that the expected pole in the production process always lies outside the ellipse.

Further it can be shown that in any case in the production processes the expected pole attains its minimum distance from the physical range when w^2 has the minimum value $(\mu'+\mu'')^2$.[5]

We have investigated the cases of type (2), $p+\pi \to p+\pi+\pi$ and $p+K \to p+K+\pi$ for the threshold value of w^2. In both cases we have found that the expected pole never lies inside the ellipse, and only for one value of W reaches the ellipse. This value is $W=m+3.5\,\mu$ and $W=M+M_K+3.6\,\mu$ for the two processes, respectively.

[5] Of course $\partial^2\sigma/\partial w^2\partial\Delta^2$ tends to zero as w^2 goes to $(\mu'+\mu'')^2$: this is due to the phase space factor of the final particles of momenta k' and k''. In this case it is more convenient to consider the quantity

$$\frac{1}{[w^4+(\mu'^2-\mu''^2)^2-2w^2(\mu'^2+\mu''^2)]^{\frac{1}{2}}}\frac{\partial^2\sigma}{\partial w^2\partial\Delta^2}$$

which in general is different from zero at $w^2=(\mu'+\mu'')^2$ and has the same analyticity properties of $\partial^2\sigma/\partial w^2\partial\Delta^2$ as function of Δ^2.

CONCLUSIONS

We may summarize the results as follows. It has not been possible to prove an analytic property of the production amplitude analogous to the one holding for the elastic amplitudes as function of $\cos\theta$. However, we have shown that some special combinations of the amplitudes [formulas (8)] are analytic within the same region of the $\cos\theta$ plane found by Lehmann in the case of elastic scattering with the same incoming particles.

Finally, we have proved that the conjectures of Chew and Low which refer to the analyticity properties of the cross section (17) as a function of $\cos\theta$ are confirmed at least within the Lehmann ellipse.

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Calculation of Single-Particle Energies in the Theory of Nuclear Matter

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The rearrangement energy corrections to the single-particle energies have been evaluated, using the procedure of Brueckner and Goldman. The shift is shown to be due largely to the second- and third-order rearrangement energy diagrams, the corrected energy at the Fermi surface now nearly agreeing with the mean binding energy.

The change of the single-particle energies of virtual excitations due to rearrangement effects is also determined and shown to shift the mean binding energy by 1.5 Mev.

I. INTRODUCTION

IT has been shown by Brueckner and Goldman[1] that a redefinition of single-particle energy within the framework of the K-matrix approximation[2] for the total energy of a Fermion many-body system leads to single-particle energies which satisfy the separation energy theorem of Hugenholtz and Van Hove.[3] It is the purpose of this paper to give the quantitative results obtained for the single-particle energies and also the ground-state energies from an application of the BG procedure.

II. SINGLE-PARTICLE ENERGY

Following BG, we start from the K-matrix expression for the ground-state energy, considered as a function of the occupation numbers of the Fermi gas, i.e.,

$$E(n_a)=\sum_i n_i\frac{p_i^2}{2M}+\tfrac{1}{2}\sum_{ij} n_i n_j(K_{ij,ij}-K_{ij,ji}), \quad (2.1)$$

[1] K. A. Brueckner and D. T. Goldman, Phys. Rev. **117**, 207 (1960); hereafter referred to as BG.
[2] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).
[3] N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).

TABLE I. Breakdown of single-particle energy. The density corresponds to $r_0=1.07\times10^{-13}$ cm and an average binding energy per particle of 15.5 Mev. The variation of corrected single-particle energy from $p=0$ to $p=p_F$ corresponds to an average effective mass of $0.73M$.

Momentum	Potential energy (Mev) K matrix	second order	third order	Single-particle energy (Mev)
$0.1p_F$	−112.0	+26.8	+9.4	−75.6
$0.9p_F$		6.9	6.4	
p_F	−70.3	4.4[a]	6.1[a]	−17.8

[a] Extrapolated from computed values at $p=0.1p_F$ and $p=0.9p_F$.

with

$$K_{kl,ij}=v_{kl,ij}+\sum_{mn} v_{kl,mn}\frac{(1-n_m)(1-n_n)}{E_i^*+E_j^*-E_m^*-E_n^*}K_{mn,ij}. \tag{2.2}$$

We have marked the single-particle energies appearing in Eq. (2.2) with an asterisk to emphasize the point that these energies are not the complex energies of real holes or particles but rather the real energies of virtual excitations. We refer to BG and also to Sec. IV for a more detailed discussion of this point.

