Effects of a Nuclear Octupole Moment on Neutron Scattering*

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Recent applications of the nuclear optical model to the description of neutron scattering by spheroidal nuclei have shown that "shape effects" are very important for highly deformed nuclei. It has also been found that the "adiabatic approximation," which assumes the nucleus to be rigidly fixed in orientation throughout the scattering, is remarkably accurate for very low-energy (S-wave) neutrons.

The first part of the present paper is devoted to a detailed investigation of this approximation, showing that the major factor determining its validity for the heavy nuclei to which it has been applied is the large size of the "effective rotating mass" of the nucleus in comparison to the neutron mass. This is analogous to the Born-Oppenheimer approximation in molecular physics, where the large ratio of nuclear to electronic mass enables one to calculate electronic wave functions by considering the slower nuclear motion to be "frozen" completely.

The second purpose of the paper is to investigate the effect of a "pear-shaped" deformation, or octupole moment, of the nuclear optical potential on the S-wave neutron strength function. (A square-edged potential well is used, which is somewhat similar in its effect to a rounded-edge well with a smaller imaginary potential.) This is done for the very heavy nuclei, $225<A<240$, where the possibility of octupole deformations has been suggested by other data. The effect of a small octupole moment for these particular nuclei is found to be largely masked by the nearly indistinguishable effect of their large quadrupole moments, and, in view of the uncertainty in their quadrupole moments, neutron scattering at this time cannot be said to provide any positive evidence of octupole moments. On the contrary, if the quadrupole moments reported from Coulomb excitation measurements are employed, the measured neutron strength function puts an upper limit on the octupole moments of about one-third the quadrupole moment. More accurate data, both on the neutron strength function (as well as R') and on the quadrupole moments, would permit a more accurate estimate of the octupole moments.

IT seems clear by now that any approximation scheme used to describe the interaction between nuclei and low-energy nucleons must include, as a central feature, a description of the "smeared out" or optical-potential aspect of this interaction. Quite aside from recent efforts to explain the origin of this independent-particle aspect of the interaction,[1] the impressive successes of the shell-model in describing the bound states, and of the optical model in describing low-energy continuum states of the system of $(A+1)$ nucleons, leave little doubt that this is a dominant characteristic of the interaction.

The complex optical model[2] has consequently served as a very natural starting-place for subsequent attempts to provide a more detailed description of the scattering of low-energy neutrons, the process with which we shall be concerned here. A simple way of summarizing what aspects of the exact neutron-nucleus interaction are neglected by the spherical optical model in its original form,[2] is to observe that such a potential-well interaction does not couple the neutron to any internal degrees of freedom of the target nucleus. The optical model, consequently, cannot provide an explicit description of direct interaction processes in which inelastic neutron scattering occurs through the excitation of a particular mode of the target nucleus; inelastic processes are all lumped together under the heading of absorption from the coherent beam, and represented by the imaginary part of the optical potential.

In response to accumulating experimental data on the details of low-energy inelastic processes,[3] considerable attention has recently been directed to generalizations of the optical model, obtained by adding to it explicitly certain degrees of freedom of the target nucleus which seem most likely to be excited by neutron bombardment.[4] The possibility of individual-particle excitations has been considered in the "distorted wave" approximation by Lamarsh and Feshbach, and by Levinson and Banerjee[5] while collective rotational motion has been investigated for spheroidal nuclei by Brink, Hayakawa and Yoshida, and Chase, Wilets, and Edmonds.[6]

The investigation of rotational effects is, of course, confined to target nuclei which are nonspherical. For such nuclei, as was emphasized particularly by Margolis and Troubetzkoy,[7] the noncentral character of the interaction with the neutron introduces observable effects into the scattering which persist even at bombarding energies much too low to produce direct excitation of

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[1] E.g., see L. C. Gomes, J. D. Walecka and V. F. Weisskopf, Ann. Phys. **3**, 241 (1958), which includes a bibliography of the previous work of K. A. Brueckner et al.
[2] H. Feshbach, C. E. Porter, and V. F. Weiskopf, Phys. Rev. **96**, 448 (1954).
[3] See, e.g., the summary by R. Sherr in the *Proceedings of the Pittsburgh Conference on Nuclear Structure, 1957*, edited by S. Meshkov (University of Pittsburgh and Office of Ordnance Research, U. S. Army, 1957).
[4] An interesting discussion and summary of recent work on direct interactions at higher energies is given by J. S. Blair and E. M. Henley, Phys. Rev. **112**, 2029 (1958).
[5] J. R. Lamarsh and H. Feshbach, Phys. Rev. **104**, 1633 (1957); C. A. Levinson and M. K. Banerjee, Ann. Phys. **2**, 471, 499 (1957); **3**, 67 (1958).
[6] D. M. Brink, Proc. Phys. Soc. (London) **A68**, 994 (1955); S. Hayakawa and S. Yoshida, Proc. Phys. Soc. (London) **A68**, 656 (1955); D. M. Chase, L. Wilets, and A. R. Edmonds, Phys. Rev. **110**, 1080 (1958).
[7] B. Margolis and E. S. Troubetzkoy, Phys. Rev. **106**, 105 (1957).

