d\tau_R. \tag{6}$$

The residual nuclei may have broad states (energywise) in which case their wave functions depend upon energy as do those of the emitted particle. We will, therefore, define the expansion coefficients to take this

[11] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd 29, No. 16 (1955).

[12] A. M. Lane, R. G. Thomas, and E. P. Wigner, Phys. Rev. 98, 693 (1955).

into account explicitly and to expose the additions of angular momentum of two particles resulting from the initial breakup. If one replaces the label λ, by the angular momentum J and writes $\psi_c(\mathbf{r})$ in the form $\psi_c(\mathbf{r})=u_c(kr)Y_c(\Omega_r)$, then

$$\Psi_J{}^M=\sum_{cp} C_{cp}{}^J(k,K)u_c(kr)v_p(KR) \times\sum_{mm'} Y_c{}^m(\Omega_r)Y_p{}^{m'}(\Omega_R)(cpmm'|JM), \quad (7)$$

where $(cpmm'|JM)$ is the Clebsch-Gordan coefficient for the addition of angular momentum c and p to that of the initial state. The wave number k is a measure of the internal energy of the residual nucleus. The coefficients $C_{cp}{}^J$ which determine the decay rate only differ from zero when energy is conserved,

$$C_{cp}{}^J(k,K)=C_{cp}{}^J(k,K)\delta(E_{ck}+E_{pk},\, E_J), \quad (8)$$

where E_{ck} and E_{pk} are the energies associated with the residual nucleus and emitted particle, respectively.

The rate of decay to a state (c,p,k,K) is given by

$$\Gamma_{cp}{}^J(k,K)/\hbar=\frac{\hbar}{2i\mu}|C_{cp}{}^J(k,K)|^2 \times\int\left(\phi_p{}^*\frac{d}{dR}\phi_p-\frac{d}{dR}\phi_p{}^*\phi_p\right)_{R=R_0}d\Omega_R \quad (9)$$

$$=(\hbar R_0/\mu)|C_{cp}{}^J(k,K)|^2|v_p(KR_0)|^2\,\mathrm{Im}f_p,$$

where

$$f_p=\frac{R_0}{v_p(KR_0)}\left(\frac{d}{dR}v_p(KR)\right)_{R=R_0}$$

and R_0 is a radius beyond which nuclear forces can be neglected. From the boundary conditions of an outgoing wave for $v_p(KR)$ one obtains

$$\mathrm{Im}f_p=KR_0W_p(KR_0), \quad (10)$$

where $W_p(KR_0)$ is the barrier transmission coefficient V_l of Blatt and Weisskopf.[13]

The energy dependence of the expansion coefficients is obtained from a knowledge of the radial dependence of the functions $u_c(kr)$ and $v_p(KR)$ inside the nucleus. Since all energies (kinetic energy of relative motion and "binding energies") are very small compared to the potential strengths which act between the particles, we can approximate the internal wave functions by

$$\begin{aligned} u_c(kr)&\cong u_c(kr_0)f_c(r),\\ v_p(KR)&\cong v_p(KR_0)g_p(R), \end{aligned} \quad (11)$$

where $f_c(r)$ and $g_p(R)$ do not depend upon k and K, respectively, and

$$u_c(kr_0)=(2/\pi)^{\frac{1}{2}}e^{i\delta}(kr_0)^{-1} \times[F_c(kr_0)\cos\delta(k)+G_c(kr_0)\sin\delta(k)] \quad (12)$$

is the normalized stationary scattering solution of two particles into which the residual nucleus decays. The functions $F_c(kr_0)$ and $G_c(kr_0)$ are the regular and irregular solutions[13] for the two particles outside their range of interaction. By means of (11) the expansion coefficients, $C_{cp}{}^J$ defined by (6) can be expressed by

$$C_{cp}{}^J(k,K)\cong u_c(kr_0)v_p(KR_0)M_{cp}, \quad (13)$$

where $M_{cp}\equiv\langle f_c(r)g_p(R)|\Psi_J\rangle$ is independent of k and K.

The decay rate, (9), becomes upon substitution of (10) and (13)

$$\Gamma_{cp}{}^J(k,K)=|v_p(KR_0)|^2|u_c(kr_0)|^2|M_{cp}|^2\Gamma_p(KR_0), \quad (14)$$

where we have defined $\Gamma_p(KR_0)$, the single-particle decay rate, by

$$\Gamma_p(KR_0)=2KR_0\gamma_{sp}{}^2W_p(KR_0), \quad (15)$$

and $\gamma_{sp}{}^2=(\hbar^2R_0/2\mu)|v_p(KR_0)|^2$ is the single-particle reduced width. The energy dependence of the decay rate (14) is determined by (a) $|u_c(kr_0)|^2$, the strength of the final state interaction of the residual nucleus, (b) $|v_p(KR_0)|^2$, the value of the outgoing particle wave functions evaluated at the radius R_0, and (c) $\Gamma_p(KR_0)$, the single-particle decay rate. The latter is proportional to the barrier transmission factor (10) and also to $|v_p(KR_0)|^2$. If we adopt for $\gamma_{sp}{}^2$, the single-particle width[14,15]

$$\gamma_{sp}{}^2\cong(\hbar^2/\mu R_0), \quad (16)$$

then this estimate requires that $|v_p(KR_0)|^2$ be given by

$$|v_p(KR_0)|^2=2/R_0{}^3, \quad (17)$$

and the final form for $\Gamma_{cp}{}^J$ is

$$\Gamma_{cp}{}^J(k,K)=(2/R_0{}^3)|u_c(kr_0)|^2|M_{cp}|^2\Gamma_p(KR_0). \quad (18)$$

IV. MOMENTUM AND ANGULAR DISTRIBUTIONS—COMPARISON WITH EXPERIMENT

The development of the previous section allows us to write for the momentum and angular dependence of the primary decay

$$\Gamma_{cp}{}^J(k,K)=2R_0{}^{-3}|M_{cp}|^2|u_c(kr_0)|^2\Gamma_p(KR_0) \times|\sum_{mm'} Y_c{}^m(\Omega_k)Y_p{}^{m'}(\Omega_K) \times(cpmm'|JM)|^2\delta(E_{ck}+E_{pK},\, E_J), \quad (19)$$

where θ_K is the angle of the direction K of the emitted particle and θ_k is the angle of the direction of the particles of the residual nucleus (see Fig. 2). If one is interested in the angular and momentum distributions in the Be^{9*} rest frame of either of the alpha particles emitted in the two successive decays (to compare with experiment), then the relative coordinates $\mathbf{k}$ and $\mathbf{K}$ (Fig. 2) must be transformed to the alpha-particle and neutron coordinates, $\mathbf{k}_{\alpha 1}$, $\mathbf{k}_{\alpha 2}$, and $\mathbf{k}_n$ and the coordinates

[13] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

[14] A. Bohr and B. Mottelson, Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd **27**, No. 16 (1953).

[15] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).

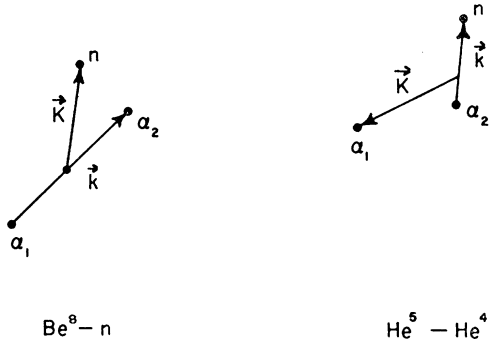

FIG. 2. Internal momentum of the particles used in computing the decay rates.

of one alpha and the neutron must be integrated out [see Appendix A].

The final state interaction is evaluated by using the scattering phase shifts of the particles involved in the residual nucleus. For the decay to the Be^8 ground state this need not be done explicitly since the extremely narrow width and the very low energy of the state causes the wave function to be approximately proportional to a delta function of the energy of the state, E_0,

$$|u_c(kr_0)|^2 \cong 2(kr_0)^{-3}E_0\delta(E_k-E_0). \tag{20}$$

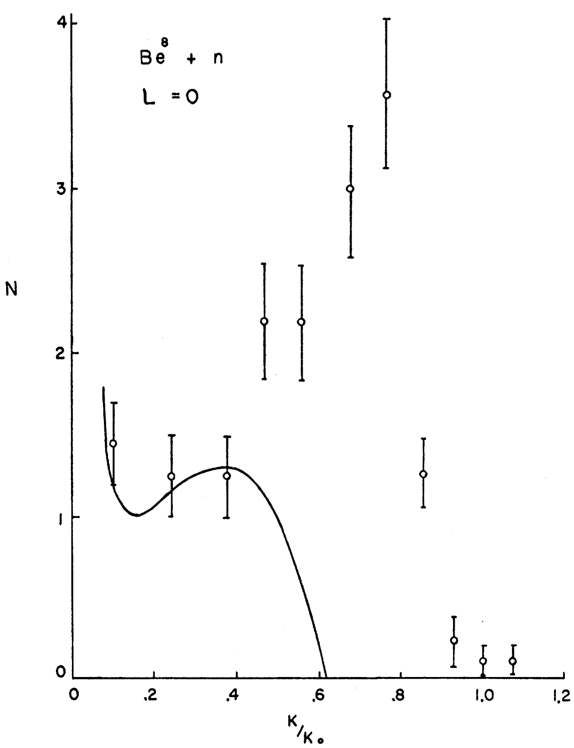

FIG. 3. Be^8+n, ground state, momentum distribution along the Be^9 recoil direction where N is the number of alpha particles per unit volume in momentum space and k/k_0 is the fraction of momentum of the alpha particle from the kinematic limit k_0. The experimental points are those of BEH.

