Puppi-Stanghellini Discrepancy*

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In order to establish the extent of the disagreement between the pion-nucleon forward-scattering-amplitude dispersion relations and experiment in a statistical sense, the uncertainty in the dispersion integrals and S-wave scattering lengths is systematically included in the analysis. To accomplish this, we fit the total cross sections below 335 Mev by a Chew-Low P-wave resonance, phenomenologically modified, and calculate the error matrix for the parameters. Our fit to the total cross sections is statistically at least as good as the Anderson parameterization used in previous work. Ignoring forward scattering amplitudes above 220 Mev because of D-wave uncertainties, we still find that there is less than a $4\frac{1}{2}\%$ probability that the published data are compatible with a unique value for the pion-nucleon coupling constant f^2, and that no adjustment of the S-wave scattering lengths can remove the discrepancy. However, if the π^- forward-scattering amplitudes measured by Ashkin *et al.* at 150 and 170 Mev are abandoned in favor of the values recently obtained by Kruse and Arnold at 130 and 152 Mev, the probability rises to 47.2% for our parameterization, or 8.6% for the Anderson parameterization.

Cini et al. have pointed out that the conventional analysis of the low-energy data to obtain the S-wave scattering lengths does not satisfy crossing symmetry, and a re-analysis by Hamilton and Woolcock gives $a_+=-0.083$, $a_-=0.088$, rather than the conventional values of -0.110 and 0.077. We obtain some independent evidence in support of this conclusion by using the dispersion relations to determine a_+, a_-, and f^2 simultaneously. We find:

Energy dependence	f^2	a_+	a_-
Anderson	0.075 ± 0.018	-0.086 ± 0.025	0.071 ± 0.020
Modified Chew-Low	0.086 ± 0.019	-0.101 ± 0.026	0.085 ± 0.020

It is clear that a better theoretical description of the energy dependence of the total cross sections will be required before further progress can be made on this problem.

I. INTRODUCTION

As was first noted by Puppi and Stanghellini,[1] the pion-nucleon forward-scattering-amplitude dispersion relations[2] are in apparent disagreement with experiment. Since a rigorous derivation of these relations has been given,[3] and electromagnetic corrections have been shown to be probably less than 5%,[4] confirmation of this result would show that at least one of the basic postulates of local field theory is incorrect. Re-evaluation of the integrals occurring in the dispersion relations using new low-energy total cross sections gave corrections which reduced the original discrepancy, but a residual discrepancy remains.[5,6] We feel that this discrepancy is best exhibited by asking the question whether the pion-nucleon coupling constant f^2, as defined by the dispersion relations, is consistent with a unique value. For this purpose, therefore, we systematically include in the analysis the statistical uncertainties due to the S-wave scattering lengths and the integrals over total cross sections, and explicitly exhibit the dependence of f^2 on these quantities. We also discuss the use of the dispersion relations to determine the S-wave scattering lengths simultaneously with f^2.

II. EXPERIMENTAL QUANTITIES

In order to calculate f^2 at a given energy, we must know (a) the real part of the forward-scattering amplitude at that energy, (b) two integrals over the total cross sections, and (c) two subtraction constants which can either be taken to be two values of the scattering amplitudes at some energy (usually the S-wave scattering lengths) or treated as two additional constants to be fitted simultaneously with f^2. Our selection of the data is given below.

(a) Forward-Scattering Amplitudes

As noted by Puppi and Stanghellini,[1] it is possible to use the observed angular distribution for elastic scattering toegther with the total cross section to calculate the forward-scattering amplitude *without* assuming charge independence, by using the relation

$$\mathrm{Re}f(0)=\pm[A+B+C+\cdots-(q\sigma_{\mathrm{tot}}/4\pi)^2]^{\frac{1}{2}}, \quad (1)$$

where

$$\sigma(\theta)=[\mathrm{Re}f(\theta)]^2+[\mathrm{Im}f(\theta)]^2 =A+B\cos\theta+C\cos^2\theta+\cdots. \quad (2)$$

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[1] G. Puppi and A. Stanghellini, Nuovo cimento **5**, 1305 (1957).
[2] M. L. Goldberger, H. Miyazawa, and R. Oehme, Phys. Rev. **99**, 986 (1955).
[3] N. Bogoliubov, B. Medvedev, and M. Polivanov, Institute for Advanced Study Notes, Princeton, 1956 (unpublished); K. Symanzik, report at the Seattle Conference, 1956 (unpublished); H. J. Bremmermann, R. Oehme, and J. G. Taylor, Phys. Rev. **109**, 2178 (1958).
[4] A. Agodi and M. Cini, Nuovo cimento **6**, 686 (1957); A. Agodi, M. Cini, and B. Vitale, Phys. Rev. **107**, 630 (1957); G. F. Chew and H. P. Noyes, Phys. Rev. **109**, 566 (1958).
[5] J. Hamilton, Phys. Rev. **110**, 1134 (1958); H-Y Chiu and J. Hamilton, Phys. Rev. Letters **1**, 146 (1959).
[6] H. J. Schnitzer and G. Salzman, Phys. Rev. **112**, 1802 (1958).

TABLE I. Forward-scattering amplitudes, in units of $(\hbar/mc)=1.413\times10^{-13}$ cm.

