# Ground-State Energy of a Many-Fermion System. II*

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(Received January 7, 1960)

The perturbation series for the ground-state energy of a many-fermion system is investigated to arbitrary order for the "isotropic" case. This is the case of over-all spherical symmetry, both in the interaction and in the unperturbed single particle energies. It is shown that for spin one-half fermions the Brueckner-Goldstone perturbation series is valid to all orders in the perturbation. For spins greater than one-half it is in general incorrect even in the isotropic case, unless the interactions are spin independent.

The discussion to arbitrary order in the interaction is carried out by means of a Feynman-like propagator formalism, which is developed in detail.

## I. INTRODUCTION

In a previous paper[1] it has been shown that the usual Brueckner-Goldstone (BG) perturbation series for the ground-state energy of a system of interacting fermions is not valid in general. By a direct calculation to terms of the second order in the strength of the perturbation, it was found that unless the unperturbed energies of the particles and also their interactions were spatially isotropic, then there were corrections to the BG series which could not vanish. This left open the question of whether or not the BG series is valid to all orders of perturbation theory in the totally "isotropic case" mentioned above. In this paper we shall show that in fact the BG series *is* valid for the isotropic case, for fermions of spin $\frac{1}{2}$.

In I, the method of calculation was the following. We computed the energy as a function of temperature (via the grand canonical distribution), and then went carefully to the limit $T=0$, in order to find the ground-state energy. The actual calculation of the grand partition function followed the method of Bloch and De Dominicis,[2] which expresses it in terms of linked diagrams analogous to those used in the BG theory. Whereas this formalism is perfectly well suited to a second order perturbation calculation, it becomes very awkward for general discussions to arbitrary order. In particular the discussion of the anomalous contributions (see I) turns out to be very closely related to the discussion of "self-energy" effects. As is well known in field theory, such effects are dealt with conveniently by means of a Feynman-like "propagator" formalism. Such formalisms for the many-body problem at nonzero temperature are enjoying a very considerable vogue these days.[3] Their charm lies in their simplicity; they represent a trivial modification of the usual Feynman technique. In Sec. II we shall develop a version of this formalism particularly suited to our problem. In Sec. III we shall show how to describe "self-energy" effects. By means of a theorem of Lee and Yang,[4] we express the grand partition function in our propagator language. Finally, in Sec. IV, we put this all together to construct a proof of the BG series for the ground-state in the isotropic, spin $\frac{1}{2}$ case.

## II. PROPAGATOR FORMALISM

Our object is to calculate the grand partition function $Z_G$. This is defined by

$$Z_G=\mathrm{Tr}\,\{e^{-\beta(H-\mu N)}\}. \tag{1}$$

In (1), $\beta=1/kT$, $\mu$ is the chemical potential of the system, $N$ is the operator giving the number of particles, and $H$ is the Hamiltonian of the system. In the notation of second quantization we may write $H$ as

$$H=H_0+H', \tag{2}$$

$$H_0=\textstyle\sum_r \epsilon_r a_r^\dagger a_r, \tag{3}$$

$$H'=\tfrac{1}{2}\sum_{rsr's'} a_r^\dagger a_s^\dagger a_{s'} a_{r'}(rs|v|r's'). \tag{4}$$

In (3) and (4), $\epsilon_r$ is the energy of the unperturbed single particle states; $a_r$, $a_r^\dagger$ are the corresponding destruction and creation operators, respectively, and finally $(rs|v|r's')$ is the ordinary matrix element of the two-body interactions between any pair of fermions. The index $r$ is a shorthand index for both the momentum of the particle and its spin state. It will be convenient to take the direction of quantization of spin along the

* This work was supported in part by the Office of Naval Research.

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momentum of the particle; then $r$ stands for $(\mathbf{p},m)$, where $m$ is the projection of the spin on $\mathbf{p}$. (We take units such that $\hbar=1$.) Finally it will be assumed that $v$ is proportional to some small dimensionless parameter $\lambda$.

If we write $Z_G$ in the form

$$Z_G=\exp\{-\beta\Omega(\beta,\mu,V)\}, \tag{5}$$

then all the thermodynamic properties of the system may be derived from $\Omega$, which we shall call the thermodynamic potential. The chemical potential is determined by solving

$$\bar{N}=-\partial\Omega/\partial\mu, \tag{6}$$

where $\bar{N}$ is the mean number of particles present. The entropy $S$ is given by

$$S=-\partial\Omega/\partial T, \tag{7}$$

and the mean energy $(E)$ by

$$E=\Omega+\mu\bar{N}+TS=\partial(\beta\Omega)/\partial\beta+\bar{N}\mu. \tag{8}$$

From (8), since $S$ approaches zero as $T$ does, we obtain for the ground-state energy $(E_0)$

$$E_0=\lim_{T\to 0}(\Omega+\mu\bar{N}). \tag{9}$$

We now have to obtain an expression for $\Omega$ in terms of propagators. To do this we write

$$F(\beta)=\exp\{-\beta(H_0-\mu N+H')\}. \tag{10}$$

Since

$$\partial F(\beta)/\partial\beta=-(H_0-\mu N+H')F(\beta), \tag{11}$$

the quantity $U(\beta)$ defined by

$$F(\beta)=[\exp\{-\beta(H_0-\mu N)\}]U(\beta) \tag{12}$$

satisfies

$$\partial U(\beta)/\partial\beta=-H'(\beta)U(\beta), \quad U(0)=1. \tag{13}$$

In (13)

$$H'(\beta)=e^{\beta H_0}H'e^{-\beta H_0}, \tag{14}$$

since $N$ commutes with $H'$.

Now

$$\begin{aligned} Z_G=e^{-\beta\Omega}&=\mathrm{Tr}\,[\exp\{-\beta(H_0-\mu N)\}U(\beta)] \\ &=e^{-\beta\Omega_0}\langle U(\beta)\rangle, \end{aligned} \tag{15}$$

where $\Omega_0$ is the thermodynamic potential for the unperturbed system, evaluated for the true chemical potential $\mu$. As is familiar from field theory, (13) may be solved in terms of the Dyson ordering operator

$$U(\beta)=1+\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\int_0^\beta\cdots\int_0^\beta du_1\cdots du_n \times P[H'(u_1)\cdots H'(u_n)], \tag{16}$$

where the operator $P$ means that the largest $u$ is always placed to the extreme left, the next largest $u$ immediately to the right of it, and so forth. Equation (15) then becomes [using (4)]

$$\begin{aligned} &\exp\{-\beta(\Omega-\Omega_0)\} \\ &=1+\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\int_0^\beta\cdots\int_0^\beta du_1\cdots du_n \\ &\qquad\times\langle P[H'(u_1)\cdots H'(u_n)]\rangle \\ &=1+\sum_{n=1}^{\infty}\frac{(-)^n}{n!}\frac{1}{2^n}\prod_{i=1}^{n}(r_is_i|v|r_i's_i') \\ &\qquad\times\int_0^\beta\cdots\int_0^\beta du_1\cdots du_n \\ &\qquad\times\langle P[a_{r1}^\dagger(u_1)a_{s1}^\dagger(u_1)a_{s'1}(u_1)a_{r'1}(u_1)\cdots]\rangle. \end{aligned} \tag{17}$$

In (17)

$$a_r^\dagger(u)=e^{uH_0}a_r^\dagger e^{-uH_0}=a_r^\dagger e^{\epsilon_r u}, \tag{18}$$

$$a_r(u)=e^{-\epsilon_r u}a_r. \tag{18'}$$

In connection with the evaluation of expressions like (17), Bloch and De Dominicis[2] have shown that the value of the indicated average (for fixed $u_1$, $u_2$, $\cdots u_n$) may be obtained by simply taking the average of all possible pairings of creation and destruction operators, and then adding the results with the proper sign. This sign is determined by the following rules. (a) To each pairing associate a plus sign if the creation operator belongs to a larger $u$ than the destruction operator, and a negative sign in the opposite case. (b) Assign to the entire term a sign given by $(-1)^{n_l}$ where $n_l$ is the number of "closed loops" associated with the pairing. The number of closed loops has a very simple geometrical significance when the different possible pairings are represented by diagrams. This is done as follows: with each interaction in (17) associate a wiggly line, and let the creation operators be represented by lines leaving the two ends, the destruction operators by entering lines. Then all possible pairings come when all lines leaving interactions are identified in all possible ways with lines entering interactions. (See Fig. 1.) Clearly an $n$th order term will have $(2n)!$ possible pairings (any creation operator can pair with any of $2n$ destruction operators), and there will therefore be $(2n)!$ diagrams representing the $n$th order term in (17). The number $n_l$ just represents the number of closed loops or paths that exist in the diagram. These numbers are given for the diagrams of Fig. 1 immediately below them.