For the $L=2$ state of Be^8, the alpha-alpha scattering phase shifts were obtained from a low-energy interpolation of an analysis of scattering data.[16] Below an E_k of 1 Mev the phase shifts were less than 1 degree but this is sufficient to cause an enhancement of the decay rate by a factor of about 4 over the free-particle wave function.

The $P_{\frac{3}{2}}$ phase shifts were obtained from the analysis of neutron-helium scattering.[17] With these values for the final state interaction and the usual form[13] for the barrier transmission factor $W_p(KR_0)$ in $\Gamma_p(KR_0)$, the momentum and angular distribution of the alpha particles can be calculated. In modes (1a) and (1b) the distribution of either alpha particle is the same since the symmetrization is taken care of in the Be^8 wave function. In the decay to mode (2) we have neglected the interference between the two alphas in He^5 and He^4 and have taken the distribution to be just the sum of the two.

In the experiment of BEH, the 2.43-Mev level of Be^9 was excited by inelastically scattered 42-Mev alpha particles. The momentum and angular dependence of alpha particles emitted by the breakup of Be^{9*}

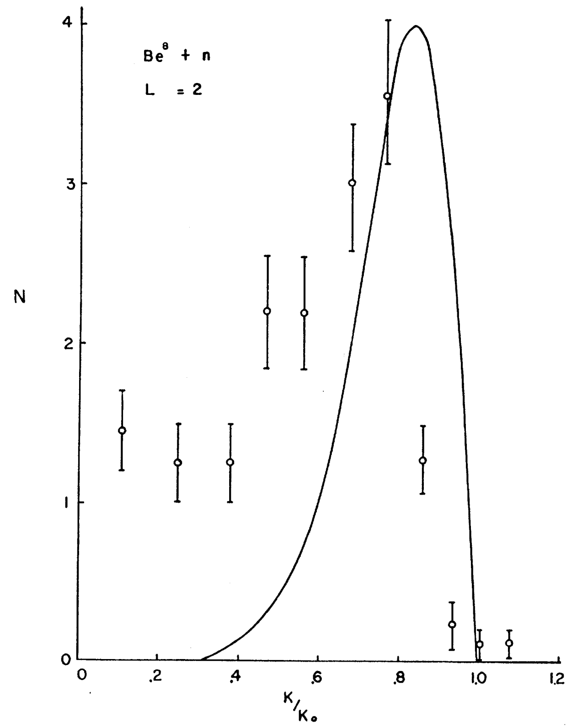

FIG. 4. Be^8+n, first excited state, momentum distribution along the Be^9 recoil direction where N is the number of alpha particles per unit volume in momentum space and k/k_0 is the fraction of momentum of the alpha particle from the kinematic limit k_0. The experimental points are those of BEH.

[16] N. P. Heydenburg and G. M. Temmer, Phys. Rev. **104**, 123 (1956); J. M. Russell, Jr., G. C. Phillips, and C. W. Reich, Phys. Rev. **104**, 135 (1956).

[17] D. C. Dodder and J. H. Gammel, Phys. Rev. **88**, 520 (1952).

were measured in coincidence with those inelastically scattered. The angular distributions of the latter[7] is in almost quantitative agreement with the predictions of simple direct interaction theories.[18] We, therefore, have used these theories to calculate the relative population of the Be^{9*} magnetic substates, M, along the recoil axis. In support of this assumption it may be worth noting that gamma-ray correlations following (P,P') and (α,α') reactions on C^{12} and Mg^{24},[19] which similarly depend upon the relative M_J population of the excited states, are in good agreement with theoretical predictions even when the inelastic angular distributions are not. Furthermore the predicted populations are not very sensitive to the nuclear model employed; both the shell model[18] and Bohr-Mottelson model[20] lead to a population proportional $(\frac{3}{2}2M0|\frac{5}{2}M)^2$ for our case. Thus the relative population of $M=\pm\frac{3}{2}$ to $M=\pm\frac{1}{2}$ is 6 to 1. The corresponding population using the alpha-particle model for Be^9 [21] is proportional to

$$[5(\tfrac{3}{2}2\tfrac{3}{2}0|\tfrac{5}{2}\tfrac{3}{2})(\tfrac{3}{2}2M0|\tfrac{5}{2}M)j_2(QR) \\ -9(\tfrac{3}{2}4\tfrac{3}{2}0|\tfrac{5}{2}\tfrac{3}{2})(\tfrac{3}{2}4M0|\tfrac{5}{2}M)j_4(QR)]^2, \quad (21)$$

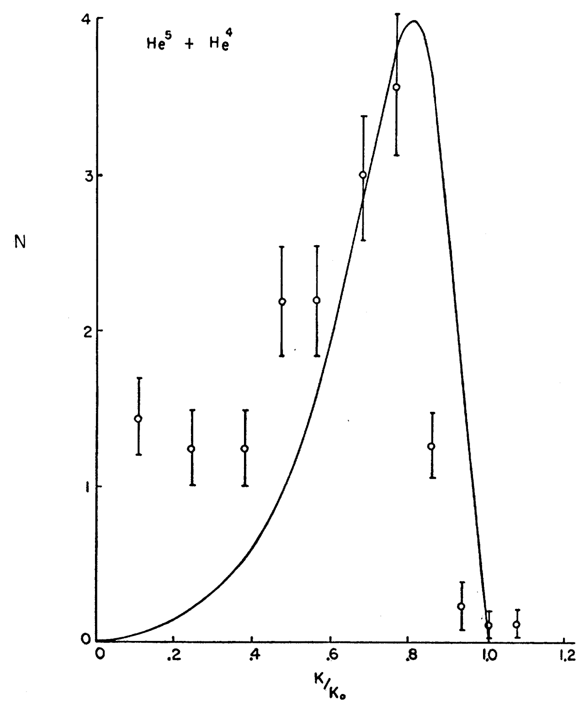

FIG. 5. He^5+He^4 momentum distribution along the Be^9 recoil direction where N is the number of alpha particles per unit volume in momentum space and k/k_0 is the fraction of momentum of the alpha particle from the kinematic limit k_0. The experimental points are those of BEH.

[18] N. Austern, S. T. Butler, and H. McManus, Phys. Rev. **92**, 350 (1953).
[19] G. B. Shook, Bull. Am. Phys. Soc. **1**, 330 (1958).
[20] S. Hayakawa and S. Yoshida, Progr. Theoret. Phys. (Kyoto) **14**, 1 (1955).
[21] J. S. Blair and E. M. Henley (private communication).

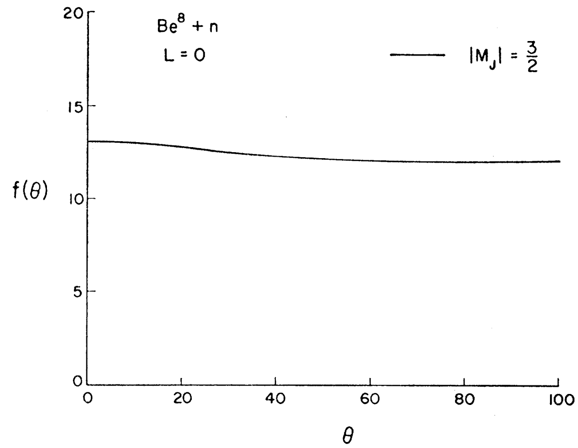

FIG. 6. Angular distribution, integrated over all momentum, of alpha particles about the recoil axis for the decay into Be^8+n, ground state.

where $j_2(QR)$ and $j_4(QR)$ are spherical Bessel functions, Q is the momentum transfer, and R is an interaction radius. For an angle of 60° of the inelastic alpha which corresponded to a maximum in the angular distribution and a radius R picked to match the angular distribution of the inelastic alphas,[7] there is almost complete cancellation of the two parts of the $M=\pm\frac{1}{2}$ term leaving the $M=\pm\frac{3}{2}$ state dominantly populated.

The calculated momentum distributions[22] of alphas emitted in the decay along the direction of recoil are compared with the results of BEH in Figs. 3–5. A separate figure is given for each breakup mode (1a), (1b), and (2) to allow detailed comparison with experiment. The cases shown are for the decay from pure $M=\pm\frac{3}{2}$ states. The addition of a $\frac{1}{6}$ fraction of the $M=\pm\frac{1}{2}$ state would make a negligible contribution along the Be^{9*} recoil direction. The theoretical angular distribution of the alphas in the Be^{9*} center-of-mass frame are shown in Figs. 6–8. The two cases shown are for $M=\pm\frac{3}{2}$ and $M=\pm\frac{3}{2}$ plus $\frac{1}{6}$ of $M=\pm\frac{1}{2}$ mixture except for the ground state of Be^8 where the difference between the two cases is negligible. The comparison of the angular distributions with the experiment of BEH is uncertain because of the large errors in the data. For the alpha group beneath the large peak in the momentum distribution, BEH obtain an angular distribution which has a minimum at an angle of 50° with the recoil direction. This minimum is of the order of the half the value at 0°. The total decay distribution is consistent with symmetry about 90°.