Energy (Mev)	Reference	$\mathrm{Re}f^+(0)$	$\mathrm{Re}f^-(0)$
41.5	a	0.102±0.034	0.104±0.014
80	b	0.261±0.065	
98	c	0.434±0.011	0.195±0.006
100	d	0.413±0.031	
113	e	0.476±0.054	
114	b	0.528±0.070	
120	b	0.584±0.048	
124	b	0.507±0.075	
150	f	0.466±0.019	0.258±0.017
170	f		0.266±0.027
217	g		−0.221±0.077
220	f	−0.536±0.061	−0.134±0.051

[a] S. Barnes, B. Rose, G. Giacomelli, J. Ring, and K. Miyake, University of Rochester Report NYO-2170 (unpublished).
[b] H. Anderson and W. Davidon, Nuovo cimento **5**, 1238 (1957): analysis of work by Puppi *et al.*
[c] D. N. Edwards, S. G. F. Frank, and J. R. Holt, Proc. Phys. Soc. (London) **73** 856 (1959) (Π^-). D. N. Edwards, S. G. F. Frank, J. R. Holt, and Massam (to be published) (Π^+).
[d] H. L. Anderson, E. Fermi, R. Martin, and D. E. Nagle, Phys. Rev. **91**, 155 (1953).
[e] J. Orear, Phys. Rev. **96**, 1417 (1954).
[f] J. Ashkin, J. P. Blaser, F. Feiner, and M. O. Stern, Phys. Rev. **101**, 1149 (1956); **105**, 724 (1957).
[g] H. D. Taft, Phys. Rev. **101**, 1116 (1956).

Although Coulomb terms must be included in the angular distribution for accurate work, it has been shown[7] that this can also be done without assuming charge independence, except for small correction terms of order $(e^2/\hbar v)^2$; in fact, the Coulomb interference term can be used to give an independent measurement of $\mathrm{Re}f(0)$. As noted in previous analyses.[5,6] values of the forward scattering amplitudes calculated from published data above 220 Mev are unreliable because the angular distributions do not go to small or large enough angles to give accurate determination of the D waves. We therefore limit our considerations to data below 220 Mev. The experimental values we have selected are given in Table I.

(b) Total Cross Sections and Dispersion Integrals

The choice of total cross sections is mainly determined by the most accurate experiments in any particular energy region. At low energies we have chosen total cross sections derived by integration of angular distributions, because $\mu\rightarrow e+\nu+\bar{\nu}$ decays necessitate large corrections to transmission measurements below about 50 Mev. At energies greater than 220 Mev we have used only transmission measurements, because of doubts about D waves. In between we have tried to strike a balance between integration and transmission measurements. Where possible we have avoided total cross sections derived from a charge-independent analysis, as they may be in error due to: (1) slight energy differences between π^+ and π^- beams, (2) consequences of deviations from charge independence,[8] (3) neglect of higher partial waves, and (4) assuming that phase shifts are real above inelastic threshold. The mean energy spread of the beam is measured either by the range of the particles or by inference from the curvature of a current-carrying wire in the magnetic fields through which the beam passes. The two methods are known to agree to within 1%.[9] The energy spread can be corrected for explicitly when the energy dependence of the total cross section is known. In recent experiments, where the accuracy justified it, such corrections have been made. The values chosen are given in Table II.

Although we have settled on the experimental values, it is still not an easy matter to decide on how to evaluate the dispersion integrals themselves, and, in particular, to assign an experimental uncertainty to them. If we make our subtractions at zero energy, these integrals have the form

$$S^{\pm}(v_0)=\mathrm{P}\int_1^\infty \frac{\sigma^{\pm}(v)dv}{(v^2-1)^{\frac{1}{2}}(v-v_0)} + \int_1^\infty \frac{\sigma^{\pm}(v)dv}{(v^2-1)^{\frac{1}{2}}(v+v_0)}. \quad (3)$$

$$v=1+(T_{\mathrm{lab}}/mc^2);\quad mc^2=139.63\ \mathrm{Mev}.$$

The difficulty arises, of course, because of the principal-value integral. While one can subtract out the singularity and obtain a reasonably accurate value for the integral, as was done by Puppi and Stanghellini, this method does not allow one to calculate the error. In order to calculate the error it is necessary to fit some analytic form to the total cross section by means of adjustable parameters and evaluate the error matrix for these parameters. The error in S is then calculable if one evaluates the derivatives of S with respect to these parameters, as can now be done.

The most extensive available parameterization of the total cross sections is due to Anderson.[10] Although he quotes an error matrix, this has negative elements on the diagonal, showing a lack of sufficient precision in his matrix inversion routine, and hence is useless for our purposes. As a first attempt we have, instead, assumed that the total cross sections can be fitted by a constant S-wave cross section and P-wave resonance of the form given by Chew and Low,[11] i.e.,

$$\sigma^+(q^2)=A^++8\pi(\hbar/mc)^2F(q^2,I^+,R^+),$$
$$\sigma^-(q^2)=A^-+8\pi(\hbar/mc)^2F(q^2,I^-,R^-)/3, \quad (4)$$

[7] H. P. Noyes, Phys. Rev. **111**, 944 (1958).
[8] D. Greenberger, thesis 1958, Massachusetts Institute of Technolocy, Cambridge; reported in *1958 Annual Conference on High-Energy Nuclear Physics at CERN*, edited by B. Ferretti (CERN Scientific Information Service, Geneva, 1958), p. 100. Preprint, Division of Physical Sciences, Department of Defense, College Park, Maryland, 1959 (unpublished).
[9] S. Lindenbaum and L. Yuan, Phys. Rev. **100**, 306 (1955).
[10] H. L. Anderson, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers Inc., New York, 1956), p. 1–43.
[11] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

TABLE II. Total cross sections.