With each line in a diagram one must associate a factor coming from the averaging of the creation and destruction operator, with the proper $u$-ordering and sign. This factor is

$$g_r(u,u')=\epsilon(u,u')\langle P(a_r^\dagger(u)a_r(u'))\rangle \tag{19}$$

for a line bearing the index $r$ and going from an interaction associated with $u$ to one associated with $u'$.

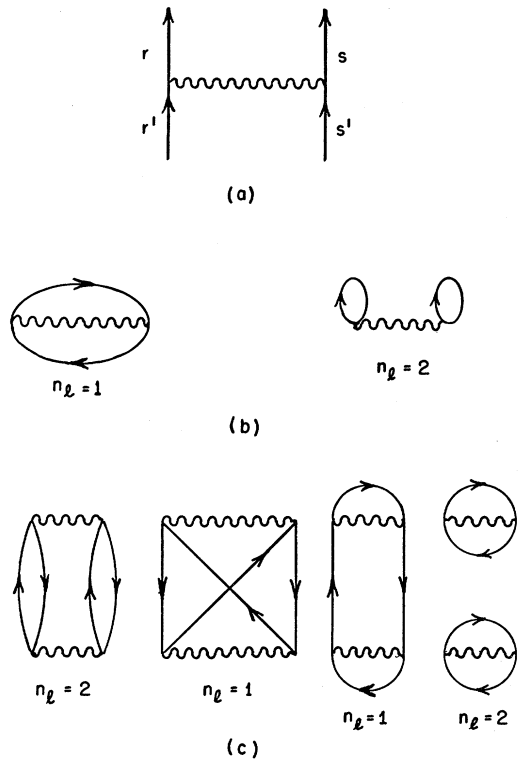


FIG. 1. Graphical representation of terms in grand partition function. (a) Representation of the interaction $(rs|v|r's')$. (b) The possible first order diagrams. (c) Typical second order diagrams.

As usual

$$\epsilon(u,u')=1, \quad u>u' \qquad =-1, \quad u<u'. \tag{20}$$

If we write out (19) a little more explicitly we obtain [using (18)]

$$g_r(u,u')=g_r(u-u')\langle a_r{}^\dagger a_r\rangle e^{\epsilon_r(u-u')}, \quad u>u' \qquad =-\langle a_r a_r{}^\dagger\rangle e^{\epsilon_r(u-u')}, \quad u<u'. \tag{21}$$

Now

$$\langle a_r{}^\dagger a_r\rangle = 1/(e^{\beta(\epsilon_r-\mu)}+1)\equiv f_r{}^-, \tag{22}$$

and

$$\langle a_r a_r{}^\dagger\rangle = \langle 1-a_r{}^\dagger a_r\rangle = 1-f_r{}^-\equiv f_r{}^+ \tag{23}$$

Therefore we may write (21)

$$g_r(u-u')=f_r{}^- e^{\epsilon_r(u-u')}, \quad u>u' \qquad =-f_r{}^+ e^{\epsilon_r(u-u')}, \quad u<u'. \tag{24}$$

There is actually one small ambiguity about (24): it is not determinate when $u=u'$. This latter case occurs whenever the pairings are associated with the same interaction. Going back to the original interaction (4) we see that in this case the averaging is always taken with the creation operator to the left. That is, for $g_r(0)$ we must take $f_r{}^-$ [or what amounts to the same thing $g_r(0^+)$, where $0^+$ is an infinitesimal positive number].

When we compute $\Omega$ we have to take the ln of the right-hand side of (17). Now it is a matter of straightforward and very simple combinatorics to show the only effect of taking the ln is to eliminate all *unlinked* diagrams from consideration (the proof is essentially identical with that of Bloch and De Dominicis). By *linked* diagrams we mean those which consist of a single piece, while by disconnected diagrams we mean those that fall into two or more pieces. In Fig. 1, all the diagrams but the last are linked, and the last is unlinked. Therefore (17) becomes

$$\Omega=\Omega_0+\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{\beta n!}\frac{1}{2^n} \times\prod_{i=1}^{n}(r_i s_i|v|r_i' s_i')\int_0^\beta\cdots\int_0^\beta du_1\cdots du_n \times\langle P(a_{r_1}{}^\dagger(u_1)a_{s_1}{}^\dagger(u_1)a_{s'1}(u_1)a_{r'1}(u_1)\cdots)\rangle_L, \tag{25}$$

where the subscript $L$ means that we are to take only those pairings which lead to linked diagrams.

The thing which makes a propagator formalism possible and useful is that there exists a very simple representation for $g_r(u-u')$ which allows the integrals over $u_i$ in (25) to be done at once. This representation is the following

$$g_r(u-u')=\frac{1}{\beta}\sum_l\frac{1}{\zeta_l-\epsilon_r}e^{\zeta_l(u-u')}, \tag{26}$$

where

$$\zeta_l=\frac{(2l+1)\pi i}{\beta}+\mu, \quad l=0,\pm1,\pm2,\cdots,\pm\infty. \tag{27}$$

The proof of (26) is elementary and is given in Appendix A. If we insert (26) into (25) for any particular diagram, we see at once that the integrals may be done. In fact, since every line starting or ending at a given interaction corresponds to the same $u$, the dependence of the integrand in (25) on that $u$ will be be of the form

$$\exp\{u(\zeta_{l_1}+\zeta_{l_2}-\zeta_{l_3}-\zeta_{l_4})\}, \tag{28}$$

where $\zeta_{l_1}$, $\zeta_{l_2}$ are associated with the lines leaving the interaction at $u$, and $\zeta_{l_3}$, $\zeta_{l_4}$ are associated with the lines entering it. If a line leaves and enters the same interaction [as in, say, diagrams like those of Fig. 1(b)], then the corresponding $\zeta$'s are the same, and such a line gives no contribution to the exponent in (28). From (27) we have for this factor

$$\exp\{u(l_1+l_2-l_3-l_4)2\pi i/\beta\}. \tag{29}$$

Integrating (29) over $u$ from 0 to $\beta$ gives

$$\beta\delta_{l_1+l_2,l_3+l_4} \quad \text{or} \quad \zeta_{l_1}+\zeta_{l_2}=\zeta_{l_3}+\zeta_{l_4}. \tag{30}$$

This has a certain analogy to the energy conservation theorem, if we think of $\zeta$ as a kind of energy associated with each line.

Putting all these results together, we may summarize the rules for computing the corrections for $\Omega$ to $\Omega_0$:

(1) Draw all possible $n$th order linked diagrams.
(2) With each such diagram associate a factor

$$\frac{1}{\beta}\frac{(-)^{n+1}}{n!}\frac{1}{(2\beta)^n}(-)^{nl}\prod_{i=1}^{n}(r_i s_i|v|r_i' s_i').$$

(3) For each line labelled by $r$ associate a factor

$$1/(\zeta l_r-\epsilon_r)\equiv S_r(\zeta l_r).$$

We shall call the quantity $S_r(\zeta l)$ the *free* propagator for a particle in state $r$.