The single-particle energy now is defined by

$$E_\alpha=\partial E/\partial n_\alpha. \tag{2.3}$$

As shown by BG, this definition leads to the Van Hove-Hugenholtz series of correction terms for the energy and also leads to a single-particle energy at the Fermi surface satisfying the separation energy theorem for a saturated system,

$$E_F=E/N. \tag{2.4}$$

The change in the single-particle energy from that given in the K-matrix approximation, which is

$$E_\alpha^*=(p_\alpha^2/2M)+\textstyle\sum_s n_s(K_{\alpha s,\alpha s}-K_{\alpha s,s\alpha}), \tag{2.5}$$

arises from the change in the propagator in the K-matrix equation. This is due to the shift in the occupation numbers as they appear explicitly in Eq. (2.2) or implicitly in the energies E_i^*. In the approximation of BG, which is based on the assumption of rapid convergence of the rearrangement energy series, the correction to the energy due to the shift in the self-consistent energies E_i^* is included only in first approximation, i.e., only through the linear change in n_α as it affects the sum in Eq. (2.5). The change in particle energy then is due to a second-order term arising from the change in exclusion effect upon hole creation,

$$(E_\alpha-E_\alpha^*)_{\text{exclusion}}=\sum_{ijm}\frac{n_in_j(1-n_m)K_{m\alpha,ij}^2}{E_i^*+E_j^*-E_m^*-E_\alpha^*}, \tag{2.6}$$

and a third-order term from the change in the spectrum

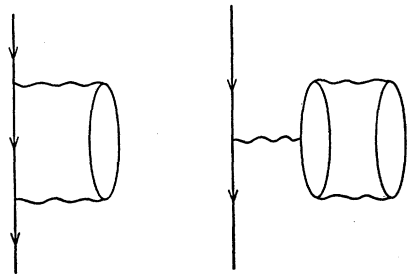

FIG. 1. Second- and third-order corrections to energies of real hole excitations.

or change in the effective mass

$$(E_\alpha-E_\alpha^*)_{M^*}=\tfrac{1}{2}\sum_{ijmn} n_in_j(1-n_m)(1-n_n)K_{mn,ij}^2 \times\frac{\partial}{\partial n_\alpha}(E_i^*+E_j^*-E_m^*-E_n^*)^{-1}. \tag{2.7}$$

These correspond to the second- and third-order diagrams given in Fig. 1 with the K matrix acting as the interaction instead of the potential.

The derivative of the total energy required in Eq. (2.3) to determine the energy has been evaluated numerically, using the K-matrix procedure previously applied by Brueckner and Gammel[4] to the determination of the properties of nuclear matter. The derivative was obtained by making a finite change Δn_α in the population of the Fermi gas and verifying that the ratio of $\Delta E/\Delta n_\alpha$ was independent of the value chosen for Δn_α.

To simplify the calculation, we have made use of the very weak dependence[5] of the K matrix on total momentum and evaluated the K matrix at the average total momentum in the Fermi gas. This leads to some complications in evaluating the change in the propagator in Eq. (2.2). We describe in Appendix A the procedure used to circumvent this problem.

Another complication arises from the occurrence of vanishing energy denominators in the K matrices determining the energies of real excitations. These occur for holes, for example, from the energy conserving transitions made possible by creation of a hole in the Fermi gas. To obtain the real part of the single-particle energy, we have taken the principal part of the singular integral encountered. In practice this was done by making the replacement:

$$1/\Delta E\rightarrow\Delta E/[(\Delta E)^2+\Gamma^2], \tag{2.8}$$

with Γ taken sufficiently small so that the result became independent of Γ. A suitable value of Γ was found to be 9 Mev, which is reasonable since typical excitation energies are of the order of 50 to 100 Mev.

The results for the corrections to the single-particle energy are summarized in Table I and their effect on the

[4] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[5] This result was obtained in the detailed numerical calculation of the K matrix in the study of nuclear matter.

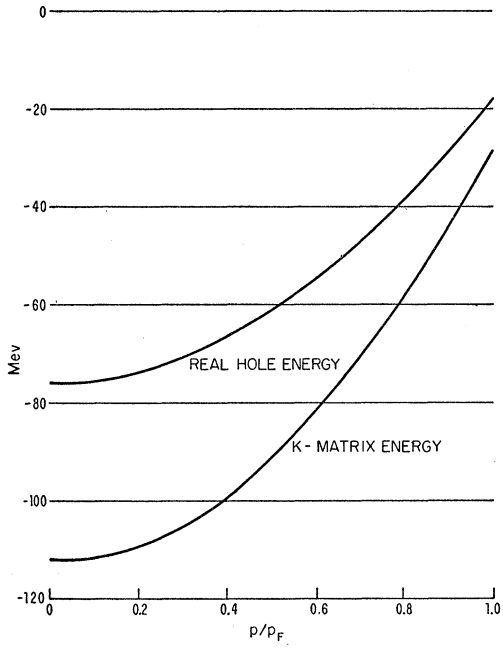

FIG. 2. Shift in single-particle energy of real excitations below the Fermi surface.

single-particle energy shown in Fig. 2. The energy spectrum shows a considerable shift, the corrected value of -17.8 Mev at the Fermi surface now nearly agreeing with the mean binding energy. The remaining small corrections are probably due to higher order effects in the rearrangement energy.