Energy (Mev)	Reference	σ^+ (mb)	σ^- (mb)	Energy (Mev)	Reference	σ^+ (mb)	σ^- (mb)
41.5	a	9.03±0.18	8.12±0.73	256	g		37.5± 1.9
58	b	15.8 ±1.5		270	l	85.2±3.0	
65	b		15.3 ±1.6	307	l	65.7±2.2	
98	c	55.6 ±0.5	21.9 ±0.7	335	n	53.0±5.0	
113	d	79.0 ±5.0		335	g		25.7± 1.0
120	e		33.0 ±3.1	363	g		26.5± 1.6
133	f		46.9 ±2.4	393	g		25.9± 1.6
135	f	126.0 ±4.0		450	o	24.8±25	28.8± 2.7
140	g		44.3 ±2.7	470	p		27.0± 5.0
143	h	140.5 ±5.0		500	o		31.5± 4.8
144	i	151.0 ±4.0		550	o	16.1±2.5	
150	j		55.3 ±1.6	550	o		37.4± 3.0
162	h	170.5 ±3.5		610	o		37.0± 2.1
164	i	169.0 ±5.0		670	o	14.5±2.0	
165	k		67.5 ±1.5	700	n		42.0±10.0
170	j	194.9 ±5.5	62.7 ±1.9	790	o	19.5±2.0	46.1± 3.4
170	h	198.0 ±3.5		840	p		47.0± 5.0
173.5	h	193.5 ±3.5		860	o		47.7± 2.7
176	l	199.4 ±4.9		900	o		44.4± 2.3
177	h	198.0 ±5.0		970	o		45.1± 2.7
183.5	h	192.0 ±3.5		1000	o	23.5±1.4	46.0± 3.0
184	g		65.7 ±2.4	1070	o	27.3±3.7	
189	k	194.1 ±5.2		1080	o		36.3± 2.6
195	h	174.0 ±4.0		1150	o	31.3±1.7	
195	f		63.1 ±2.5	1250	o	38.8±2.5	29.2± 3.7
200	l	177.9 ±3.7		1350	o		30.1± 2.8
205	h	178.0 ±4.5		1380	o	41.5±3.0	30.8± 2.8
209	m		57.2 ±2.9	1470	o		31.4± 1.8
216	g		57.3 ±2.5	1500	o	35.3±2.5	30.0± 2.0
220	j	140.9 ±4.2	52.1 ±2.3	1670	o	32.6±1.8	31.4± 3.9
240	l	125.6 ±2.5		1779	o	31.7±2.4	

[a] S. Barnes, B. Rose, G. Giacomelli, J. Ring, and K. Miyake, University of Rochester Report NYO-2170 (unpublished).
[b] D. Bodansky, A. M. Sachs, and J. Steinberger, Phys. Rev. **93**, 1367 (1954).
[c] D. N. Edwards, S. G. F. Frank, and J. R. Holt, Proc. Phys. Soc. (London) **73**, 856 (1959) (Π^-). D. N. Edwards, S. G. F. Frank, J. R. Holt, and T. Massam (to be published) (Π^+).
[d] J. Orear, Phys. Rev. **96**, 1417 (1954).
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[m] M. Glicksman, Phys. Rev. **94**, 1335 (1954).
[n] S. Lindenbaum and L. Yuan, Phys. Rev. **100**, 306 (1955).
[o] R. Cool, O. Piccioni, and D. Clark, Phys. Rev. **103**, 1082 (1956).
[p] A. M. Shapiro, C. P. Leavitt, and F. F. Chen, Phys. Rev. **92**, 1073 (1953).

where

$$F(q^2,I,R)=q^4/[q^6+(3w^*I(1-Rw^*)/4)^2];$$
$$q^2=(v^2-1)/(1+2vm/M+m^2/M^2);$$
$$w^2=1+q^2;\quad m/M=0.14882; \tag{5}$$
$$8\pi(\hbar/mc)^2=501.88\text{ mb};$$
$$w^*=w+[(M/m)^2+q^2]^{\frac{1}{2}}-(M/m).$$

The S-wave constant contribution to the cross section is assumed known from the scattering lengths (see Sec. IIc below). After this term has been subtracted, the data up to 335 Mev in Table II are then fitted by the method of least squares. The parameters and error matrices are given in Table III, and the χ^2 values in Table IV. To take partial account of possible departures

TABLE III. Parameterization of the total cross sections.

A. Chew-Low shape

	A (mb)	I	R	$\langle\delta I\delta R\rangle$
σ^+	3.04±0.23	12.28±0.14	0.4680±0.0010	0.000110
σ^-	5.88±1.48	12.60±0.33	0.4639±0.0023	0.000584

B. Chew-Low shape+Bq^4

	A (mb)	I	R	B (mb)	$\langle\delta I\delta R\rangle$	$\langle\delta I\delta B\rangle$	$\langle\delta R\delta B\rangle$
σ^+	3.04±0.23	11.39±0.16	0.4641±0.0012	−0.6344±0.0082	0.000144	−0.000711	−0.0000422
σ^-	5.88±1.48	11.47±0.47	0.4590±0.0034	−0.1264±0.0050	0.00142	0.001692	0.00001228

from charge independence,[8] we fit the positive and negative cross sections separately. We see from the goodness-of-fit parameter that the Chew-Low shape is not a good representation of the data. This is due to the well-known fact that the cross section given by this formula does not fall off rapidly enough beyond the first resonance. If the discrepancy were in the opposite direction, we could ascribe it to nonresonant P states and account for it by adding a term proportional to q^4. As it is, we are forced to subtract such a term and refit the parameters. Adding additional powers of q^2 does not improve the fit or change the sign of the q^4 term. The fit is still not good in a statistical sense, but this is due to inconsistencies among the data themselves, as can be seen from a glance at Fig. 1. For comparison with other analyses that make use of the Anderson curve, we also give the statistical comparison of his curve with the data. Although that formula has 15 parameters, our 6-parameter fit is better statistically. This is due to the fact that we have used more recent data than Anderson, and have not attempted to readjust his parameters to fit them. In particular, the poor fit of his formula to the π^- data is due almost entirely to the 335-Mev point. If this is dropped (figures in parentheses in Table IV), the fits are comparable.