(4) Restrict the number of independent $\zeta$'s by the conservation theorem (30), which is equivalent to putting the total $\zeta$ in lines leaving any interaction, equal to the total $\zeta$ for lines entering it. There will in general be $n+1$ independent $\zeta$'s in an $n$th order diagram.

(5) Lines labelled by $r$ entering and leaving the *same* interaction give a factor of

$$[1/(\zeta l_r-\epsilon_r)]\exp\{\zeta l_r 0^+\}$$

the $\zeta l_r$ being freely summed over.

(6) Sum over all the indices $r$, $s$ etc. and over all the independent $\zeta$'s.

As examples of these rules we give the contributions of some of the diagrams of Fig. 1: the contribution to $\Omega$ of the first diagram of Fig. 1(b) and the first and third diagrams of Fig. 1(c) are, respectively,

$$-\frac{1}{2\beta^2}\sum_{l_1 l_2}\sum_{r_1 r_2}(r_1 r_2|v|r_2 r_1)\frac{\exp\{\zeta l_1 0^+\}}{\zeta l_1-\epsilon_{r1}}\frac{\exp\{\zeta l_2 0^+\}}{\zeta l_2-\epsilon_{r2}} \tag{31}$$

$$-\frac{1}{2^2 2!\beta^3}\sum_{l_1 l_2 l_3 l_4}\sum_{r_1 r_2 r_3 r_4}|(r_1 r_2|v|r_3 r_4)|^2 \times\frac{\delta_{l_1+l_2,l_3+l_4}}{(\zeta l_1-\epsilon_{r1})(\zeta l_2-\epsilon_{r2})(\zeta l_3-\epsilon_{r3})(\zeta l_4-\epsilon_{r4})} \tag{32}$$

$$\frac{1}{2^2 2\beta^3}\sum_{l_1 l_2 l_3 l_4}\sum_{r_1 r_2 r_3 r_4}\frac{(r_1 r_4|v|r_4 r_2)(r_3 r_2|v|r_1 r_3)\delta_{l_1,l_2}}{(\zeta l_1-\epsilon_{r1})(\zeta l_2-\epsilon_{r2})} \times\frac{\exp(\zeta l_3 0^+)}{(\zeta l_3-\epsilon_{r3})}\frac{\exp(\zeta l_4 0^+)}{(\zeta l_4-\epsilon_{r4})}. \tag{33}$$

In (33) if we assume isotropic, translationally invariant forces between the particles, $r_1=r_2$ by momentum and angular momentum conservation, and the expression simplifies somewhat.

The sums over $l$ which come into $\Omega$ are actually not difficult to do explicitly. If one does them, one recovers the general formulas of Bloch and De Dominicis in terms of energy denominators and $f_r^{\pm}$ functions. The usefulness of the formalism, however, lies in just avoiding this summation now, and first grouping together self-energy terms before we carry it out.

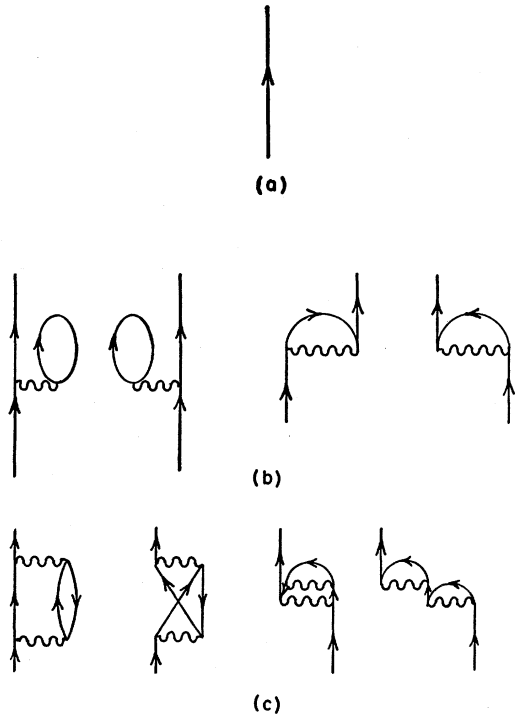


FIG. 2. Diagrams for the propagator. (a) Zeroth order diagram. (b) All possible first order diagrams. (c) Some second order diagrams.

## III. REDUCTION OF SELF-ENERGY DIAGRAMS

In considering the expression for $\Omega$ we have only had to deal with linked closed diagrams. In considering self-energy effects it is convenient to study another class of linked diagrams: those for which a particle line enters, things of arbitrary complexity happen, and then a single particle line emerges. To avoid complications which are unnecessary in the establishment of the BG series, we shall assume at the outset that the interaction is translationally invariant and isotropic. Then, by conservation of linear and angular momentum, the emerging line must represent the same state as the entering line. We call the object which is obtained by calculating the contribution of all such diagrams according to the rules[5] (1) through (6), the *propagator* $S_r'(\zeta l)$[6] for a particle in state $r$. In Fig. 2, some diagrams contributing to $S_r'$ are given. The contributions of Fig. 2(a), Fig. 2(b) and the first diagram of Fig. 2(c)

[5] Rule (2) has to be multiplied by $(-\beta)$ to correct for the factor $-1/\beta$ which comes from the definition of $\Omega$.

[6] This quantity is exactly the single particle Green's function (in the momentum-energy representation) for our problem, in the sense of Fradkin, Abrikosov et al. For many purposes it is very convenient to make use of its definition as a Green's function, but not in the present context.

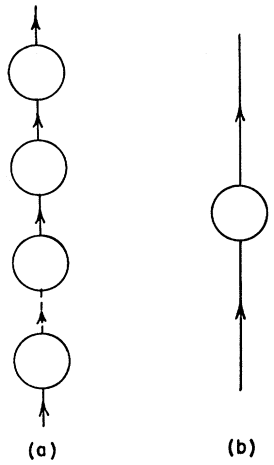


FIG. 3. Structure of propagator. (a) Structure of a general diagram for the propagator. (b) Structure of a proper propagator diagram.

are, respectively,

$$1/(\zeta_l-\epsilon_r) \tag{34}$$

$$\frac{1}{(\zeta_l-\epsilon_r)^2}\frac{1}{2\beta}\sum_{r_1}\sum_{l_1}[(rr_1|v|rr_1)+(r_1r|v|r_1r) -(rr_1|v|r_1r)-(r_1r|v|rr_1)]\frac{\exp(\zeta_{l_1}0^+)}{\zeta_{l_1}-\epsilon_{r_1}} \tag{35}$$

$$\frac{1}{(\zeta_l-\epsilon_r)^2}\left(-\frac{1}{2!2^2\beta^2}\right)\sum_{r_1r_2r_3}\sum_{l_1l_2l_3l_4} \times\frac{|(r_1r_2|v|rr_3)|^2\delta_{l_1+l_2,l_3+l}}{(\zeta_{l_1}-\epsilon_{r_1})(\zeta_{l_2}-\epsilon_{r_2})(\zeta_{l_3}-\epsilon_{r_3})}. \tag{36}$$

The general structure of the diagrams contributing to $S_r'$ is given in Fig. 3(a). Such a diagram will be called *improper* whenever it can be made to fall into two parts by cutting a single particle line. It is called *proper* when this is not so. Thus in Fig. 2(b), 2(c), all the diagrams but the last are proper, and the last is improper. The contribution of improper propagator diagrams is easily expressed in terms of proper ones. By their structure the contribution of all possible proper diagrams [Fig. 3(b)] to $S_r'(\zeta_l)$ may be written,

$$\frac{1}{\zeta_l-\epsilon_r}G_r(\zeta_l)\frac{1}{\zeta_l-\epsilon_r}. \tag{37}$$