[22] We have consistently assumed He^5 to be in the $P_{\frac{3}{2}}$ state. Decay through a $P_{\frac{1}{2}}$ state of He^5 might be expected to compete with the $P_{\frac{3}{2}}$ state, but the energy available for decay is several Mev from the $P_{\frac{1}{2}}$ resonance energy. This reason alone reduces the ratio of the $P_{\frac{1}{2}}$ to $P_{\frac{3}{2}}$ decay to less than 8%, and the alpha-particle and Nilsson models predict very little $P_{\frac{1}{2}}$ state to be present in the initial wave function which further lowers the ratio.

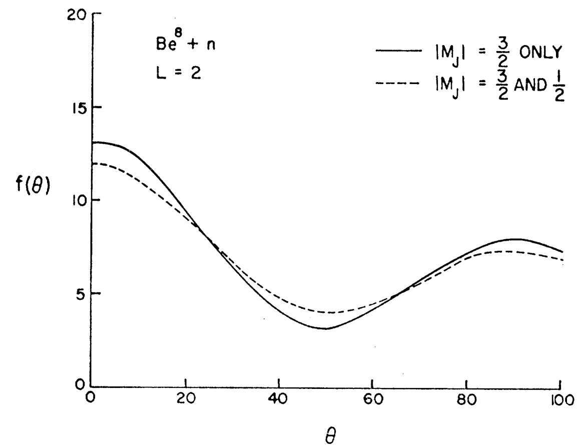

FIG. 7. Angular distribution, integrated over all momentum, of alpha particles about the recoil axis for the decay into Be^8+n, first excited state.

V. ESTIMATES OF DECAY RATES; REDUCED WIDTHS

We have been able to obtain both the energy and angular distributions of some of the final decay products from the 2.43-Mev state of Be^9 without knowledge of the explicit internal wave function for that state. In order to estimate the absolute decay rate into the various two-body channels available, we now use an alpha-particle model to describe the motion of the initial state. Aside from simplicity, this model has the advantage that it easily explains[8] the position and excitation of rotational states such as the $\frac{5}{2}$ level under discussion.

In the alpha-particle model of Be^9 the motion of the neutron is assumed to be much more rapid than that of the alpha particles. To lowest order, this motion can be considered to take place about two alpha particles separated by an equilibrium distance. The effective potential in which the neutron moves is, therefore, not spherically symmetric, and only the projection of the neutron angular momentum on the alpha-particle symmetry axis is a good quantum number. We have here a mechanism for the decay to the ground state of Be^8.

The wave functions for a state specified by a total spin J with projection M on a space-fixed axis and K on a body $(\alpha-\alpha)$ fixed axis is given by[23]

$$\Phi(JMK)=[(2J+1)/16\pi^2]^{\frac{1}{2}}R(\xi)\times[\phi_\Omega(\rho)D_{MK}{}^J(\alpha_i)+(-)^{J-j}\phi_{-\Omega}(\rho)D_{M-K}{}^J(\alpha_i)]. \quad (22)$$

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the coordinates of the centers of mass of the two alpha particles and $\mathbf{r}_n$ the neutron coordinate, then

$$\boldsymbol{\xi}=(\mathbf{r}_1-\mathbf{r}_2)/2, \quad \boldsymbol{\rho}=\mathbf{r}_n-(\mathbf{r}_1+\mathbf{r}_2)/2, \quad (23)$$

[23] This formula assumes that the states are stable. We shall assume that the life time of the 2.43-Mev state is sufficiently long so that this relation is still approximately valid inside a region of radius R_0.

(unprimed quantities signify that angles are referred to the body axis) and α_i represents the Eulerian angles which specify the orientation of the body axis in space. The wave function is separated into the neutron motion [specified by $\phi_\Omega(\rho)$] with a magnetic substate projection on the $\alpha-\alpha$ body axis of Ω, the vibrational motion of the alphas [specified by $R(\xi)$], and rotational motion [specified by the symmetric top function, $D_{MK}{}^J(\alpha_i)$]. The second term in (22) is required for the Bose-Einstein statistics of the alpha particles.

With the assumption of small vibrational motion of the two alphas, one obtains the usual formula for the spacing of the rotational energy

$$E=[J(J+1)-J_0(J_0+1)](\hbar^2/2I_0), \quad (24)$$

where J and J_0 are the spin of the excited state and ground state, respectively, $I_0=2M\xi_0{}^2$, and $2\xi_0$ is the equilibrium spacing of the two alpha particles. The ground state of Be^9 is specified by the quantum numbers

$$J_0=K=\tfrac{3}{2},$$

and the first rotational state at 2.43 Mev by

$$J=\tfrac{5}{2}, \quad K=\tfrac{3}{2}.$$

The neutron wave function satisfies the wave equation

$$[K+V(|\boldsymbol{\rho}-\boldsymbol{\xi}|)+V(|\boldsymbol{\rho}+\boldsymbol{\xi}|)]\phi_\Omega(\boldsymbol{\rho},\xi)=E(\xi)\phi_\Omega(\boldsymbol{\rho},\xi), \quad (25)$$

where K is the kinetic energy of the neutron and $V(|\boldsymbol{\rho}+\boldsymbol{\xi}|)$ and $V(|\boldsymbol{\rho}-\boldsymbol{\xi}|)$ are the neutron-alpha interactions. These potentials were assumed to be of a Gaussian form

$$V(x)=-A_j\exp[-\beta x^2], \quad (26)$$

where x represents the neutron-alpha distance and j is the total spin of the neutron.

The parameters which give a best fit of the p-wave scattering of protons on alphas are[24]

$$(1/\beta)^{\frac{1}{2}}=2.3\times10^{-13}\text{ cm}, \quad A_{\frac{3}{2}}=53.17\text{ Mev}, \quad A_{\frac{1}{2}}=35.61\text{ Mev}. \quad (27)$$

To solve the Schrödinger equation of the neutron, we take a trial wave function

$$\phi_{\frac{3}{2}}(\rho,\xi)=a_1[\phi_1(\boldsymbol{\rho}+\boldsymbol{\xi})+\phi_1(\boldsymbol{\rho}+\boldsymbol{\xi})]+a_0\phi_0(\boldsymbol{\rho}), \quad (28)$$

where

$$\phi_1(x)=\phi_0(x)=[8(2\alpha)^{\frac{5}{2}}/3\pi^{\frac{1}{2}}]^{\frac{1}{2}}x\exp(-\alpha x^2)\mathcal{Y}_{\frac{3}{2}}{}^{\frac{3}{2}}(\theta_x).$$

The form of (28) was chosen so as to mix in implicitly higher than $l=1$ angular momentum states by means of the noncentral wave functions $\phi_1(\boldsymbol{\rho}\pm\boldsymbol{\xi})$. The ratio of the coefficients a_0/a_1, and the oscillator strength α

[24] S. Sack, L. C. Biedenharn, and G. Breit, Phys. Rev. **93**, 321 (1954).

were determined by minimizing the energy $E(\xi)$ with respect to these constants [see Appendix B].

The absolute decay rates of the various modes can now be calculated. From (19) we obtain the total decay rate

$$\Gamma_{cp}{}^{J}=4|M_{cp}|^2(\pi R_0{}^3r_0{}^3)^{-1}\times\int[F_c(z)\cos\delta+G_c(z)\sin\delta]^2\Gamma_p(KR_0)dz, \quad (29)$$

where $z=kr_0$ and k and K are connected by energy conservation. The decay rate will be dependent upon the radius of interaction for the outgoing wave. R_0, since the barrier transmission depends upon this quantity. For the two Be^8 decay modes this parameter is the radius outside of which no interaction with the neutron occurs and for the He^5 decay it is the He^5-He^4 interaction distance. The radius of interaction r_0 is presumed to be fixed by analysis of scattering data.

The reduced width amplitudes are denoted by γ_{cp}, where[12]

$$\gamma_{cp}(k,K)=(R_0\hbar^2/2\mu)^{\frac{1}{2}}\int\psi_c Y_p(\Omega_R)\Psi d\tau_r d\Omega_R. \quad (30)$$

In this expression the total nuclear wave function is evaluated at the radius $R=R_0$. From the above equation and (2), the reduced width amplitude can be given as

$$\gamma_{cp}(k,K)=(R_0\hbar^2/2\mu)^{\frac{1}{2}}C_{cp}(k,K)v_p(KR_0). \quad (31)$$

From the single-particle behavior of the coefficients $C_{cp}{}^J(k,K)$, their approximate form (13), and the value of $v_p(kR_0)$, (17), the reduced widths are given by

$$\gamma_{cp}{}^2(k,K)=2R_0{}^{-3}|M_{cp}|^2|u_c(kr_0)|^2(\hbar^2/\mu R_0{}^2). \quad (32)$$

We shall define a total reduced width for the state (c,p) to be (32) integrated over all energies available for decay. We denote this by

$$\gamma_{cp}{}^2=4|M_{cp}|^2(\pi R_0{}^3r_0{}^3)^{-1}(\hbar^2/\mu R_0{}^2)\times\int[F_c(z)\cos\delta+G_c(z)\sin\delta]^2dz, \quad (33)$$

where again $z=kr_0$. The values of the total decay rate $\Gamma_{cp}{}^J$ and total reduced width divided by $\hbar^2/\mu R_0{}^2$ are shown in Table II for various values of the parameter R_0. The evaluation of M_{cp} is in Appendices C and D.