We now calculate the contribution to $S^{\pm}$ below 335 Mev, and the error, by numerical integration. This is done with a 1-Mev grid after all singular terms have been subtracted out and evaluated analytically; this integration routine was shown to be accurate to at least 0.01%. Above 335 Mev we simply use trapezoidal integration on the experimental points, and assume the cross section constant above the highest value. Previous discussions[5,6] have shown that the discrepancy is insensitive to this region; data showing the higher $I=\frac{1}{2}$ resonances[12] also introduce negligible changes in our region of interest. The trapezoidal integration we have used is adequate for the purpose of this paper, but the reader should be warned that it rapidly becomes unreliable for evaluating the principal-value integral when the singularity lies much above 220 Mev. This is particularly true for S^+ because of the large experimental uncertainty in the total cross section for $\pi^+ + p$ scattering at 335 Mev and the absence of data between 335 and 450 Mev. For example, if we try to use values of S^+ and S^- computed by our prescription at 310 Mev, we obtain $f^2=0.04$, rather than the value of $f^2=0.08$ obtainable by integrating under a reasonable smooth curve drawn between 335 and 450 Mev. It is clear that the parameterization must be extended to give an accurate representation of the fall of the positive cross section and the beginning of the second resonance in the negative cross section before accurate values of f^2 can be computed from forward scattering amplitudes in the 300–400 Mev energy range. The values of the integrals for various shapes are compared in Table V. As will be seen in detail in the next section, an error of 2 to 3 millibarns in S would not prevent us from making a significant test of the dispersion relations, but the difference in the integrals of up to 17 millibarns for different forms for the resonance-energy dependence is serious. Although it would not prevent us from establishing the existence of a discrepancy, it makes the value of f^2 calculated from any individual experiment untrustworthy.

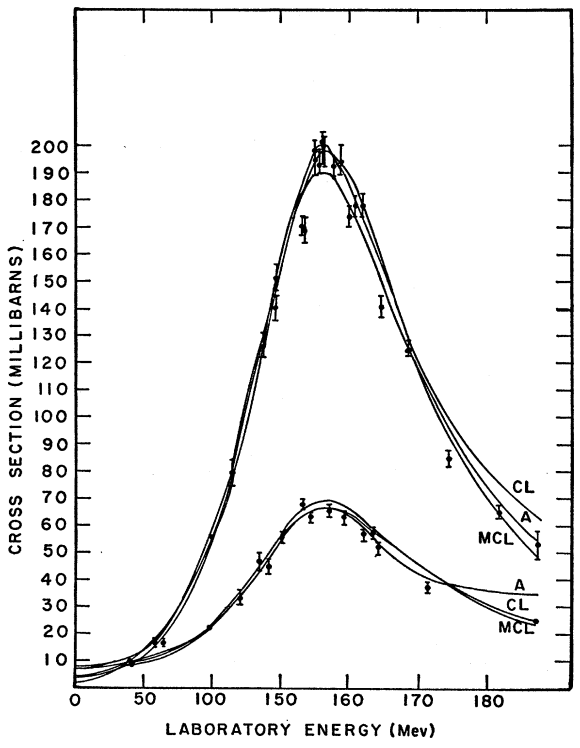

FIG. 1. Comparison of curves fitted to the $\pi^{\pm}+p$ total cross sections with the experimental points. A = Anderson parameterization; CL = Chew-Low + const; MCL = Chew-Low + const + Bq^4.

(c) The S-Wave Scattering Lengths

Since experiments at very low pion energy are difficult, it is customary to obtain the S-wave scattering lengths by assuming an energy dependence for the phase shifts, and charge independence.[13] The most recent analysis of this type gives $a_+ = -0.110 \pm 0.004$

TABLE IV. Comparison of different shapes with experiment.

Shape	$(\chi^2)_+$	Expected	$(\chi^2)_-$	Expected
Chew-Low	166.9	22	49.3	14
Chew-Low + Bq^4	69.5	21	36.8	13
Anderson	84.0	16	111.8	9
			(32.4)	(8)

[12] H. C. Burrowes, D. O. Caldwell, D. H. Frisch, D. A. Hill, D. M. Ritson, R. A. Schluter, and M. A. Wahlig, Phys. Rev. Letters 2, 119 (1959); R. R. Crittenden, J. H. Scandrett, W. D. Shephard, W. D. Walker, and J. Ballam, Phys. Rev. Letters 2, 121 (1959).

[13] J. Orear, Phys. Rev. 96, 176 (1954).

TABLE V. Values of the dispersion integrals for various parameterizations of the total cross sections.