The quantity $G_r(\zeta_l)$ is called the *proper self-energy*, for reasons which become clear below. In terms of $G_r(\zeta_l)$ we may write for the entire propagator

$$S_r'(\zeta_l)=\frac{1}{\zeta_l-\epsilon_r}+\frac{1}{\zeta_l-\epsilon_r}G_r(\zeta_l)\frac{1}{\zeta_l-\epsilon_r} +\frac{1}{\zeta_l-\epsilon_r}G_r(\zeta_l)\frac{1}{\zeta_l-\epsilon_r}G_r(\zeta_l)\frac{1}{\zeta_l-\epsilon_r}+\cdots. \tag{38}$$

Summing (38) we obtain

$$S_r'(\zeta_l)=1/[\zeta_l-\epsilon_r-G_r(\zeta_l)]. \tag{39}$$

If we regard $G_r(\zeta_l)$ as a modification of $\epsilon_r$, then (39) has the same general form as the expression for $S_r(\zeta_l)$; hence the name proper self-energy for $G_r(\zeta_l)$.

The $G_r(\zeta_l)$ are given by diagrams which have the structure of a closed linked diagram with one line simply removed. Now let us call a diagram that has no self-energy insertions a *skeleton* diagram. Then clearly all diagrams for $G_r(\zeta_l)$ may be obtained by drawing all skeleton diagrams and then inserting all possible proper self-energy parts. This is equivalent to

$$G_r(\zeta_l)=[\text{All possible skeleton diagrams with } S_r \text{ replaced by } S_r'.] \tag{40}$$

Equation (40) is actually an implicit equation for $G_r(\zeta_l)$ since the $S_r'$ also contain the $G_r$.

We now come to the question of expression the thermodynamic potential in terms of the propagators or the proper self-energies. At first glance one might think that all that is necessary is to write down all skeleton closed linked diagrams, and replace $S_r$ by $S_r'$ everywhere. This is not so, however, since it would result in an overcounting of diagrams. The reason is that the prescription for computing $\Omega$ is to take each possible closed linked diagram that can be drawn just once. However, if we insert self-energy parts into all closed linked skeleton diagrams, we get the same diagram several times. This is illustrated in Fig. 4 in a very simple case. Inserting a self-energy part into the line $r$ of Fig. 4(a) gives 4(b); inserting it into the line $S$ of 4(a) gives 4(c). However, this closed linked diagram occurs only once and not twice. Another way of saying this is that the reduction to skeleton diagrams for closed linked diagrams is not unique; we can regard either the upper or the lower interaction line in [say Fig. 4(b) as contained in the self-energy part. This nonuniqueness does not enter into the expression for $G_r$ since some interactions (the ones at which the lines enter and leave] are singled out and are by definition to remain in the skeleton diagrams. Hence we have no such counting difficulty for $G_r$.

To get a correct expression for $\Omega$ we proceed as follows. Suppose we consider any $n$th order diagram for $\Omega$. If we break open any of its $2n$ lines we obtain a possible $n$th order diagram (not proper in general) for the propagator. Let us call $G_{rn}'(\zeta_l)$ the total self-energy part of the $n$th order, proper or improper. Then we may

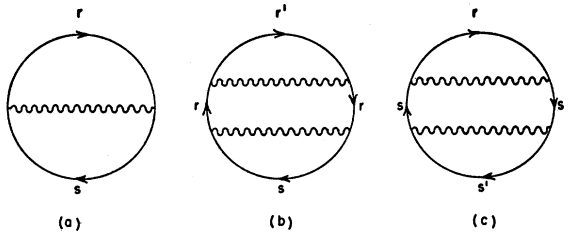


FIG. 4. Effect of inserting self-energy parts in skeleton diagram.

write that the $n$th order contribution to the thermodynamic potential ($\Omega_n$) is

$$\Omega_n=\frac{1}{2n}\frac{1}{\beta}\sum_r\sum_l\frac{1}{\zeta_l-\epsilon_r}G_{rn}'(\zeta_l). \tag{41}$$

The factor $1/(2n)$ is due to the overcounting difficulty: we get the same closed linked diagram $2n$ times when we close all possible $n$th order propagator diagrams. Equation (41) may also be easily established directly by combinatorics on the numbers of different possible diagrams. The factor $1/(2n)$ in (41) makes it difficult to carry out the summation over $n$. As is usual in such cases, we employ the following artifice. Imagine that we compute everything for an interaction strength $\lambda'$ instead of $\lambda$, but imagining the chemical potential $\mu$ as given. Then

$$(1/n)G_{rn}'(\zeta_l;\lambda)=\int_0^\lambda G_{rn}'(\zeta_l;\lambda')(d\lambda'/\lambda'). \tag{42}$$

Using (42) and (41) we obtain

$$\Omega=\Omega_0+\frac{1}{2\beta}\sum_r\sum_l\int_0^\lambda\frac{1}{\zeta_l-\epsilon_r}G_r'(\zeta_l;\lambda')\frac{d\lambda'}{\lambda'}, \tag{43}$$

where $G_r'$ is the sum of all possible self-energy parts, proper or improper. This is in turn given by

$$G_r'=G_r+G_r\frac{1}{\zeta_l-\epsilon_r}G_r+G_r\frac{1}{\zeta_l-\epsilon_r}G_r\frac{1}{\zeta_l-\epsilon_r}G_r+\cdots=\frac{(\zeta_l-\epsilon_r)G_r}{\zeta_l-\epsilon_r-G_r}. \tag{44}$$

Therefore we obtain

$$\Omega=\Omega_0+\frac{1}{2\beta}\sum_r\sum_l\int_0^\lambda\frac{d\lambda'}{\lambda'}G_r(\zeta_l;\lambda')S_r'(\zeta_l,\lambda'). \tag{45}$$

We may also write

$$\lambda\frac{\partial\Omega}{\partial\lambda}=\frac{1}{2\beta}\sum_r\sum_l G_r(\zeta_l;\lambda)S_r'(\zeta_l;\lambda). \tag{46}$$

The expression (45) for $\Omega$ is still very awkward because it contains an integration over the interaction strength. We now transform this into a formula much more convenient for our (and many other) purposes.

Consider the following expression[7]

$$Y=-\frac{1}{\beta}\sum_r\sum_l\exp(\zeta_l 0^+)\{\ln(\epsilon_r+G_r(\zeta_l)-\zeta_l)+S_r(\zeta_l)G_r(\zeta_l)\}+Y', \tag{47}$$

[7] The argument of the ln in (47) is a complex number in general. We mean by it that branch of the ln which is real if the argument is a real positive number.

$$Y'\equiv\left[\begin{array}{l}\text{Contribution of all closed linked skeleton}\\ \text{diagrams computed according to rules}\\ \text{(1)–(6) but with } S_r(\zeta_l) \text{ replaced by } S_r'(\zeta_l).\end{array}\right] \tag{48}$$

The quantity $Y$ may be regarded as a function of all the numbers $G_r(\zeta_l)$. Suppose we consider

$$\frac{\partial Y}{\partial G_r(\zeta_l)}=-\frac{1}{\beta}\sum_r\sum_l G_r(\zeta_l)[S_r'(\zeta_l)]^2+\frac{\partial Y'}{\partial G_r(\zeta_l)}. \tag{49}$$