It remains to consider the decay through channel (3). A calculation with the alpha-particle model of the magnetic dipole transition width, which is the dominant one, gives $\Gamma_\gamma=1.3$ ev. The single-particle estimate[13] of this width is approximately twice this value.

VI. DISCUSSION

In the last section we made use of an alpha-particle model to estimate the two-body decay rate of the 2.43 state of Be^9 through channels (1)–(3). The reduced width $\gamma_{cp}{}^2$ divided by $\hbar^2/\mu R_0{}^2$ are approximately independent of the nuclear interaction radii R_0, and for the particle decays are $\approx0.015(Be^8,n)$, $\approx0.00020\times(Be^{8*},n)$, and $\approx0.32(He^5,He^4)$.

TABLE II. Total decay rate and reduced width for the decay modes as a function of the interaction radius R_0.

	R_0 (10^{-13} cm)	Γ (ev)	$\gamma^2/(\hbar^2/\mu R_0{}^2)$
Be^8+n	3.0	3.6	0.008
	3.5	15	0.011
	4.0	37	0.015
	4.5	68	0.016
$Be^{8*}+n$	3.0	4	0.00029
	3.5	5	0.00024
	4.0	6	0.00020
	4.5	7	0.00016
He^5+He^4	3.5	33	0.32
	4.0	73	0.32
	4.5	150	0.32
	5.0	280	0.32

The preliminary qualitative discussion given in Sec. II indicated that decay to the ground state of Be^8 should be considerably inhibited. It should, in fact, be forbidden if center-of-mass effects and single-particle configuration mixing are neglected. In our deteailed calculation we were able to introduce in a natural manner configuration mixing for the neutron. The above results indicate that even with such mixing the reduced width for neutron emission is less than 5% of the total width. Emission through the first excited state of Be^8 is computed to be negligibly small; this comes about because the energy available to the Be^{8*} state is at least three half-widths from the resonance peak. Our calculations predict that most of the decay proceeds by means of He^5+He^4. It should be noted that the angular momentum and Coulomb barrier transmission factors do not essentially change the conclusions, but for reasonable R_0 do predict that approximately 5–20% of the decay will occur to the ground state of Be^8. This is also in accord with the momentum distribution of alpha particles as measured by BEH.

Our computations of the momentum and angular distributions of alphas emitted are relatively independent of any specific model for the description of Be^9. They make use, principally, of a generalization which we have developed along the lines suggested by Watson[4] to discuss the decay to and through an intermediate state, the energy of which is not well defined. The other information needed in the calculation of the distributions are (a) the spin and parity of the state, and (b) the relative population of the magnetic substates of Be^{9*} along its recoil direction. We have argued that any direct excitation of this state populates mainly the $M=\pm\frac{3}{2}$ states, independent of any nuclear model.

A detailed comparison (see Fig. 9) of the calculated momentum spectra for modes (1)–(3) with the experi-

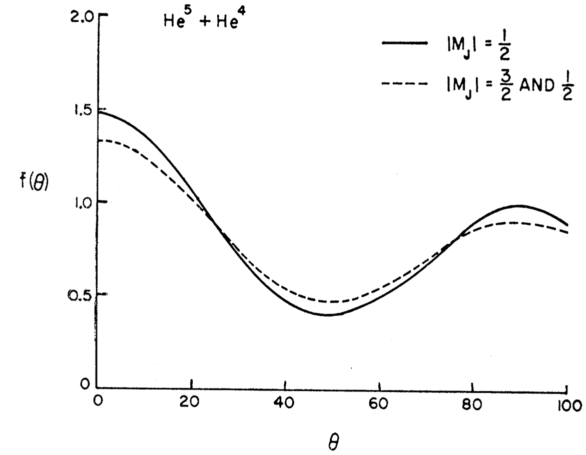

FIG. 8. Angular distribution, integrated over all momentum, of alpha particles about the recoil axis for the decay into He^5+He^4.

mental findings of BEH, indicate that a good fit can be obtained by admixing approximately 7% decay to the ground state of Be^8 to the momentum distribution of the He^5+He^4 decay. As indicated earlier this ratio of decay rates is consistent with our calculation and can be obtained by requiring the radius of the Be^8+n and He^5+He^4 systems to be related as shown in Fig. 10. If one chooses the He^5+He^4 radius to be 4.6×10^{-13} cm which is consistent with the alpha particle model rota-

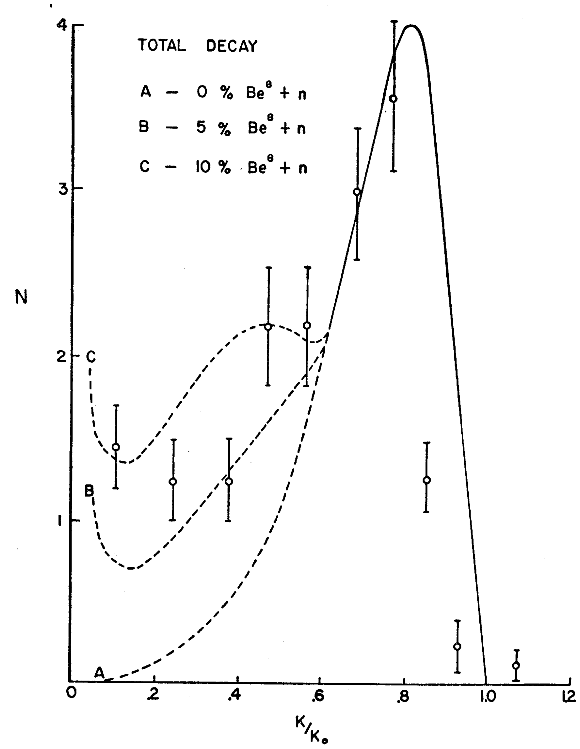

FIG. 9. Total decay of Be^9 assuming the decay is He^5+He^4 with the following percentages of Be^8+n, ground state admixed: *A*—0%, *B*—5%, and *C*—10%.

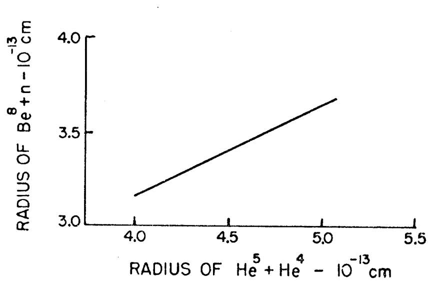

FIG. 10. Relationship between the interaction radius of He^5+He^4 and Be^8+n to obtain a 7% admixture of Be^8+n, ground state, in the decay.

tional energy spacings in Be^9, then the radius of Be^8+n must be approximately 3.5×10^{-13} cm. In view of the approximate nature of the decay rate calculations, these sizes are not unreasonable.

From the momentum distribution alone it is not possible to exclude appreciable decay to $Be^{8*}+n$. We do not believe this to be the case for two reasons: First, in spite of inaccuracies of the decay rate calculations, we expect the relative magnitude of the various processes to be given accurately, and second, the momentum distributions for the He^5+He^4 case fits the experimental data better, especially near the kinematic limit where purely kinematical considerations should determine the shape of the curve.

The experimental angular distribution of BEH has a minimum at about 50° where the magnitude is about half the value at 0°. These results agree with our previous conclusion that there is little decay to the Be^8 ground state and the direct interaction prediction that the magnetic substates $M=\pm\frac{3}{2}$ are the ones principally populated. (The other magnetic substates contribute a maximum at about 50°.)

In order to satisfy the experimental result of BEH of an upper limit of 1% for the decay by γ emission to the ground state of Be^9, it is only necessary to choose an interaction radius of He^5+He^4 that is at least 4.3×10^{-13} cm. However, the magnetic dipole transition rate calculation should not be taken too literally, since the alpha-particle model gives a value for the ground-state magnetic moment of -0.85 nm as compared with -1.18 nm experimentally observed.[25]

In conclusion, with interaction radii indicated, the total width of the 2.43-Mev state of Be^9 is of the order of 200 ev, and this is consistent with present knowledge.[26] The decay is expected to occur predominantly to He^5 and He^4.

APPENDIX A

From Eq. (19), the decay rate per unit volume in momentum space for one of the alphas is

[25] H. Kopfermann, *Kernmomente* (Akademische Verlags Gesellschaft, Frankfurt am Main, 1956).

[26] C. P. Browne, R. M. Williamson, D. S. Craig, and D. J. Donahue, Phys. Rev. **83**, 179 (1951).

$$d\Gamma_{cp}{}^J/d^3k_{\alpha 1}$$
$$=4(\pi R_0{}^3)^{-1}|M_{cp}|^2\int F_{cp}(k,K)$$
$$\times|\sum_{mm'} Y_c{}^m(\Omega_k)Y_p{}^{m'}(\Omega_K)(cpmm'|JM)|^2$$
$$\times\delta(E_J-E_{\alpha 1}-E_{\alpha 2}-E_n)\delta(\mathbf{k}_{\alpha 1}+\mathbf{k}_{\alpha 2}+\mathbf{k}_n)$$
$$\times d^3k_{\alpha 2}d^3k_n, \quad \text{(A1)}$$

where the symbols $\mathbf{k}_{\alpha 1}$, $\mathbf{k}_{\alpha 2}$ and $\mathbf{k}_n$ are the momenta of the two alphas and the neutron, the axis of quantization is the direction of recoil, and

$$F_{cp}(k,K)=(kr_0)^{-2}[F_c(kr_0)\cos\delta+G_c(kr_0)\sin\delta]^2$$
$$\times(2E_K/K^3)\Gamma_p(KR_0). \quad \text{(A2)}$$

In order to carry out the integrations in Eq. (A1) we express $\mathbf{k}$ and $\mathbf{K}$ in terms of $\mathbf{k}_{\alpha 1}$, $\mathbf{k}_{\alpha 2}$, and $\mathbf{k}_n$ (Fig. 2). The procedure is given below for the open channels.