Energy (Mev)	(3.04±0.23) +Chew-Low $+Bq^4$ to 335 Mev	(3.04±0.23) +Chew-Low to 335 Mev	Anderson to 335 Mev	Trapezoidal above 335 Mev
A. Parts of S^+ in millibarns				
41.5	143.97±2.42	143.50±1.74	144.93	22.28±0.52
80	181.94±2.42	183.12±1.83	174.46	22.89±0.56
98	193.40±2.05	197.11±1.55	183.92	23.24±0.59
100	193.98±1.99	198.07±1.48	184.54	23.28±0.59
113	192.21±1.41	198.62±1.08	184.96	23.59±0.60
114	191.59±1.39	198.72±1.03	184.67	23.61±0.60
120	186.05±1.13	194.60±0.91	181.67	23.77±0.60
124	175.23±1.11	189.94±0.85	178.32	23.88±0.61
150	103.39±2.01	114.92±1.99	120.90	24.70±0.66
220	−122.27±0.85	−121.28±0.79	−123.66	28.64±0.91
B. Parts of S^- in millibarns				
Energy (Mev)	(5.88±1.48) +Chew-Low $+Bq^4$ to 335 Mev	(5.88±1.48) +Chew-Low to 335 Mev	Anderson to 335 Mev	Trapezo d above 335 Mev
41.5	67.02±2.42	66.70±2.20	72.46	22.69±0.54
98	80.30±2.17	81.29±2.06	81.56	23.62±0.58
130	72.67±2.09	76.04±1.83	75.84	24.38±0.62
150	52.74±2.91	56.51±2.20	58.64	24.86±0.62
152	49.97±3.13	55.61±2.22	55.96	24.94±0.62
170	24.09±3.08	26.21±2.28	29.88	25.47±0.64
217	−22.90±2.02	−23.16±1.83	−21.71	27.41±0.71
220	−24.24±2.02	−24.45±1.83	−22.93	27.56±0.71

$(\hbar/mc)$.[14] Since we wish to avoid the assumption of charge independence, we take the value of $a_-=0.077\pm0.011$ $(\hbar/mc)$ from an analysis of 10–30 Mev *elastic* scattering data,[15] rather than using $\frac{1}{3}(a_3+2a_1)$. The constants in our fit to the total cross sections are

$$A^+=4\pi a_+^2=3.04\pm0.23 \text{ mb}$$

and

$$A^-=4\pi[\tfrac{1}{9}a_-^2+\tfrac{1}{2}(a_--a_+)^2]=5.88\pm1.48 \text{ mb}.$$

The constant A^- is poorly enough known and makes so small a contribution to the dispersion integrals that we need not fear the charge-independence assumption used in evaluating it. As was pointed out by Cini et al.,[16] the conventional analysis is incorrect in that the assumed S-wave energy dependence does not satisfy crossing symmetry. Hamilton and Woolcock[17] have recently carried out a reanalysis of the low-energy scattering data along these lines, and found $a_+=-0.083$, $a_-=0.088$. We will return to this point in the next section.

III. COMPARISON OF THE DISPERSION RELATIONS WITH EXPERIMENT

(a) Calculation of f^2

It has been customary in discussions of the Puppi-Stanghellini discrepancy to assume that the dispersion relations predict a value of $\mathrm{Re}f(0)$, and to compare

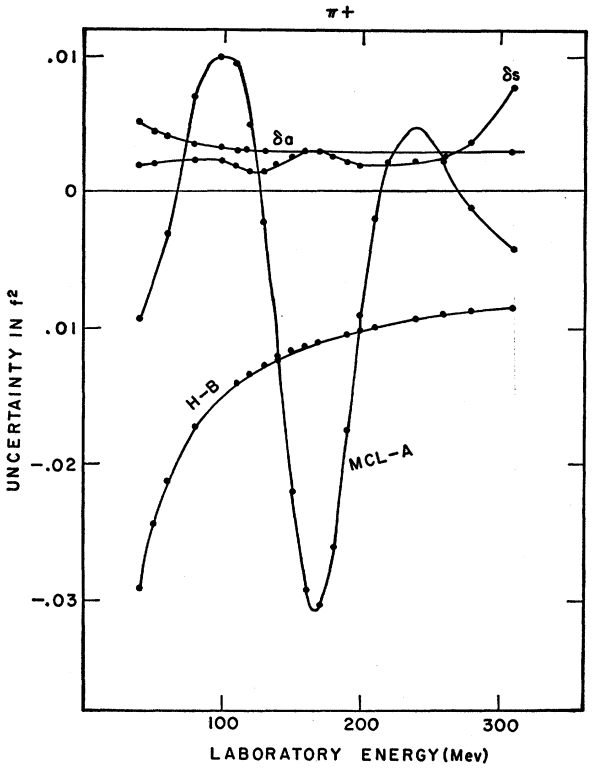

FIG. 2. Uncertainty in f^2 calculated from π^+ data due to uncertainties in the scattering lengths and dispersion integrals. $\delta a=\langle(\alpha_+\delta a_+)^2+(\alpha_-\delta a_-)^2\rangle^{\frac{1}{2}}$; $\delta S=\langle(\delta S)^2\rangle^{\frac{1}{2}}$; $H-B$ = difference between Hamilton and Barnes scattering lengths; MCL$-A$ = difference between S evaluated using the modified Chew-Low or the Anderson shape.