To compute $\partial Y'/\partial G_r(\zeta_l)$ we proceed as follows. Call $\nu$ the number of interactions which appear explicitly in some skeleton diagram corresponding to $Y'$. Then by the same reasoning that leads to (41), we may write

$$Y'=\frac{1}{\beta}\sum_\nu\sum_r\sum_l\frac{1}{2\nu}S_r'(\zeta_l)G_{r\nu}''(\zeta_l). \tag{50}$$

$G_{r\nu}''(\zeta_l)$ is the total $\nu$th order self-energy part according to (40), where only the $\lambda$ occurring in the explicit interactions of the skeleton diagram are used to determine the order. Differentiating (50) with respect to $G_r(\zeta_l)$ has the effect of "opening" any of the $2\nu$ lines of a $\nu$th order diagram. Each will give the same contribution after summation on $r$, so we obtain

$$\frac{\partial Y'}{\partial G_r(\zeta_l)}=\frac{1}{\beta}\sum_\nu\sum_r\sum_l[S_r'(\zeta_l)]^2G_{r\nu}''=\frac{1}{\beta}\sum_r\sum_l[S_r'(\zeta_l)]^2G_r(\zeta_l). \tag{51}$$

Combining (51) and (49) we see that

$$\partial Y/\partial G_r(\zeta_l)=0. \tag{52}$$

If we make any first order change in the $G_r(\zeta_l)$ around their correct values, the quantity $Y$ does not change, i.e., $Y$ is *stationary* with respect to changes of the $G_r(\zeta_l)$. [It can also be shown to have its maximum value for the correct $G_r(\zeta_l)$. We may also say that the condition (52) gives us the Eq. (40) for $G_r(\zeta_l)$, so that (52) provides us with a variational principle for determining $G_r(\zeta_l)$. We shall not, however, exploit these properties here.]

Now consider $\lambda(\partial Y/\partial\lambda)$, with $\mu$ held fixed. By the stationary property (52) we can ignore the dependence of the $G_r(\zeta_l)$ on $\lambda$. Therefore we only obtain something from the $\lambda$'s in the explicit interactions in $Y'$. By (50) this is

$$\lambda\frac{\partial Y}{\partial\lambda}=\frac{1}{2\beta}\sum_\nu\sum_r\sum_l S_r'(\zeta_l)G_{r\nu}''(\zeta_l)=(1/2\beta)\sum_r\sum_l S_r'(\zeta_l)G_r(\zeta_l)=\lambda\partial\Omega/\partial\lambda \tag{53}$$

from (46). On the other hand,

$$Y=(\lambda=0)=-(1/\beta)\sum_r\sum_l \exp(\zeta_l 0^+)\ln(\epsilon_r-\zeta_l)$$

$$=-\frac{1}{2\pi i}\sum_r\int_{\Gamma 0}d\zeta\,[\ln(\epsilon_r-\zeta)]\frac{\exp(30^+)}{e^{\beta(\zeta-\mu)}+1}$$

$$=-\frac{1}{\beta}\frac{1}{2\pi i}\sum_r\int_{\Gamma 0}d\zeta\frac{\exp(30^+)}{\zeta-\epsilon_r}\times\ln(1+e^{\beta(\zeta-\mu)}) \tag{54}$$

$$=-(1/\beta)\sum_r\ln(1+e^{-\beta(\epsilon_r-\mu)})$$

$$=\Omega_0.$$

(See Appendix A for the discussion of such contour integral techniques.) From (53) and (54) we see that

$$\Omega=Y, \tag{55}$$

which is the desired expression. It is just the translation into propagator language of a theorem of Lee and Yang[4] which expresses $\Omega$ in terms of mean occupation numbers.

## IV. PROOF OF THE BRUECKNER-GOLDSTONE SERIES FOR THE ISOTROPIC CASE

It will be recalled that in I the BG expression was established for the isotropic case to the second order in $\lambda$ by showing that the modifications of the ground state energy which arise from changes in the chemical potential are compensated for by the contributions of certain "anomalous" diagrams. In order to extend this proof to arbitrary order it is necessary to have an expression for the shift in chemical potential to arbitrary order, and also to have a general classification of anomalous contributions.

To obtain an expression for the chemical potential shift, we make use of (6) for determining $\mu$. If we use (55) as our expression for $\Omega$, then we must be careful to distinguish several sources of $\mu$ dependence. There is, first of all, the dependence which arises from the various $\zeta_l$ which occur explicitly in the $G_r$ and in the propagators of $Y'$, and which must be summed over. Differentiation with respect to $\mu$ for such terms is the same as differentiation with respect to the corresponding $\zeta_l$. Secondly, the various $G_r(\zeta_l)$ really have an additional dependence on $\mu$ since they in turn were obtained from diagrams containing propagators depending on $\mu$. Now in differentiating $\Omega$ with respect to $\mu$ we may neglect this second dependence, since by the stationary property *any* first order changes in $G_r$ do not affect $\Omega$. Therefore we may write

$$\frac{\partial\Omega}{\partial\mu}=-\frac{1}{\beta}\sum_l\sum_r\left\{\frac{\partial}{\partial\zeta_l}\ln[\epsilon_r+G_r(\zeta_l)-\zeta_l]+\frac{\partial}{\partial\zeta_l}[G_r(\zeta_l)S_r'(\zeta_l)]\right\}\exp(\zeta_l 0^+)+\frac{1}{\beta}\sum_l\sum_r G_r(\zeta_l)\frac{\partial S_r'(\zeta_l)}{\partial\zeta_l}, \tag{56}$$

the last term arising from the usual argument of opening $2\nu$ lines, similar to that leading to (51). Combining, we get

$$\bar{N}=-\frac{\partial\Omega}{\partial\mu}=\frac{1}{\beta}\sum_l\sum_r\left\{\frac{\partial}{\partial\zeta_l}\ln[\epsilon_r+G_r(\zeta_l)-\zeta_l]+S_r'(\zeta_l)\frac{\partial G_r(\zeta_l)}{\partial\zeta_l}\right\}\exp(\zeta_l 0^+). \tag{57}$$

To analyze (57) further, we need to know just a little bit about the analytic properties of $G_r(\zeta_l)$ viewed as a function of $\zeta_l$, in the limit of zero temperature.[8]

From the definition of $G_r(\zeta_l)$ it follows at once that it is regular everywhere except on the real axis. Immediately above the real axis let us write

$$G_r(x+i0^+)=K_r(x)-iJ_r(x);\quad K_r,\ J_r \text{ real}. \tag{58}$$

Then from the definitions it also follows that

$$G_r(x-i0^+)=K_r(x)+iJ_r(x), \tag{59}$$

so that in general $G_r$ has a jump when we cross the real axis. Further it is not difficult to see that

$$J_r(x)\gtrless 0 \tag{60}$$

Lastly, inspection of the perturbation series for indicates that, as $x$ approaches $\mu$,

$$J_r(x)=C_r(x-\mu)^2,\quad C_r>0, \tag{61}$$

which arises from the fact that the exclusion principle severely limits the possible momentum space regions that can give rise to singularities, when $x$ is near $\mu$.

Now let us consider the second term of (57). In passing to the limit $T=0$ here, we can simply make the replacement

$$\frac{1}{\beta}\sum_l=\frac{1}{2\pi i}\int d\zeta, \tag{62}$$

where $C$ is the vertical line from $\mu-i\infty$ to $\mu+i\infty$, since the separation between different $\zeta_l$ values is

[8] Some of the properties which follow are particularly easy to obtain from the Green's function point of view. See for example, Fradkin, reference 3. The analytic property expressed in (61) has been noticed by several authors. N. M. Hugenholtz, Physica **23**, 533 (1957) (for somewhat different propagators). D. F. DuBois, Ann. Phys. **8**, 24 (1959). We hope to return to a complete demonstration (in the sense of perturbation theory) at a later time.