These quantitative results do not agree with previous estimates of Hugenholtz and Van Hove[3] and of Thouless.[6] The discrepancy arises from an overestimate of the second-order term and an underestimate of the third-order term, the latter coming from the shift in energy spectrum or effective mass of the bound particles in the presence of an excitation.

III. EFFECT ON THE GROUND-STATE ENERGY

We now turn to the determination of the effects of the changed single-particle energies on the total energy of the system. To do so, as BG have shown, it is essential to take into account the differences between the energies of virtual excitations as they appear in Eq. (2.2) and the complex energies of real particles and holes. The

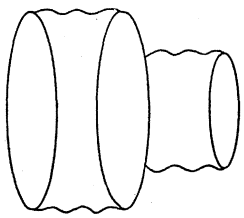

FIG. 3. Second-order rearrangement energy correction to ground-state energy.

[6] D. J. Thouless, Phys. Rev. **112**, 906 (1958).

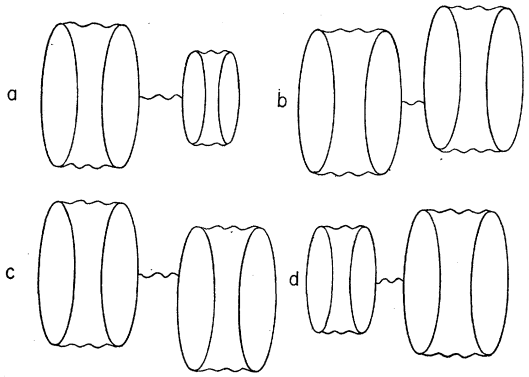

FIG. 4. Third-order rearrangement energy correction to ground-state energy.

second-order correction to the single-particle energy appears as an insertion into the ground-state energy as shown in lowest order in Fig. 3. The excitation energy of the virtual particles and holes now enters into the energy denominator of the correction and, as shown by BG, markedly reduces the size of the shift in energy. We have approximated to the "off-energy-shell" effects by introducing into the second-order correction to the single-particle energy a mean excitation energy equal to the excitation energy of the Fermi gas. The third-order correction to the single-particle energy appears as an insertion into the ground-state energy as shown in Fig. 4(a). This term is the only correction included by Hugenholtz in his definition of reducible contributions to the single-particle energy. There are, however, the completely equivalent diagrams, irreducible according to the definition of Hugenholtz, shown in Figs. 4(b), (c), and (d). These combine very simply, as shown by BG, to give a change in ground-state energy equivalent to one-half that resulting from the "on-energy-shell" insertion of the third-order correction to the single-particle energy.

The change in the ground-state energy then is equivalent to that resulting from the change in single-particle energy due to the "off-energy-shell" second-order term and one-half of the "on-energy-shell" third-order term. These shift the single-particle energy upward by about 5 Mev at the Fermi surface and 7 Mev at zero momentum. The resulting change in the ground-state energy has been computed and found to be 1.5 Mev. This correction is of the same magnitude as other uncertainties in the K-matrix method[7] as well as that arising from "off-energy-shell" propagation, as discussed by Brueckner and Gammel.[4]

APPENDIX

In determining the contribution to the rearrangement energy from the second-order term, an expression of the

[7] These are discussed in detail particularly by H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

form

$$I=\frac{1}{8}\int\frac{d\mathbf{k}_m}{E_i+E_j-E_m-E_n}f(\tfrac{1}{2}|\mathbf{k}_m-\mathbf{k}_\alpha|) \quad \text{(A.1)}$$

must be evaluated. This is a function of the relative momenta

$$\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_i-\mathbf{k}_j),\quad \mathbf{k}'=\tfrac{1}{2}(\mathbf{k}_m-\mathbf{k}_\alpha), \quad \text{(A.2)}$$

and the total momentum

$$\mathbf{P}=\mathbf{k}_i+\mathbf{k}_j. \quad \text{(A.3)}$$

We wish to define an appropriate mean value of the total momentum and suppress in Eq. (A.1) the explicit dependence on total momentum. To do this, we first ignore the dependence of the energy denominator on total momentum. This dependence is zero, for example, in the effective mass approximation. We then write Eq. (A.1) in relative and total momentum coordinates as