this with the experimental value. Although the influence of uncertainties in the scattering lengths and coupling constant have been discussed qualitatively, it is not easy, in such a comparison, to see their effect simultaneously at different energies. We therefore propose, to take the point of view that the dispersion relations define a value of f^2 at each energy in terms of the four experimental quantities given in the last section, and ask whether the values so determined are consistent with a unique f^2. That is, we write the dispersion relations in the form

$$f^2=\alpha_+a_++\alpha_-a_-+\beta_\pm S^\pm+\gamma_\pm \mathrm{Re}f^\pm(0), \tag{6}$$

where

$$\begin{aligned}\alpha_+&=(1+m/M)(\mp v-m/2M)(1\pm v)/4k^2,\\ \alpha_-&=(1+m/M)(\mp v-m/2M)(1\mp v)4k^2,\\ \beta_\pm&=(\mp v+m/2M)/8\pi^2(\hbar/mc)^2,\\ \gamma_\pm&=-(\mp v+m/2M)/2kq(\hbar/mc),\\ k^2&=v^2-1,\end{aligned} \tag{7}$$

and $\pm$ refers to positive and negative pions, respectively. We can now see clearly the effect of the experimental uncertainties in S and the scattering lengths by plotting, as a function of energy, the errors in these quantities given in Sec. II, as is done in Figs. 2 and 3. Taking, as

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[15] D. E. Nagle, R. H. Hildebrand, and R. J. Plano, Phys. Rev. **105**, 718 (1957).
[16] M. Cini, R. Gatto, E. L. Goldwasser, and M. Rudermann, Nuovo cimento **10**, 243 (1958).
[17] J. Hamilton and W. S. Woolcock, communication to the Kiev Conference on High-Energy Nuclear Physics, 1959 (unpublished); I am indebted to Dr. Hamilton for private discussions and permission to publish these results.

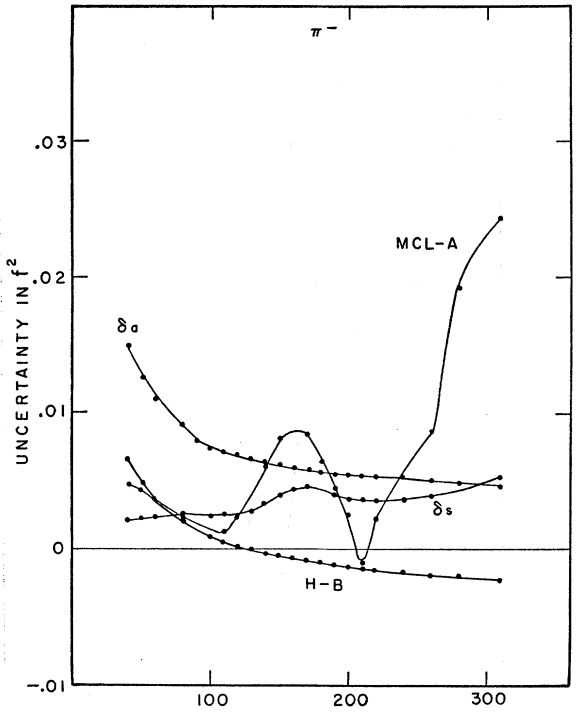

FIG. 3. Same as Fig. 2, except for π^- data.

an example, the scattering of negative pions at 150 Mev, we find that the combined uncertainty in S and a makes f^2 uncertain by ± 0.0065. By computing γ_- at this energy, we find that a measurement of $\mathrm{Re}f^-(0)$ to better than ± 0.015 cannot appreciably increase our knowledge of f^2 at this energy. We note that if this were the only source of uncertainty, a significant test of the dispersion relations to a few percent accuracy would be possible with the present data.

Unfortunately, there is another source of uncertainty which is not statistical. As already noted, the energy dependence assumed in evaluating the scattering lengths can change their values appreciably. In Figs. 2 and 3 we plot the difference between the contribution of f^2 computed from Hamilton and Woolcock's[17] values for the scattering lengths and the contribution computed from Barnes'[14] values, we see that, although insignificant for negative pions, this difference could significantly lower the value of f^2 computed from positive-pion data. We also plot the difference between the values of βS as computed from the Anderson curve and as computed from the modified Chew-Low curve, and note that in the resonance regions this can change f^2 by as much as 0.03! Clearly a theoretically reliable energy dependence for the S-wave phase shifts and for the total cross sections must be derived before individual values of f^2 can be considered reliable. For convenience in calculating f^2 from our integrals, we also give plots of βS for both cases, in Figs. 4 and 5.

Fortunately, although individual f^2 values are unreliable, we can still obtain significant statistical information. The values of f^2 computed from the data given in Sec. II are plotted in Fig. 6, and the χ^2 computed from their least-squares adjusted average values are given in Table VI. We see from the table that even if the uncertainties in the scattering lengths and the dispersion integrals are included in the analysis, the value of χ^2 is 25.50 as compared with an expected value of 14. This corresponds to a probability of only $4\frac{1}{2}\%$ that the discrepancy is due to statistical errors. If we leave out the errors in the scattering lengths, the results shown in the second and third columns of Table VI are obtained. Treating the scattering lengths as free parameters by the method described in the next section and adjusting them to give a best fit leads to a χ^2 value of 24.47 in the best case, compared to an expected value of 12. Since this corresponds to a probability of only 2.87%, we conclude that no choice of scattering lengths could reconcile this set of data with the dispersion relations. We therefore confirm the existence of the discrepancy discovered by Puppi and Stanghellini. By examining Fig. 6, however, we note that, as in previous discussions, the negative-pion values at 150 and 170 Mev are most out of line. If these are dropped, the remaining data are statistically consistent with a unique f^2. We conclude that either these points are in error, or that there is at best a $4\frac{1}{2}\%$ probability that the