$2\pi i/\beta$. In general a great deal of care must be exercised in using (62) [we shall see below that the existence of the anomalous terms is just connected with corrections to (62)] when the integrand can have singularities on the path of integration. However, in the second term of (57) there is at most a simple pole [from $S_r'(\zeta)$; $\partial G_r(\zeta)/\partial\zeta$ is regular on $C$] and it is easy to see that this leads to no difficulty.[9] Therefore for this term we may write

$$-\frac{1}{2\pi i}\sum_r\int_C d\zeta\, S_r'(\zeta)\frac{\partial G_r(\zeta)}{\partial\zeta}$$
$$=\frac{1}{2\pi i}\sum_r\int_C d\zeta\, G_r(\zeta)\frac{\partial S_r'(\zeta)}{\partial\zeta}, \quad (63)$$

if we integrate by parts. This is exactly the same as the last term of (56), if we had used (62). Therefore (63) represents the result of differentiating $Y'$ with respect to every explicit $\zeta_l$ and then replacing the sums by integrals. In other words, we need to differentiate every skeleton diagram with respect to all explicit $\zeta_l$'s, and replace sums by integrals. From this point of view we see very easily that (63) is in fact zero. The result would be trivial if all the $\zeta_l$'s were independent; then integration by parts on any of them would give the desired result. Therefore we need only consider those $\zeta_l$'s which are connected by the "energy conservation" condition (30). In the integrals this is equivalent to factors of $\delta(\zeta_1+\zeta_2-\zeta_3-\zeta_4)$ for each explicit interaction. Now if we integrate the corresponding term of (63) by parts with respect to $\zeta_1$, $\zeta_2$, $\zeta_3$, and $\zeta_4$ we will get a factor in the integrand of

$$\left(\frac{\partial}{\partial\zeta_1}+\frac{\partial}{\partial\zeta_2}+\frac{\partial}{\partial\zeta_3}+\frac{\partial}{\partial\zeta_4}\right)\delta(\zeta_1+\zeta_2-\zeta_3-\zeta_4)=0. \quad (64)$$

Therefore (63) yields nothing.

We are finally left with

$$\bar{N}=\frac{1}{\beta}\sum_r\sum_l\left\{\frac{\partial}{\partial\zeta_l}\ln(\epsilon_r+G_r(\zeta_l)-\zeta_l)\right\}\exp(\zeta_l 0^+)$$
$$=\frac{1}{2\pi i}\sum_r\int_{\Gamma_0}d\zeta\exp(\zeta 0^+)$$
$$\times\left\{\frac{\partial}{\partial\zeta}\ln[\epsilon_r+G_r(\zeta)-\zeta]\right\}\frac{1}{e^{\beta(\zeta-\mu)}+1} \quad (65)$$

[9] Consider for example $(1/\beta)\sum_l\exp(\zeta_l 0^+)[1/(\zeta_l-\epsilon_r)]=f_r^-$. If we evaluate this for $T=0$ by using (62) we obtain

$$\frac{1}{2\pi i}\int_C d\zeta\exp(\zeta 0^+)\frac{1}{\zeta-\epsilon_r}=\begin{cases}1, & \epsilon_r<\mu\\ 0, & \epsilon_r>\mu\end{cases}$$

since we can always close to the left. However this is just the limit of $f_r^-$ as $T$ approaches zero.

from Appendix A. Integrating by parts, we obtain

$$\bar{N}=\frac{1}{2\pi i}\sum_r\int_{\Gamma_0}d\zeta\exp(\zeta 0^+)$$
$$\times\ln[\epsilon_r+G_r(\zeta)-\zeta]\frac{e^{\beta(\zeta-\mu)}}{e^{\beta(\zeta-\mu)}+1}. \quad (66)$$

For $\zeta$ on the real axis the last factor becomes, in the limit $T=0$, simply $\delta(\zeta-\mu)$ and therefore we have

$$\bar{N}=\sum_r\frac{1}{2\pi i}\{\ln[\epsilon_r+K_r(\mu)-\mu+i\eta]-\text{c.c.}\}, \quad (67)$$

where $\eta$ is an infinitesimal positive quantity, and we have made use of the properties (58)–(61). Since we are on the principal branch of the ln, we may write

$$\lim_{\eta\to 0}\ln(a+i\eta)=\ln a, \quad a>0 \quad (68)$$
$$=\ln|a|+i\pi, \quad a<0.$$

Using (68), (67) becomes

$$\bar{N}=\sum_{\substack{r\\ \mu-\epsilon_r-K_r(\mu)>0}}1, \quad (69)$$

as the equation for determining $\mu$.

Now the unperturbed chemical potential ($\mu_0$), is determined by

$$\bar{N}=\sum_{\substack{r\\ \mu_0-\epsilon_r>0}}1. \quad (70)$$

Since by isotropy $\epsilon_r$ depends only on the magnitude of $\mathbf{p}$, this determines a certain momentum $p_F$ (the Fermi momentum) up to which we sum, given by

$$\epsilon_{pF}=\mu_0. \quad (71)$$

Comparison of (69) with (70) shows that to get the same number of terms we must sum (69) to the same $p_F$. Therefore we obtain (again by isotropy $K_r(\mu)$ doesn't depend on spin direction[10])

$$\epsilon_{pF}+K_{pF}(\mu)=\mu,^{11}$$

or

$$\delta\mu\equiv\mu-\mu_0=K_{pF}(\mu)=G_{pF}(\mu). \quad (72)$$

We now return to the consideration of the ground-

[10] This is only true for spin less than or equal to $\frac{1}{2}$. There is no symmetry reason why $K_r(x)$ cannot depend on an even power of $m$, and in general it will, as direct calculation shows. For spin $\frac{1}{2}$, however, $m=\pm\frac{1}{2}$ so that any dependence on an even power of $m$ means no dependence on $m$ at all. The proof of BG we are giving therefore is only valid for fermions with spin $\leq\frac{1}{2}$. One sees easily by a second order calculation (as in I) that in fact for spin greater than $\frac{1}{2}$ BG is not even valid in the general isotropic case. It is valid, however, if the forces are spin-independent.

[11] Formulas of this type have been given previously by L. Van Hove and N. M. Hugenholtz, Physica 24, 363 (1958). Their propagator, however, has a different meaning than ours.

state energy. By (9) this is

$$E_0=\lim_{T\to 0}\left[-\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\{\ln[\epsilon_r+G_r(\zeta_l)-\zeta_l] +G_r(\zeta_l)S_r'(\zeta_l)\}+Y'+\bar{N}\mu\right]. \quad (73)$$

It is easy to see (Appendix B) that an alternative expression for $\bar{N}$ is

$$\bar{N}=-\frac{1}{\beta}\sum_r\sum_l \frac{\exp(\zeta_l 0^+)}{\zeta_l-\epsilon_r-G_r(\zeta_l)}. \quad (74)$$

Therefore we may write

$$E_0=\lim_{T\to 0}\left[-\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\{\ln[\epsilon_r+G_r(\zeta_l)-\zeta_l] +S_r'(\zeta_l)(G_r(\zeta_l)-\delta\mu)\}+Y'\right]+\bar{N}\mu_0. \quad (75)$$

Let us put

$$\zeta_l=Z_l+\mu, \quad Z_l=(2l+1)\pi i/\beta. \quad (76)$$

In every propagator in (75), only the function $S'$ occurs, and therefore only combinations like

$$\begin{aligned}\epsilon_r+G_r(\zeta_l)-\zeta_l&=\epsilon_r+G_r(Z_l+\mu)-Z_l-\mu\\&=\epsilon_r+G_r(Z_l+\mu)-\delta\mu-Z_l-\mu_0\\&=\epsilon_r+\bar{G}_r(Z_l)-Z_l-\mu_0, \quad (77)\end{aligned}$$

$$\begin{aligned}\bar{G}_r(Z_l)&\equiv G_r(Z_l+\mu)-\delta\mu\\&=G_r(Z_l+\mu)-G_{pF}(\mu). \quad (78)\end{aligned}$$

Therefore we may write for (75)

$$E_0=\lim_{T\to 0}\Omega(\mu_0,\bar{G})+\bar{N}\mu_0, \quad (79)$$

where $\Omega(\mu_0,\bar{G})$ means we are to calculate $\Omega$ using $\mu_0$ as the chemical potential, and $\bar{G}$ as the proper self-energy part.