$$I=\int d\mathbf{k}'\frac{M^*}{k^2-k'^2}f(k'), \quad \text{(A.4)}$$

with the integration taken so that the limits imposed by the exclusion principle are satisfied, i.e.,

$$\begin{aligned}|\mathbf{k}_i|&=|\mathbf{k}+\tfrac{1}{2}\mathbf{P}|\leqq k_F,\\ |\mathbf{k}_j|&=|\mathbf{k}-\tfrac{1}{2}\mathbf{P}|\leqq k_F,\\ |\mathbf{k}_m|&=|\mathbf{k}'+\tfrac{1}{2}\mathbf{P}|\geqq k_F.\end{aligned} \quad \text{(A.5)}$$

The relation of $\mathbf{k}'$ to $\mathbf{k}_\alpha$ also imposes the condition

$$\mathbf{k}_\alpha=\mathbf{k}'-\tfrac{1}{2}\mathbf{P}. \quad \text{(A.6)}$$

The result for the rearrangement energy does not depend on the direction of $\mathbf{k}_\alpha$, so we replace Eq. (A.6) by the equivalent condition

$$|\mathbf{k}_\alpha|=|\mathbf{k}'-\tfrac{1}{2}\mathbf{P}|. \quad \text{(A.7)}$$

We then can evaluate the angular integral in Eq. (A.4), with the result

$$I=\frac{4\pi k_\alpha}{P}\int_{L\{[\frac{1}{2}(k_F^2+k_\alpha^2)-\frac{1}{4}P^2]^{\frac{1}{2}},|\mathbf{k}_\alpha-\frac{1}{2}\mathbf{P}|\}}^{k_\alpha+\frac{1}{2}P}k'dk'f(k'), \quad \text{(A.8)}$$

where $L\{\ \}$ we near the lesser of the two terms. We now define the mean value of $\bar{P}$ so that it gives correctly the average of Eq. (A.8) if the variation of $f(k')$ is ignored. This approximation derives its approximate validity from the strong dependence on P as it appears explicitly in Eq. (A.8), and the weak dependence on $\bar{P}$ through its implicit appearance in $f(k')$. In this approximation, Eq. (A.8) becomes

$$I=4\pi k_\alpha^2 H(k_\alpha,P)[f(k')]_{\text{av}}, \quad \text{(A.9)}$$

with

$$\begin{aligned}H(k_\alpha,P)&=\frac{(k_\alpha+P)^2-k_F^2}{4k_\alpha P}, && |\mathbf{k}_\alpha-P|\leqq k_F\\ &=1, && |k_\alpha-P|\geqq k_F.\end{aligned}$$

We next evaluate the average of I over $\mathbf{P}$, since it is this average we wish to approximate by introduction of an average of $\mathbf{P}$ into Eq. (A.8). The result is, after evaluation of the angular integrals,

$$[H(k_\alpha,P)]_{\text{av}}=\int P^2dPG(P)H(k_\alpha,P)\Big/\int P^2dPG(P), \quad \text{(A.11)}$$

with

$$\begin{aligned}G(P)&=1, && 0<P<2(k_F-k)\\ &=\frac{k_F^2-k^2-\frac{1}{4}P^2}{Pk}, && 2(k_F-k)<P<2(k_F^2-k^2)^{\frac{1}{2}}\\ &=0, && 2(k_F^2-k^2)^{\frac{1}{2}}<P.\end{aligned} \quad \text{(A.12)}$$

Our final value of P_{av} then is defined by

$$[H(k_\alpha,P)]_{\text{av}}=H(k_\alpha,P_{\text{av}}) \quad \text{(A.13)}$$

The evaluation of Eq. (A.11) is straightforward but tedious; we do not give details here. The result is given in Fig. 5.

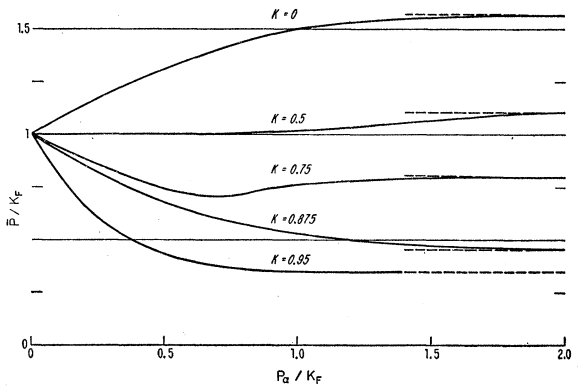

FIG. 5. Average value of total momentum as a function of relative momentum, as defined by Eq. (A.13). The dotted lines indicate the average approached by large values of P_α.