1. Be^8+n, Ground State

The ground state of Be^8 is spin zero and the neutron is emitted with spin $\frac{5}{2}-$. In Eq. (A1) (c,k) refer to Be^8 and (p,K) to the relative motion of the neutron with respect to Be^8. We obtain

$$d\Gamma_{cp}{}^J/d^3k_{\alpha 1}=4(\pi R_0{}^3)^{-1}|M_{cp}|^2I, \quad \text{(A3)}$$

where (hereafter we set $\hbar=1$)

$$I=(4\pi)^{-1}\int F_{cp}(k,K)|\sum_{mm'} Y_3{}^m(\theta_n)\chi_{\frac{1}{2}}{}^{m'}$$
$$\times(3\tfrac{1}{2}mm'|\tfrac{5}{2}M)|^2\delta(E-E_{\alpha 1}-E_{\alpha 2}-E_n)$$
$$\times\delta(\mathbf{k}_{\alpha 1}+\mathbf{k}_{\alpha 2}+\mathbf{k}_n)d^3k_{\alpha 2}d^3k_n. \quad \text{(A4)}$$

The angular part of the neutron wave function in (A4) is transformed from a system in which the recoil axis is the z axis to one in which the z axis is $k_{\alpha 1}$. The spin quantization axis is also changed to the new axis. The transformation is performed for ease in evaluating the integrals in (A4) and is given by

$$\mathcal{Y}_J{}^M(\theta')=\sum_{M'}\mathcal{Y}_J{}^{M'}(\theta)(M'|D^J(\theta_{\alpha 1},\varphi_{\alpha 1})|M), \quad \text{(A5)}$$

where $\theta_{\alpha 1}$ and $\varphi_{\alpha 1}$ are the angles of $\mathbf{k}_{\alpha 1}$ with respect to the recoil axis and $(M'|D^J|M)$ is the symmetric top function.[27] Carrying out this transformation and integrating over d^3k_n and $d(\cos\theta_{\alpha 2})d\phi_{\alpha 2}$ we obtain for I

$$I=m(4\pi k_{\alpha 1})^{-1}\int F_{cp}(kK)$$
$$\times\sum_{M'}\sum_{mm'}|Y_3{}^m(\theta_n{}')(3\tfrac{1}{2}mm'|\tfrac{5}{2}M')|^2$$
$$\times|(M'|D^{\frac{5}{2}}|M)|^2k_{\alpha 2}dk_{\alpha 2}, \quad \text{(A6)}$$

where the functions $Y_3{}^m(\theta_n{}')$ of the neutron coordinates referred to the $\mathbf{k}_{\alpha 1}$ axis are expressed in terms of $\mathbf{k}_{\alpha 1}$ and $\mathbf{k}_{\alpha 2}$ by energy and momentum conservation. The

[27] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957).

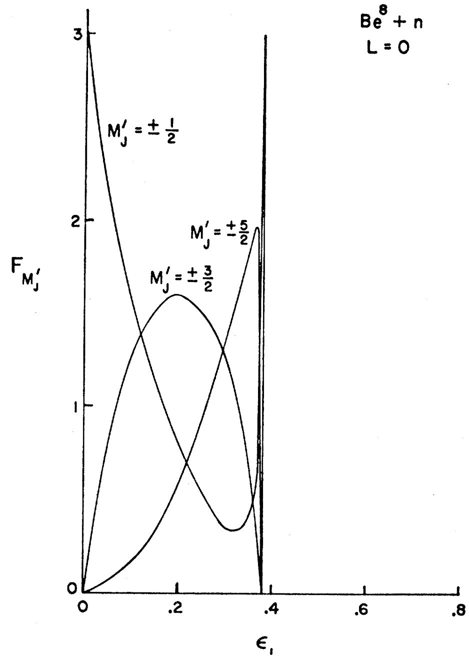

FIG. 11. The functions $F_{MJ'}$ used in momentum distribution of Be^8+n, ground state, plotted against ϵ_1, the normalized energy.

remaining integration over $dk_{\alpha 2}$ is trivial in this case because of the long life time of the Be^8 state. The consequent sharply defined energy allows us to use the approximate relation Eq. (20). This relation is derived by neglecting the regular solution in the expression for $|u_c(kr_0)|^2$ and using

$$\sin^2\delta=\frac{(\Gamma/2)^2}{(E_k-E_0)^2+(\Gamma/2)^2}\cong\tfrac{1}{2}\pi\Gamma\delta(E_k-E_0), \quad \text{(A7)}$$

where Γ is the width of the Be^8 state, E_k is the energy of the state in terms of $k_{\alpha 1}$ and $k_{\alpha 2}$, and $E_0=0.096$ Mev.[28]

Carrying out the integration, we obtain

$$I=mME_0E_k(8\pi k_{\alpha 1}K^3)^{-1}\Gamma_p(KR_0)$$
$$\times\sum_{M'}F_{M'}|(M'|D^{\frac{5}{2}}|M)|^2, \quad \text{(A8)}$$

where

$$F_{M'}=4\pi\sum_{mm'}|Y_3{}^m(\theta_n{}')(3\tfrac{1}{2}mm'|\tfrac{5}{2}M')|^2,$$
$$=(15/8)\sin^4\theta_n{}', \qquad M'=\pm\tfrac{5}{2},$$
$$=6\sin^2\theta_n{}'-(45/8)\sin^4\theta_n{}', \qquad M'=\pm\tfrac{3}{2},$$
$$=3-6\sin^2\theta_n{}'+(15/4)\sin^4\theta_n{}', \qquad M'=\pm\tfrac{1}{2},$$

and

$$\sin\theta_n{}'=-(k_{\alpha 2}/k_n)\sin\theta_{\alpha 2}.$$

From energy and momentum conservation one finds

$$\cos^2\theta_{\alpha 2}=(E-10E_0)^2/36E_{\alpha 1}(8E_0+E-9E_{\alpha 1}).$$

Using these relations the $F_{M'}$ can be evaluated and are shown in Fig. 11. The momentum distribution along

[28] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **21**, 77 (1955).

the recoil axis for a particular magnetic quantum substate is then

$$d\Gamma_{cp}{}^{JM}/d^3k_{\alpha 1}=mME_0E_KE|M_{cp}|^2 \times(2\pi^2R_0{}^3k_{\alpha 1}K^3)^{-1}\Gamma_p(KR_0)F_M.$$

The angular distribution integrated over all energies of α_1, is

$$d\Gamma_{cp}{}^{JM}/d\Omega_{\alpha 1}=4|M_{cp}|^2(\pi R_0{}^3)^{-1}\int Ik_{\alpha 1}{}^2dk_{\alpha 1}$$
$$=10mM^2E_0E_KE|M_{cp}|^2(9\pi^2K^3R_0{}^3)^{-1} \times\Gamma_p(KR_0)\textstyle\sum_{M'}a_{M'}|(M'|D^{\frac{5}{2}}|M)|^2,$$

where $a_{M'}=\int F_{M'}d\epsilon_1$, and $\epsilon_1=9E_{\alpha 1}/5E$. The results of the integration are shown in Table III. As indicated in the text the $M=\pm\frac{3}{2}$ states are the main ones populated and give for the angular dependence of $d\Gamma_{cp}/d\Omega_{\alpha 1}$

$$d\Gamma_{cp}/d\Omega_{\alpha 1}\approx\textstyle\sum_{M'}a_{M'}|(M'|D^{\frac{5}{2}}|\tfrac{3}{2})|^2 \approx3.70[1+0.015\cos^2\theta_{\alpha 1}+0.020\cos^4\theta_{\alpha 1}]. \quad \text{(A11)}$$

This distribution, plotted in Fig. 6, is almost isotropic. If one wishes to add a contribution from the $M=\pm\frac{1}{2}$ state, one will find for the maximum contribution of $\frac{1}{6}$

$$d\Gamma_{cp}/d\Omega_{\alpha 1}\approx\textstyle\sum_{M'}a_{M'}[6/7|(M'|D^{\frac{5}{2}}|\tfrac{3}{2})|^2 +1/7|(M'|D^{\frac{5}{2}}|\tfrac{1}{2})|^2] \approx3.64[1+0.034\cos^2\theta_{\alpha 1}+0.019\cos^4\theta_{\alpha 1}]. \quad \text{(A12)}$$

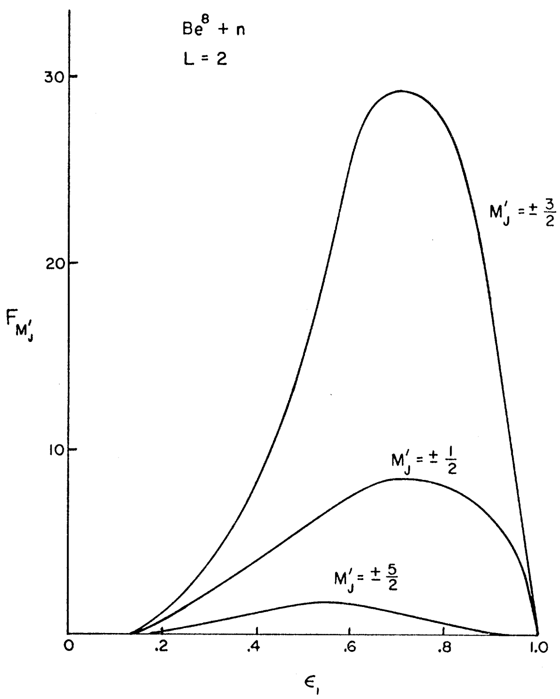

FIG. 12. The functions $F_{MJ'}$ used in momentum distribution of Be^8+n, excited state, plotted against ϵ_1, the normalized energy.