From (78) and (79) we obtain the following alternative way of computing $\Omega(\mu_0,\bar{G})$: write down all closed linked graphs for $\Omega$ as a power series in $\lambda$, using as free propagators $1/(Z_l+\mu_0-\epsilon_r)$. Whenever a diagram has a self-energy part, however, simply subtract from that self-energy part its value at $p=p_F$, $Z_l=0$.

The importance of this formulation is that by means of it we see at once that (79) can no longer contain any "anomalous" terms. The anomalous diagrams (in the graphical representation of Bloch and De Dominicis) were those which gave rise to contributions of the form $\delta(\mu-\epsilon_r)$, $\delta'(\mu-\epsilon_r)$, etc., in the zero temperature limit. In our propagator formalism, where all possible $\mu$ orderings of the interactions have been summed over, there is no longer in general a distinction between anomalous and regular diagrams. Rather a general diagram when evaluated will give rise to both regular and anomalous contributions. It is easy to see where the anomalous contributions come from. Consider, for example, the third diagram of Fig. 1(c). From (33), with isotropy, the $l$-sums to be performed are

$$\frac{1}{\beta}\sum_{l_1}\frac{1}{(\zeta_{l_1}{}^0-\epsilon_{r1})^2}\frac{1}{\beta}\sum_{l_2}\frac{\exp(\zeta_{l_2}{}^0 0^+)}{(\zeta_{l_2}{}^0-\epsilon_{r2})}\frac{1}{\beta}\sum_{l_3}\frac{\exp(\zeta_{l_3}{}^0 0^+)}{(\zeta_{l_3}{}^0-\epsilon_{r3})} \quad (80)$$

$$\zeta_l{}^0\equiv Z_l+\mu_0.$$

The second and third factors of (80) just give $f_{r2}^-$, $f_{r3}^+$, respectively. These factors are discontinuous in the zero temperature limit, but they contain no $\delta$-functions. They may be obtained by using (62), i.e., there is no trouble in going to the zero temperature limit. On the other hand, the first factor gives (using Appendix A)

$$\begin{aligned}\frac{1}{\beta}\sum_l\frac{1}{(\zeta_l{}^0-\epsilon_r)^2}&=\frac{1}{2\pi i}\int_{\Gamma 0}\frac{1}{(\zeta-\epsilon_r)^2}\frac{1}{e^{\beta(\zeta-\mu_0)}+1}d\zeta\\&=\frac{\partial}{\partial\zeta}\left(\frac{1}{e^{\beta(\zeta-\mu_0)}+1}\right)\Bigg|_{\zeta=\epsilon_r}\\&=\frac{-\beta e^{\beta(\epsilon_r-\mu_0)}}{(e^{\beta(\epsilon_r-\mu_0)}+1)^2}\\&=-\delta(\epsilon_r-\mu_0) \quad (81)\end{aligned}$$

in the zero temperature limit. If we had used (62) immediately, we would have gotten

$$\frac{1}{\beta}\sum_l\frac{1}{(\zeta_l{}^0-\epsilon_r)^2}=\frac{1}{2\pi i}\int_C\frac{d\zeta}{(\zeta-\epsilon_r)^2}=0 \quad (82)$$

by closing either to the left or the right, there being no simple pole within the integration contour. The result (82) is incorrect, as is seen from (81). It is correct, however, if $\epsilon_r$ is not right in the neighborhood of $\mu_0$. The mistake in using (62) comes because when $\epsilon_r$ is near $\mu_0$ the integrand becomes too singular for $\zeta^0$ near $\mu_0$. The anomalous contribution (81) therefore comes from the region where $\epsilon_r=\mu_0$ (or $p=p_F$) and $Z=0$. This is clearly general: when $n$ self-energy parts are present in a line of a diagram there will be a factor in the denominator of $(\zeta_l{}^0-\epsilon_r)^{n+1}$, which will give anomalous contributions up to the $(n-1)$st derivative of the function $\delta(\epsilon_r-\mu_0)$, these contributions arising from the neighborhood of $p=p_F$, $Z=0$. Therefore we see that with the prescription given immediately below (79) there are no anomalous contributions, since the self-energy parts always have their values at $p=p_F$, $Z=0$ subtracted from them. In other words, a factor of $(\zeta_l{}^0-\epsilon_r)^{n+1}$ in the denominator always comes with a factor $[\bar{G}_r(Z_l)]^n$ in the numerator, so that we have from (78) at worst a simple pole in the summand. However, for a simple pole we may use (62) and never get contributions of the form (say) of (81).

We have shown that the thermodynamic potential when expressed in terms of $\mu_0$, has no anomalous contributions. In order to establish the BG series only one point remains: we must show that the *only* effect of replacing $G$ by $\bar{G}$ is to remove the anomalous contributions. In other words, we have shown that we may use (62) in the evaluation of the thermodynamic potential; we now must show that *after* this is done we get in fact no contributions from the $\delta\mu$ corrections to the self-energy parts. This is *not* true if we consider the effects of $\delta\mu$ in a single diagram; if we expand out $[\bar{G}_r]^n=[G_r-\delta\mu]^n$ we get in general nonvanishing contributions from all the terms, whereas what we want is the contribution from the first term only. On the other hand, the total effect of the $\delta\mu$ shift *must* be zero after we have used (62). This comes about as follows. Clearly, the effect of the $\delta\mu$ correction to $G_r$ may be described by saying that we are modifying the chemical potential from $\mu$ to $\mu_0$, but not taking any anomalous contributions which arise due to this modification. On the other hand the *only* effect of modifying the chemical potential is just to produce anomalous contributions. The chemical potential only occurs in expressions like $f_r^{\pm}(\mu)$. Consider say

$$f_r^-(\mu)=f_r^-(\mu_0+\delta\mu)$$
$$=f_r^-(\mu_0)+\delta\mu\frac{\partial f_r^-(\mu_0)}{\partial\mu_0}+\frac{(\delta\mu)^2}{2!}\frac{\partial^2 f_r^-(\mu_0)}{\partial\mu_0^2}+\cdots. \quad (83)$$

This becomes, in the zero temperature limit,

$$f_r^-(\mu)=f_r^-(\mu_0)+(\delta\mu)\delta(\mu_0-\epsilon_r)$$
$$+[(\delta\mu)^2/2!]\delta'(\mu_0-\epsilon_r)+\cdots. \quad (84)$$

Therefore the effect of a shift in chemical potential is to produce anomalous contributions. We saw, however, that there are in fact no anomalous contributions to the thermodynamic potential, and so all these corrections from $\delta\mu$ must cancel out against each other. (One can actually see how this happens in detail by studying some low order diagrams. To get the cancellation one must add together diagrams which have the same self-energy insertions put into each line in all possible ways.)

Thus we may write

$$E_0=\Omega_0(\mu_0)+\bar{N}\mu_0+\Omega_{\mathrm{BG}}(\mu_0), \quad (85)$$

where $\Omega_{\mathrm{BG}}(\mu_0)$ is just the contribution of all diagrams leaving out anomalous contributions. As already discussed in I, and as may very easily be seen again in our propagator formalism, this gives rise exactly to the BG series.