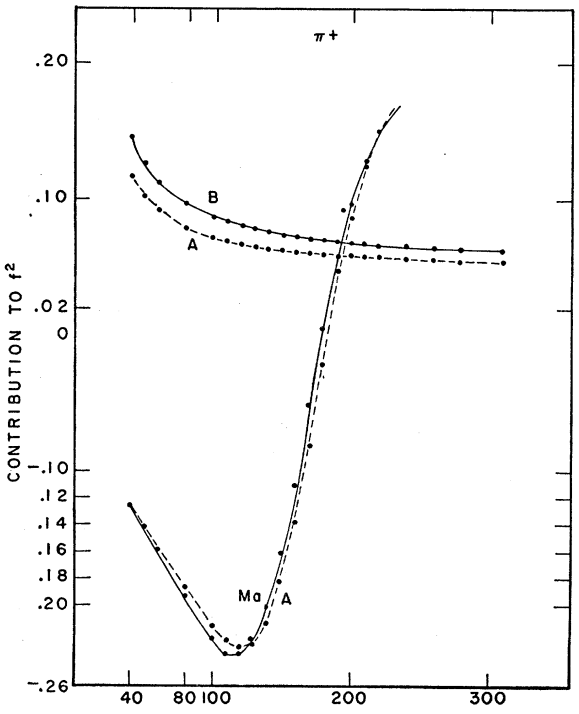

FIG. 4. Contribution to f^2 from the scattering lengths and dispersion integrals for π^+ data. B: $a_+=-0.110$, $a_-=0.077$; H: $a_+=0.085$, $a_-=0.085$; A=Anderson parameterization; MCL =modified Chew-Low parameterization.

TABLE VI. Comparison of the dispersion relations with experimental data below 220 Mev.

a_+ / a_- / Energy dependence of the total cross sections	−0.110±0.004 / 0.077±0.011 f^2	χ^2	% Prob.	−0.110 / 0.077 f^2	χ^2	% Prob.	−0.085 / 0.085 f^2	χ^2	% Prob.
Anderson	0.0844±0.0033	30.49	1.04	0.0832±0.0023	37.06	0.18	0.0790±0.0023	32.88	0.71
Modified Chew-Low	0.0829±0.0033	27.67	2.54	0.0813±0.0023	34.88	0.46	0.0771±0.0023	28.83	1.73
Modified Chew-Low (including parameter error)	0.0832±0.0034	25.50	4.54	0.0818±0.0026	30.81	0.97	0.0768±0.0026	25.63	4.51

forward-amplitude pion-nucleon dispersion relations are in agreement with experiment.

After the completion of this analysis, we learned of two new measurements of $\mathrm{Re}f^-(0)$ at 132 and 152 Mev, by Kruse and Arnold.[18] These values are 0.243±0.015 and 0.218±0.016, respectively, in clear disagreement with the older values. If we abandon the old values at 150 and 170 Mev and substitute these two measurements, the probabilities rise to 8.65% for the Anderson curve, 43.5% for the modified Chew-Low curve, and 47.2% if the parameterization error is included in the latter case. We therefore agree with Kruse[18] that the pion-nucleon forward-dispersion relations are now in agreement with experiment, with the reservation that more theoretical work is needed on the energy dependence of the total cross sections before all lingering doubts can be removed.

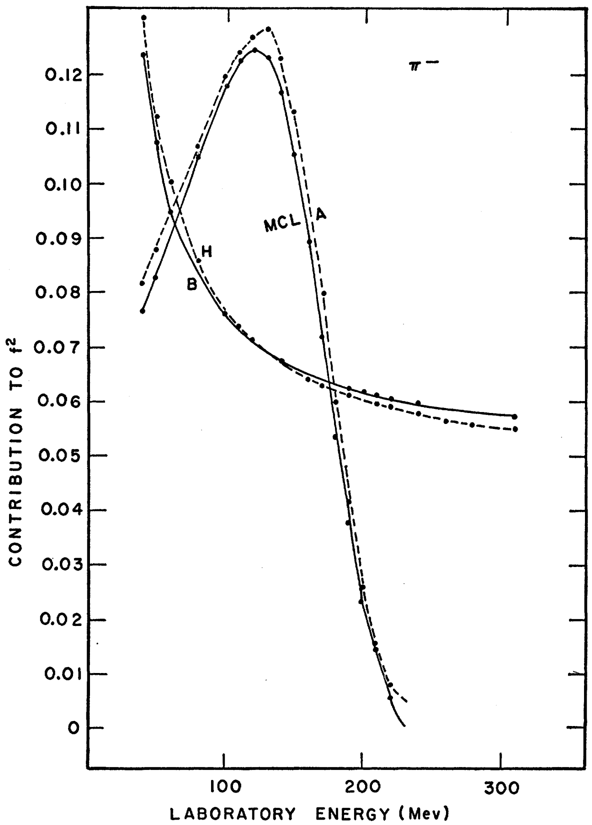

FIG. 5. Same as Fig. 4, except for π^- data.