2. $Be^{8*}+n$, First Excited State

The subscripts (c,k) now refer to the $L=2$ state of Be^8 and (p,K) apply to the neutron in a $j=\frac{3}{2}-$ state. The method of calculation is the same as for the ground-state case except for the difference in angular momenta and the integration over $dk_{\alpha 2}$ must be done with the appropriate scattering solution substituted in $F_{cp}(k,K)$ which is given by (A2).

The momentum distribution is given by

$$d\Gamma_{cp}/d^3k_{\alpha 1}=4(\pi R_0{}^3)^{-1}|M_{cp}|^2I, \quad \text{(A13)}$$

where

$$I=5mME(144\pi^2k_{\alpha 1})^{-1}\textstyle\sum_{M'}F_{M'}|(M'|D^{\frac{5}{2}}|M)|^2.$$

The functions $F_{M'}$ are shown in Fig. 12. The results obtained for the angular dependence integrated over all energies for α_1 is

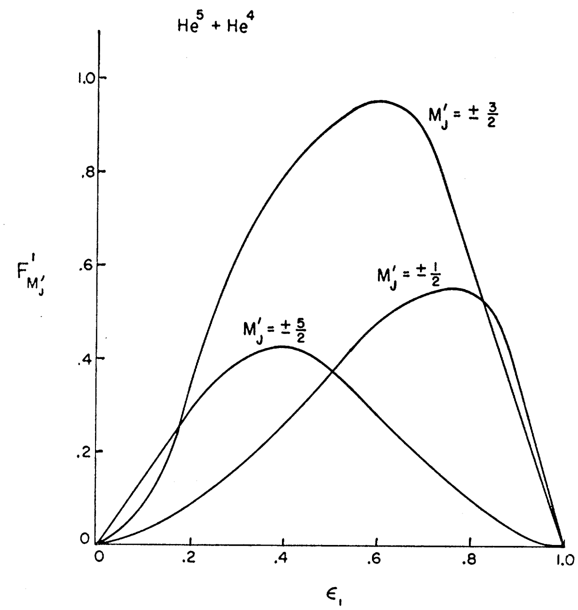

FIG. 13. The functions $F_{MJ'}{}^1$ used in momentum distribution of the alpha particle from He^5 plotted against ϵ_1, the normalized energy.

$$d\Gamma_{cp}/d\Omega_{\alpha 1}\approx\textstyle\sum_{M'}a_{M'}|(M'|D^{\frac{5}{2}}|M)|^2,$$

where the $a_{M'}$ are defined by (A10) and tabulated in Table III. The angular dependence for the $M=\pm\frac{3}{2}$ case is

$$d\Gamma_{cp}/d\Omega_{\alpha 1}\approx8.0[1-2.9\cos^2\theta_{\alpha 1}+3.5\cos^4\theta_{\alpha 1}], \quad \text{(A14)}$$

and with a $\frac{1}{6}$ fraction of $M=\pm\frac{1}{2}$ added, the angular dependence is

$$d\Gamma_{cp}/d\Omega_{\alpha 1}\approx7.3[1-2.3\cos^2\theta_{\alpha 1}+2.8\cos^4\theta_{\alpha 1}]. \quad \text{(A15)}$$

3. He^5+He^4

The decay into He^5 [subscripts (c,k)] and He^4 [subscripts (p,K)] differs somewhat from the decay into Be^8+n, and we will go into somewhat more detail. The distribution of alpha particles for this case contains the sum of those alphas emitted in the primary decay and those emitted by He^5.

$$d\Gamma_{cp}/d^3k_\alpha=d\Gamma_{cp}/d^3k_{\alpha1}+d\Gamma_{cp}/d^3k_{\alpha2}=4(\pi R_0^3)^{-1}|M_{cp}|^2 I, \quad \text{(A16)}$$

where

$$I=\int F_{cp}(k,K)|\textstyle\sum_{mm'} Y_{\frac{3}{2}}{}^m(\Omega_k)Y_2{}^{m'}(\Omega_K) \times(\tfrac{3}{2}2mm'|\tfrac{5}{2}M)|^2\delta(E-E_{\alpha1}-E_{\alpha2}-E_n) \times\delta(\mathbf{k}_{\alpha1}+\mathbf{k}_{\alpha2}+\mathbf{k}_n)d^3k_{\alpha2}d^3k_n+\int F_{cp}(k,K) \times|\textstyle\sum_{mm'} Y_{\frac{3}{2}}{}^m(\Omega_k)Y_2{}^{m'}(\Omega_K)(\tfrac{3}{2}2mm'|\tfrac{5}{2}M)|^2 \times\delta(E-E_{\alpha1}-E_{\alpha2}-E_n)\delta(\mathbf{k}_{\alpha1}+\mathbf{k}_{\alpha2}+\mathbf{k}_n) \times d^3k_{\alpha1}d^3k_n, \quad \text{(A17)}$$

and where

$$\mathbf{k}=\tfrac{1}{5}\mathbf{k}_{\alpha1}-\tfrac{4}{5}\mathbf{k}_n\equiv\mathbf{k}_1, \quad \mathbf{K}=5/9\mathbf{k}_{\alpha2}-4/9(\mathbf{k}_{\alpha1}+\mathbf{k}_n)\equiv\mathbf{K}_2. \quad \text{(A18)}$$

TABLE III. Values of $a_{MJ'}$ used in the angular distribution of the decay modes.

M'	$\pm\frac{5}{2}$	$\pm\frac{3}{2}$	$\pm\frac{1}{2}$
$Be^{8*}+n$	0.27	0.41	0.46
Be^8+n	0.7	13.1	4.0
$(He^5+He^4)_2$	0	0.92	0.15
$(He^5+He^4)_1$	0.23	0.55	0.39

If in the second part of I we make a change of variables,

$$\mathbf{k}_{\alpha1}\rightarrow\mathbf{k}_{\alpha2}, \quad \mathbf{k}_{\alpha2}\rightarrow\mathbf{k}_{\alpha1},$$

then I becomes

$$I=\int[F_{cp}(k_1,K_2)|\textstyle\sum_{mm'} Y_{\frac{3}{2}}{}^m(\Omega_{k1})Y_2(\Omega_{k2}) \times(\tfrac{3}{2}2mm'|\tfrac{5}{2}M)|^2+F_{cp}(k_2,K_1) \times|\textstyle\sum_{mm'} Y_{\frac{3}{2}}{}^m(\Omega_{k2})Y_2(\Omega_{k1})(\tfrac{3}{2}2mm'|\tfrac{5}{2}M)|^2] \times\delta(E-E_{\alpha1}-E_{\alpha2}-E_n)\delta(\mathbf{k}_{\alpha1}+\mathbf{k}_{\alpha2}+\mathbf{k}_n) \times d^3k_{\alpha2}d^3k_n, \quad \text{(A19)}$$

where k_2 and K_1 are defined by (A18) with subscripts α_1 and α_2 interchanged. As in the two Be^8 cases we transform the angular and spin functions to the $\mathbf{k}_{\alpha1}$ axis, and the integrations are performed in the same manner as in the preceding two cases. One then finds

$$I=125M^2E(6048\pi^2k_{\alpha1})^{-1} \times\textstyle\sum_{M'}(F_{M'}{}^1+F_{M'}{}^2)|(M'|D^{\frac{5}{2}}|M)|^2,$$

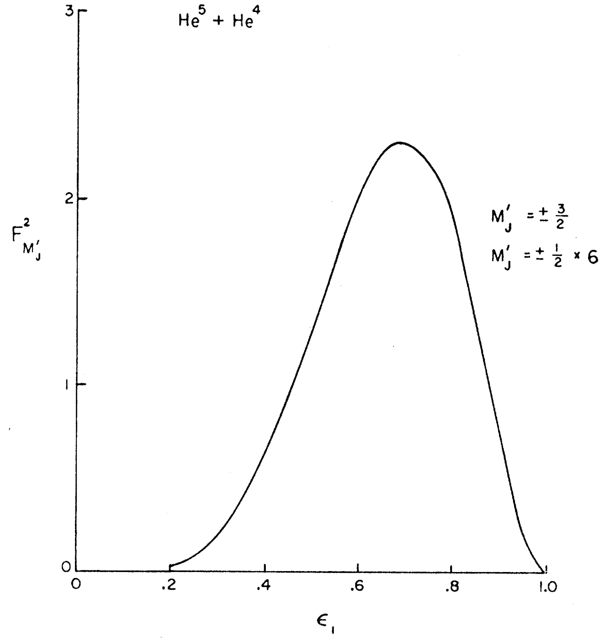

FIG. 14. The functions $F_{MJ'}{}^2$ used in momentum distribution of the emitted alpha particle plotted against ϵ_1, the normalized energy.