In summarizing, we can say that we have shown that for the totally isotropic case and for spin $\frac{1}{2}$ fermions, we have established the Brueckner-Goldstone form of the perturbation series.[12] For spin not equal to $\frac{1}{2}$, or for a non-isotropic situation the series is not valid in general.

[12] The theorem is also true formally for spin zero fermions, but they do not exist in nature.


## ACKNOWLEDGMENTS

One of us (JML) would like to thank Professor C. N. Yang for a very helpful discussion of the material in reference 4. We would also like to thank Professor W. Kohn for innumerable valuable discussions of the entire contents of this paper.


## APPENDIX A. THE PROPAGATOR REPRESENTATION

In this appendix we prove Eq. (26). Let us first consider the case $u-u'>0$. Then we may write

$$\frac{1}{\beta}\sum_l e^{\zeta_l(u-u')}\frac{1}{\zeta_l-\epsilon_r}=\frac{1}{2\pi i}\int_\Gamma\frac{e^{\zeta(u-u')}}{(\zeta-\epsilon_r)(e^{\beta(\zeta-\mu)}+1)}d\zeta, \quad (A.1)$$

where $\Gamma$ is the contour indicated in Fig. 5, since the function $[e^{\beta(\zeta-\mu}+1]^{-1}$ has simple poles at the points $\zeta=\zeta_l$ with residues $-1/\beta$. Since $u-u'>0$, $\Gamma$ may be deformed into $\Gamma_0$ of Fig. 5. Using the Cauchy theorem and noticing that the only singularity inside $\Gamma_0$ is a simple pole at $\zeta=\epsilon_r$ we obtain

$$\frac{1}{\beta}\sum_l e^{\zeta_l(u-u')}\frac{1}{\zeta_l-\epsilon_r}=f_r^-e^{\epsilon_r(u-u')}, \quad u>u'. \quad (A.2)$$

On the other hand, if $u-u'<0$, it is convenient to write

$$\frac{1}{\beta}\sum_l e^{\zeta_l(u-u')}\frac{1}{\zeta_l-\epsilon_r}$$
$$=-\frac{1}{2\pi i}\int_\Gamma\frac{e^{\zeta(u-u')}e^{\beta(\zeta-\mu)}}{(\zeta-\epsilon_r)(e^{\beta(\zeta-\mu)}+1)}d\zeta, \quad (A.3)$$

since this again allows us to deform the contour $\Gamma$ into $\Gamma_0$. Cauchy's theorem then gives

$$\frac{1}{\beta}\sum_l e^{\zeta_l(u-u')}\frac{1}{\zeta_l-\epsilon_r}=-\frac{e^{\beta(\epsilon_r-\mu)}}{e^{\beta(\epsilon_r-\mu)}+1}e^{\epsilon_r(u-u')}$$
$$=-f_r^+e^{\epsilon_r(u-u')}, \quad u<u'. \quad (A.4)$$

Comparison of (A.2) and (A.4) with (24), establishes (26).

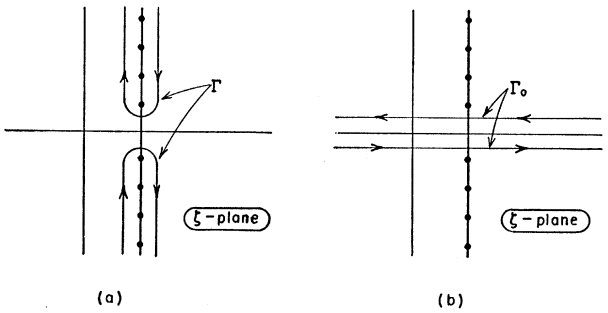


FIG. 5. Contours for propagators. The indicated points are the points $\zeta_l=[(2l+1)\pi i/\beta]+\mu$.

## APPENDIX B. ALTERNATIVE EXPRESSION FOR $\bar{N}$

Consider the expression for $\Omega$,

$$\Omega=\Omega_0+\sum_{n=1}^{\infty}\Omega_n, \tag{B.1}$$

where the $\Omega_n$ are given by (41). If we differentiate this with respect to $\mu$ to obtain $\bar{N}$, we see at once

$$\bar{N}=-\frac{\partial\Omega_0}{\partial\mu}-\frac{1}{\beta}\sum_n\sum_r\sum_l G_{rn}'(\zeta_l)\frac{\partial}{\partial\mu}\frac{1}{\zeta_l-\epsilon_r}, \tag{B.2}$$

since in differentiating a closed linked $n$th order diagram is equivalent to differentiating any of its $2n$ lines. When we sum over $r$, all of these give the same contribution. Therefore

$$\bar{N}=-\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\frac{1}{\zeta_l-\epsilon_r}$$
$$+\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\frac{G_r(\zeta_l)}{(\zeta_l-\epsilon_r)(\zeta_l-\epsilon_r-G_r(\zeta_l))}, \tag{B.3}$$

using (54) and (44). Combining

$$\bar{N}=-\frac{1}{\beta}\sum_r\sum_l \exp(\zeta_l 0^+)\frac{1}{\zeta_l-\epsilon_r-G_r(\zeta_l)}. \tag{B.4}$$

This actually corresponds to the result that the mean number of particles is just the sum of the mean occupation number of each state.

PHYSICAL REVIEW VOLUME 118, NUMBER 5 JUNE 1, 1960

# Example of a Soluble Field Theory with Finite Charge Renormalization

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A soluble field theory suggested by the Lee and Machida models is described in which coupling constant renormalization arises from a dressed boson and is finite if the contributing fermions are assumed nonrelativistic. For the unrenormalized charge to be real, the renormalized charge must satisfy a certain inequality depending on the boson and fermion mass ratios; if this inequality is violated a single boson ghost state occurs, as expected.

## 1. INTRODUCTION

EVER since the appearance of the Lee model[1] there has been much interest in obtaining examples of field theories wherein quantities of interest may be derived in closed form; and of those theories which have been found, several[2,3] are essentially extensions of the Lee model. A variation of Lee's procedure was discussed by Machida[4] who considered the soluble problem of a dressed boson, rather than a dressed fermion; and more recently Goldstein[5] has presented a sort of combination of the two models. In each of these theories the renormalization constants are infinite, i.e., cutoff dependent, implying an imaginary value for the unrenormalized coupling constant (charge) as the cutoff exceeds a certain critical value. For this latter situation, the analysis of Källén and Pauli[6] indicates that a ghost state is to be expected.

Although nothing basically new is to be learned from the following discussion, it may nevertheless be of some interest to examine an exactly soluble theory with finite charge renormalization. The rather trivial remark to be made in this connection is the observation that in all such previous models infinite renormalization constants are obtained as a result of adhering to the relativistic energy-momentum relation for that particle whose momentum appears as the variable of integration in the definition of the renormalization constants. In the original Lee model, for example, replacing the boson energy $\omega(\mathbf{k})=(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}$ by $\mu+\mathbf{k}^2(2\mu)^{-1}$ when integrating over the boson momentum $\mathbf{k}$, yields a finite value for $Z_2^{-1}$; and similarly the replacement of $E(\mathbf{p})=(\mathbf{p}^2+m^2)^{\frac{1}{2}}$ by $m+\mathbf{p}^2(2m)^{-1}$ provides a finite value for the renormalization constant $Z_3^{-1}$ of Machida. Since one has already mutilated the physically correct interaction Hamiltonian in order to obtain a set of exactly soluble equations, little further rigor is lost in assuming nonrelativistic particles in intermediate states; and the advantage of doing this is that one may then obtain, within certain well-defined limits, a quite respectable field theory (at the possible risk, of course, of being even further removed from physical reality). These statements will be illustrated briefly by consider-

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