(b) Calculation of a_+, a_-, and f^2

As has been pointed out,[19,20] it is possible to consider the dispersion relations as defining the three parameters a_+, a_-, and f^2 rather than f^2 alone. Schnitzer and Salzman[20] have carried through such a calculation by fixing f^2 and adjusting a_3 and a_1. They obtained $f^2=0.08\pm0.01$, $a_3=-0.089\pm0.048$, $a_1=0.193\pm0.050$ (or $a_+=-0.089$, $a_-=0.099$). This method conceals the corre-

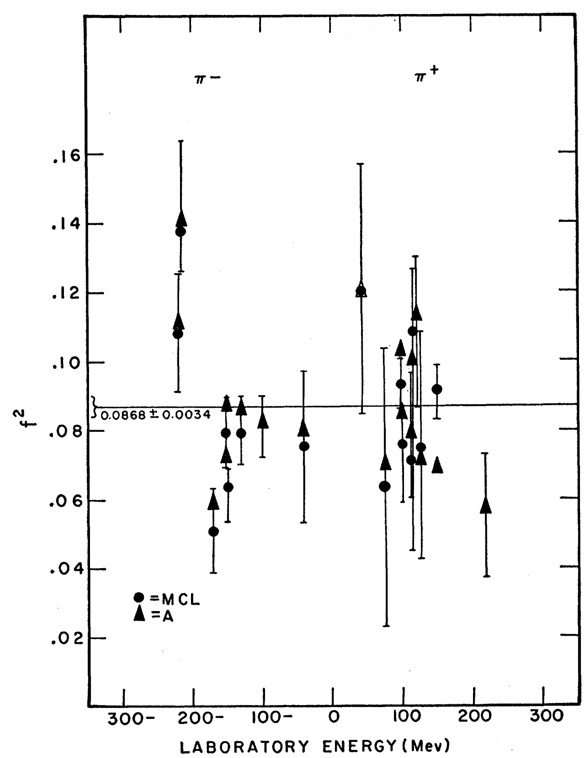

FIG. 6. Comparison of values of f^2 with the average value. Circles give values for the modified Chew-Low parameterization, and triangles for the Anderson parameterization.

[18] U. E. Kruse and R. C. Arnold, Phys. Rev. **116**, 1008 (1959). We are indebted to Dr. Kruse for receipt of this data prior to publication and for interesting discussions.

[19] H. P. Noyes and D. N. Edwards (unpublished).

[20] H. J. Schnitzer and G. Salzman, Phys. Rev. **113**, 1153 (1959).

TABLE VII. Values of f^2 and χ^2 for various scattering lengths, using the Kruse data at 130 and 152 Mev.

a_+ / a_- / Energy dependence of the total cross sections	Best fit f^2	Best fit χ^2	Best fit % Prob.	-0.085 / 0.085 f^2	-0.085 / 0.085 χ^2	-0.110 / 0.077 f^2	-0.110 / 0.077 χ^2
Anderson	0.0755	25.30	2.16	0.0824	28.03	0.0862	27.89
Modified Chew-Low	0.851	14.89	22.2	0.0802	16.37	0.0841	18.56
Modified Chew-Low (including parameter error)	0.0861	14.45	25.4	0.0800	15.89	0.0847	17.29

TABLE VIII. Values of f^2, a_-, and a_+ and their errors for best fit to the data below 220 Mev, using the Kruse data at 130 and 152 Mev.

Energy dependence of the total cross sections	f^2	a_-	a_+	Error matrix$\times 10^4$ $\langle\delta f^2\rangle$	$\langle\delta a_-^2\rangle$	$\langle\delta a_+^2\rangle$	$\langle\delta f^2\delta a_-\rangle$	$\langle\delta f^2\delta a_+\rangle$	$\langle\delta a_-\delta a_+\rangle$
Anderson	0.0755	0.0714	-0.0861	3.28	3.82	6.21	3.44	-4.33	-4.63
Modified Chew-Low	0.0851	0.0854	-0.0990	3.28	3.82	6.21	3.44	-4.33	-4.63
Modified Chew-Low (including parameter error)	0.0861	0.0861	-0.1007	3.56	4.12	6.74	3.70	-4.69	-4.67

lation in error between the three parameters, so we prefer to adjust all three simultaneously. The values of f^2 and χ^2 for the Barnes, Hamilton, and best-fit values for a_+ and a_- are given in Table VII, and the parameters and error matrices in Table VIII. We note that the Anderson curve favors the Hamilton rather than the Barnes value for a_+, as is also true of the values found by Schnitzer and Salzmann, but that with our assignment of errors, does not give a good statistical fit. Conversely, the modified Chew-Low shape gives a better fit, but a higher value for a_+. This again points up the need for a better parameterization before further progress can be made. We also note that when the correlation in error is included, the present data do not determine these three quantities well enough to allow a definite conclusion as to the values of a_+ and a_-.

IV. CONCLUSION

We have evaluated the statistical uncertainty in the dispersion integrals by parameterizing the energy dependence of the total cross sections. The uncertainty from this cause is shown to be small enough to allow a significant comparison of the forward-amplitude pion-nucleon dispersion relations with experiment. Although individual values of f^2 are sensitive to the parameterization, we show that the published data have at best a $4\frac{1}{2}\%$ probability of being consistent with the dispersion relations, even if we ignore measurements of the forward amplitudes above 220 Mev because of uncertainties due to D waves. However, the discrepancy is removed completely for one assumption as to the energy dependence of the total cross sections if the values of the negative-pion forward-scattering amplitudes at 150 and 170 Mev are abandoned in favor of the values recently obtained by Kruse and Arnold at 130 and 152 Mev. Calculation of the scattering lengths, as well as f^2, from the dispersion relations gives some slight evidence for a value of a_+ lower than the value -0.110 given by Barnes.[14] Unfortunately, the uncertainty in this result is too large to allow a definite conclusion to be drawn, and, is sensitive to the parameterization. We conclude that additional theoretical work on the energy dependence of the S phases and the total cross sections will be required in order for further progress to be made.

V. ACKNOWLEDGMENTS

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