where $F_{M'}{}^1$ and $F_{M'}{}^2$ are obtained from integrating the two parts of (A19), respectively, and are shown in Figs. 13 and 14. The angular distribution is given by

$$d\Gamma_{cp}/d\Omega_\alpha\approx\textstyle\sum_{M'}(a_{M'}{}^1+a_{M'}{}^2)|(M'|D^{\frac{5}{2}}|M)|^2,$$

where again $a_{M'}{}^1$ and $a_{M'}{}^2$ are defined in terms of $F_{M'}{}^1$ and $F_{M'}{}^2$ similarly to (A10) and are tabulated in Table III. For the case of $M=\pm\frac{3}{2}$

$$d\Gamma_{cp}/d\Omega_\alpha\approx0.96[1-2.7\cos^2\theta_\alpha+3.2\cos^4\theta_\alpha],$$

and with a $\frac{1}{6}$ fraction of $M=\pm\frac{1}{2}$ mixed in

$$d\Gamma_{cp}/d\Omega_\alpha\approx0.88[1-2.2\cos^2\theta_{\alpha1}+2.6\cos^4\theta_{\alpha1}].$$

APPENDIX B

The minimization of the energy with respect to the constants a_0 and a_1 of Eq. (28) gives the usual secular determinant

$$\begin{vmatrix} 2H_{11}+2H_{12}-2(1+a)E(\xi) & 2H_{10}-2bE(\xi) \\ 2H_{10}-2bE(\xi) & H_{00}-E(\xi) \end{vmatrix}=0, \quad \text{(B1)}$$

where

$$H=K+V(|\varrho-\xi|)+V(|\varrho+\xi|),$$
$$H_{11}=\langle\phi_1(\varrho\pm\xi)|H|\phi_1(\varrho\pm\xi)\rangle,$$
$$H_{12}=\langle\phi_1(\varrho\pm\xi)|H|\phi_1(\varrho\mp\xi)\rangle,$$
$$H_{10}=\langle\phi_0(\varrho)|H|\phi_1(\varrho\pm\xi)\rangle,$$
$$H_{00}=\langle\phi_0(\varrho)|H|\phi_0(\varrho)\rangle,$$
$$a=\langle\phi_1(\varrho\pm\xi)|\phi_1(\varrho\mp\xi)\rangle,$$
$$b=\langle\phi_1(\varrho\pm\xi)|\phi_0(\varrho)\rangle.$$

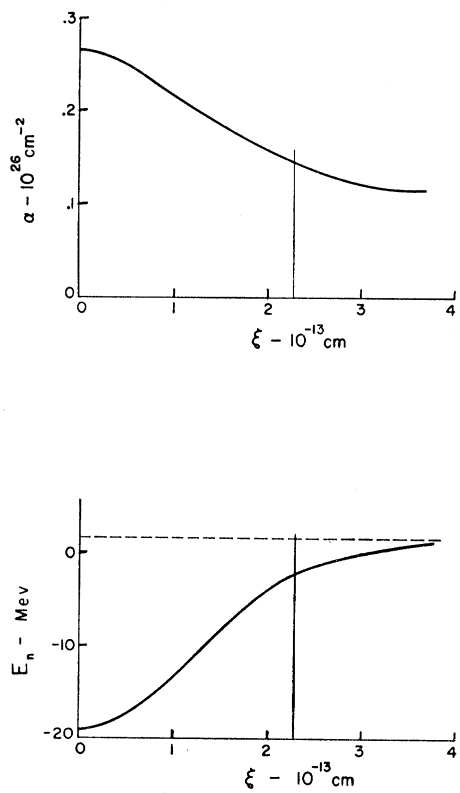

FIG. 15. Oscillator strength function, α, and neutron energy, $E(\xi)$, from the variational calculation as a function of ξ.

For the especially simple case of

$$\phi_1(\mathbf{x})=\phi_0(\mathbf{x})=(2\alpha)^{5/4}(\tfrac{3}{8}\sqrt{\pi})^{-\frac{1}{2}}x\exp(-\alpha x^2)\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\Omega_x),$$

and for the gaussian form of the neutron-alpha potential, these integrals reduce to

$$H_{11}=5\alpha\hbar^2/2m-\bar{V}-\bar{V}\exp(4j^2-k^2),$$

$$H_{12}=(5\alpha\hbar^2/2m)(1-\tfrac{2}{5}k^2)\exp(-k^2)-2\bar{V}\exp(j^2-2k^2),$$

$$H_{10}=(5\alpha\hbar^2/2m)(1-k^2/10)\exp(-k^2/4)$$
$$-\bar{V}\{\exp[(j^2/4)-(k^2/2)]$$
$$+\exp[(9j^2/4)-(5k^2/4)]\},$$

$$H_{00}=(5\alpha\hbar^2/2m)-2\bar{V}\exp(j^2-k^2),$$

$$a=\exp(-k^2),$$

$$b=\exp(-k^2/4),$$

where

$$\bar{V}=A_{\frac{3}{2}}/(1+\beta/2\alpha)^{\frac{5}{2}},$$

$$k^2=2\alpha\xi^2,$$

$$j^2=2\alpha\xi^2/(1+\beta/2\alpha).$$

The minimum energy with respect to the constants a_0 and a_1 was obtained by solving the secular determinant (B1) for each value ξ. This energy was again minimized by varying the neutron oscillator strength α. Since the minimum energy $E(\xi)$ is not a simple function of α, the work must be done numerically. The values of $E(\xi)$, a_0/a_1 and α as a function of ξ are shown in Figs. 15 and 16.

The equilibrium distance of the two alpha particles is found by calculating the moment of inertia of the Be^8 system assuming the alpha particles to be mass points. The energy level formula in this case is

$$E_J=(\hbar^2/2I_0)J(J+1),$$

and matching the $J=2$, 2.9-Mev state, to the first rotational level one obtains the equilibrium distance $\xi_0=2.3\times10^{-13}$ cm. At this separation of the two alphas, the values of the calculated quantities are

$$a_0/a_1=0.57,$$

$$\alpha=0.147\times10^{26}\ \text{cm}^{-2},$$

$$E(\xi_0)=-2.11\ \text{Mev}.$$

The total energy of the Be^9 nucleus is given by

$$E=E_R+E_n,$$

where E_n is the neutron energy given approximately by $E(\xi_0)$ and E_R is the rotational energy given by[14]

$$E=(\hbar^2/2I_0)[J(J+1)+j(j+1)-K^2-m^2],$$

where J is the total spin of the nucleus, j is the neutron spin, and K and m are the projection on the body axis of the total spin and neutron spin, respectively.

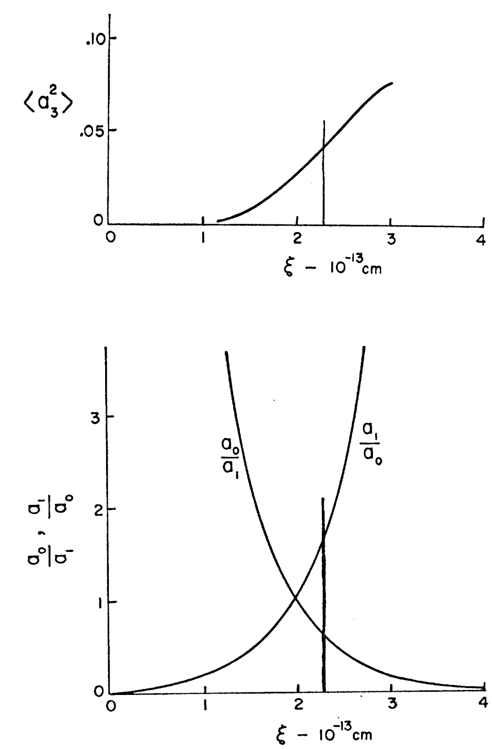

FIG. 16. Expectation value of the $l=3$ neutron wave function and the ratio of the noncentral wave function to central wave function as a function of ξ.

Normally, the j^2, K^2, and m^2 terms are omitted from the expression since only rotational levels are treated, but for the total energy of the system, these must be included. The ground state of Be^9 is given by

$$J=j=K=m=\tfrac{3}{2},$$

and for the given value of $\hbar^2/2I_0$ one obtains $E_R=1.45$ Mev. Together with the value of $E(\xi_0)$ from the variational calculation one finds for the ground-state energy $E=-0.66$ Mev. This figure is to be compared with the actual ground-state energy $E=-1.57$ Mev. Since E_n is the difference of large energies, the agreement is rather good.

This calculation means the following in terms of $l=3$ mixing for the neutron wave functions. If we expand the neutron wave function about the center of mass in spherical harmonics

$$\phi_{\frac{3}{2}}(\rho)=[a_1(\rho)Y_1^1(\theta)+a_3(\rho)Y_3^1(\theta)+\cdots]\chi_{\frac{1}{2}}^{\frac{1}{2}},$$

with $\sum_i\langle|a_i|^2\rangle=1$ then the amount of $l=3$ wave is $\langle|a_3|^2\rangle$. This quantity is plotted in Fig. 16. At the equilibrium distance the amount of $l=3$ wave is about 4%.

APPENDIX C

The initial wave function of Be^9 is given by Eq. (22). To perform the overlap integrals M_{cp} in the total decay rate Eq. (29), we expand the neutron wave function into spherical harmonics about the alpha-alpha center of mass for the Be^8+n decay and about each alpha for the He^5+He^4 decay. The expansion of the neutron state, Eq. (28), is accomplished by first using the law of sines on the angular part of the function (see Fig. 17).

$$\rho_1\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\theta_1)=\rho_2\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\theta_2)=\rho\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\theta). \tag{C1}$$

1. Be^8+n

Using (C1), we obtain for the noncentral portions of the neutron wave function

$$\phi_1(\boldsymbol{\rho}\pm\boldsymbol{\xi})=N_\varphi^{-1}\rho\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\theta)\exp(-\alpha\xi^2-\alpha\rho^2)\exp(\mp2\alpha\boldsymbol{\xi}\cdot\boldsymbol{\rho}),$$

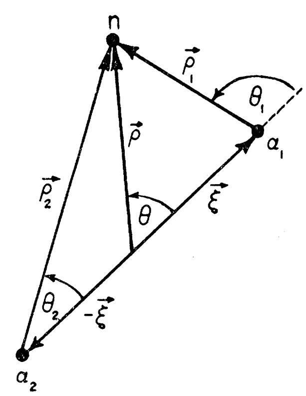

FIG. 17. Description of coordinate system used in the calculation of the total decay rate.

where $N_\varphi^{-2}=(2\alpha)^{\frac{5}{2}}8/3\sqrt{\pi}$. Using the expansion

$$\exp(-m\alpha\boldsymbol{\xi}\cdot\boldsymbol{\rho})=\sum_{l=0}^{\infty}(2l+1)i^l j_l(im\alpha\rho\xi)P_l(\cos\theta),$$

one obtains for the $j=\frac{3}{2}-$ projection of the neutron wave function about the center of the two alphas to be the following

$$(NN_\varphi)^{-1}\{\rho\exp(-\alpha\rho^2)[a_0+2a_1\exp(-\alpha\xi^2)(j_0(2i\alpha\xi\rho) + j_2(2i\alpha\xi\rho))]\}\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\theta), \tag{C2}$$

and similarly for the $j=\frac{2}{5}-$ projection one obtains

$$(NN_\varphi)^{-1}\{(12/7)a_1\rho\exp(-\alpha\xi^2-\alpha\rho^2)[j_2(2i\alpha\xi\rho) + j_4(2i\alpha\xi\rho)]\}\mathcal{Y}_{\frac{5}{2}}^{\frac{3}{2}}(\theta), \tag{C3}$$

where $N^2=\langle\varphi|\varphi\rangle$ is the normalizing constant for the total neutron wave function.

2. He^5-He^4

By means of the procedure illustrated above, one obtains the $j=\frac{3}{2}-$ projection of the neutron wave function centered about α_1 to be

$$(NN_\varphi)^{-1}\rho_1\exp(-\alpha\rho_1^2)\{a_1[1+\exp(-4\alpha\xi^2) \times(j_0(4i\alpha\xi\rho_1)+j_2(4i\alpha\xi\rho_1))]+a_0\exp(-\alpha\xi^2) \times(j_0(2i\alpha\xi\rho_1)+j_2(2i\alpha\xi\rho_1))\}\mathcal{Y}_{\frac{3}{2}}^{\frac{3}{2}}(\theta_1). \tag{C4}$$

APPENDIX D

To evaluate M_{cp}, Eq. (13), we use the variational wave function inside the range of forces and the set $\psi_c\phi_p$ of which we need $f(r)$ and $g(R)$.

1. Be^8+n, Ground State

In this case $f(r)$ is the relative alpha-particle wave function with $r\equiv2\xi$, and $g(R)$ is the neutron wave function with $R\equiv\rho$. The latter is approximated by an $l=3$ harmonic oscillator function

$$g(\rho)=\frac{\rho^3\exp(-\delta\rho^2)}{\rho_0^3\exp(-\delta\rho_0^2)}, \tag{D1}$$

and δ is adjusted to match the slope and value of $g(\rho)$ to an outgoing wave at the radius ρ_0. The values of $\delta\rho_0^2$ required are shown in Table IV. The relative alpha-particle interior wave function is

$$f(2\xi)=R(\xi)/R(\xi'), \tag{D2}$$

TABLE IV. Values of ρ_0 and $\delta\rho_0^2$ used to match the outgoing wave to the interior wave function at ρ_0.

	ρ_0	$\delta\rho_0^2$
Be^8+n	3.0	3.43
	4.0	3.41
	5.0	3.40
$Be^{8*}+n$	3.0	1.38
	4.0	1.34
	5.0	1.25

where $R(\xi)$ is the Be^8 radial function which matches onto the outside scattering solution at ξ'. Using the relations (C2), (D1), and (D2), one obtains for $|M_{cp}|^2$,

$$|M_{cp}|^2=\frac{3\sqrt{\pi}}{8}|C_{\frac{5}{2}0}{}^{\frac{5}{2}}|^2\frac{a_1{}^2}{N^2}\exp\left(2\delta\rho_0{}^2-\frac{\delta}{\alpha+\delta}\alpha\xi^2\right)$$
$$\times\left(\frac{2\alpha}{\alpha+\delta}\right)^{7/2}\frac{1}{(\alpha+\delta)^{9/2}}\frac{1}{\rho_0{}^6}\frac{(2\alpha\xi^2)^2}{|R(\xi')|^2}. \quad \text{(D3)}$$

The values of the constants used in evaluating this expression are

(1) From the alpha-particle model and variation calculation

$$|C_{\frac{5}{2}0}{}^{\frac{5}{2}}|^2=\tfrac{1}{3},$$
$$\xi_0=2.3\times10^{-13}\text{ cm},$$
$$\alpha=0.147\times10^{26}\text{ cm}^{-2},$$
$$a_0/a_1=0.57,$$

(2) From matching the outgoing wave, $\delta\rho_0{}^2$, which is tabulated in Table IV,

(3) From the single-particle estimate of the nuclear wave function at the boundary

$$|R(\xi')|^2=2/(2\xi')^3.$$

These values substituted into (D3) give the total decay rate Γ_{cp}, Eq. (29), and the reduced width $\gamma_{cp}{}^2$, Eq. (33). The values are shown in Table III.

2. Be^8+n, First Excited State

The calculation is similar to the ground state except now the neutron wave function is approximated by an $l=1$ harmonic oscillator wave function.

$$g(\rho)=\frac{\rho\exp(-\delta\rho^2)}{\rho_0\exp(-\delta\rho_0{}^2)}. \quad \text{(D4)}$$

Whereas the radial alpha-alpha wave function is taken to be the same as for the ground-state case, the constant δ is again evaluated by matching an outgoing wave onto $g(\rho)$ at ρ_0; the results are shown in Table IV. Performing the overlap integral M_{cp} one obtains

$$|M_{cp}|^2=\tfrac{3}{8}(\pi)^{\frac{1}{2}}N^{-2}|C_{\frac{3}{2}2}{}^{\frac{5}{2}}|^2\left(\frac{2\alpha}{\alpha+\delta}\right)^{\frac{5}{2}}\frac{\exp(2\delta\rho_0{}^2)}{(\alpha+\delta)^{\frac{5}{2}}\rho_0{}^2}$$
$$\times\frac{1}{|R(\xi')|^2}\left[a_0+2a_1\exp\left(-\frac{\delta}{\alpha+\delta}\alpha\xi^2\right)\right], \quad \text{(D5)}$$

where the value of $|C_{\frac{3}{2}2}{}^{\frac{5}{2}}|^2=6/7$, and the other constants (except for δ) are the same as the ground-state case. The values of the decay rate and reduced width are shown in Table III.

3. He^5+He^4

The interior function $f(r)$ and $g(R)$ refer to the He^5 and relative He^5+He^4 systems, respectively. We approximate $f(r)$ by an $l=1$ harmonic oscillator function with the oscillator strength adjusted to match the scattering solution at a radius of 2.9×10^{-13} cm, the radius used in the energy and angular distributions, and a value of $\delta\rho_0{}^2=0.74$.

The relative He^5-He^4 radial function is approximated by that of Be^8, Eq. (D2). This approximation is in the spirit of the alpha-particle model, where the neutron mass is taken to be small compared to that of the alphas. One simply obtains

$$|M_{cp}|^2=\tfrac{3}{8}(\pi)^{\frac{1}{2}}|C_{\frac{3}{2}2}{}^{\frac{5}{2}}|^2N^{-2}\left(\frac{2\alpha}{\alpha+\delta}\right)^{\frac{5}{2}}\frac{\exp(2\delta\rho_0{}^2)}{(\alpha+\delta)^{\frac{5}{2}}\rho_0{}^2}$$
$$\times\frac{1}{|R(\xi')|^2}\left[a_1+a_1\exp\left(-\frac{4\delta}{\alpha+\delta}\alpha\xi^2\right)\right.$$
$$\left.+a_0\exp\left(-\frac{\delta}{\alpha+\delta}\alpha\xi^2\right)\right]. \quad \text{(D6)}$$

The values of the parameters in the above expression are the same as in the Be^8 excited-state case except for δ and ρ_0. The evaluation of the total decay rate and reduced width for this case is listed in Table III.