rotational levels. In other words, at such energies it is predominantly the *shape* of the nuclear surface rather than the rotational degree of freedom associated with this shape, which influences the scattering. This suggests the use of an "adiabatic approximation"[8] to describe the scattering, in which the nonspherical nucleus is not allowed to rotate at all, but is held in a fixed orientation by giving it an infinite moment of inertia. (Such a nucleus can absorb angular momentum but not energy, just as the "static nucleon" of the Chew-Low model can absorb linear momentum but not energy.) The purpose of the present paper is to investigate the applicability of this approximation to the case of S-wave neutron scattering, and, within the approximation, to consider the effect of an octupole ("pear shaped") moment of the target nucleus on the low-energy scattering.

We shall be most interested in the adiabatic approximation in the $k\rightarrow 0$ limit, i.e., as the energy of the bombarding neutrons tends to zero. In this long wavelength limit, it may at first seem surprising that any "shape effects" at all should appear in the cross section. Although it is true that the neutron's *external* wavelength is much greater than the dimensions characterizing the nuclear deformation, the well depth is so great ($\sim$40 Mev) that the neutron's *internal* wavelength is short enough to "feel" the shape of the nucleus, even when the external wavelength is infinite. In other terms, even though only S waves are incident upon the nucleus, the noncentral force it exerts on the neutron, together with the neutron's large internal kinetic energy, generates higher partial waves in the neutron's wave function in the internal region of the nucleus. These higher waves will, of course, leak out, but even in the $k=0$ limit all except the outgoing S wave (which leaves the nucleus in its rotational ground state) are attenuated faster than $1/r$, so that in the asymptotic region again only S waves are detectable. The effects of the nonspherical shape then appear only in the dependence of the S-wave phase shift on the nuclear "radius," which dependence is markedly different for the spherical and nonspherical cases.

I. INTERNAL COORDINATES AND DIRECT INTERACTIONS[9]

The imaginary part of the optical potential is one means of representing the difference between the actual neutron-nucleus interaction and that described by the real optical potential, i.e., the "residual forces" of the nuclear shell model. In low-energy scattering processes, these residual forces cause the formation of a compound nucleus as an intermediate state, as indicated, e.g., by the symmetry of the angular distributions of such processes about 90°. However, direct processes, which do not proceed through an intermediate compound-nucleus state, may begin to appear as the bombarding energy is increased, and if we are to have a model which can describe them, as well as the compound-nucleus type of inelastic process, we must include the effects of at least a portion of the residual forces by something more explicit than an absorptive potential. That is, in molecular physics terms, we must include specific internal coordinates of the target nucleus, and couple them directly to the incident neutron.

If the Hamiltonian describing these internal coordinates ξ in a target nucleus of A nucleons is $H_A(\xi)$, the total Hamiltonian for the problem is (taking $\hbar=1$)

$$H=-\nabla^2/2m+V(\mathbf{r},\xi)+H_A(\xi), \tag{1}$$

where r is the position-vector of the neutron relative to the center of the nucleus and $V(\mathbf{r},\xi)$ is the interaction potential between the neutron and the nucleus. If H_A has a complete set of eigenstates in ξ space,

$$H_A\varphi(\xi)=\epsilon_n\varphi(\xi), \tag{2}$$

we can employ them to expand the wave function $\psi(\mathbf{r},\xi)$, following the standard procedure employed, e.g., in the atomic scattering case,[10]

$$\psi(\mathbf{r},\xi)=\textstyle\sum_n(f_n(\mathbf{r})\varphi_n(\xi)). \tag{3}$$

It is then clear that the asymptotic behavior of $f_0(\mathbf{r})$ determines the cross section for elastic scattering (i.e., no excitation of the ξ coordinate), and in general $f_n(\mathbf{r})$ determines the scattering with excitation of the internal state $\varphi_n(\xi)$. We shall still take $V(\mathbf{r},\xi)$ complex, in order to account for all other types of inelastic processes.

We note the following asymptotic properties of the solution. Using the Hamiltonian (1) for the Schrödinger equation, we can insert the expansion (3) and use Eq. (2) as well as the orthogonality of the φ_n's, to see that the Schrödinger equation in the *external* region ($V=0$) is

$$(-\nabla^2+2m\epsilon_n)f_n(\mathbf{r})=k^2f_n(\mathbf{r}), \tag{4}$$

for each n. But, introducing the effective wave number k_n,

$$k_n^2=k^2-2m\epsilon_n, \tag{5}$$

this is just the wave equation,

$$(\nabla^2+k_n^2)f_n(\mathbf{r})=0. \tag{6}$$

Consequently if we expand $f_n(\mathbf{r})$ in spherical harmonics, we get for the asymptotic behavior of each outgoing partial wave, $R_{nl}(r)\rightarrow e^{ik_nr}/r$. Provided the energy $E=k^2/2m$ is high enough to excite the level ϵ_n, k_n is real and this is indeed an outgoing spherical wave. If $E<\epsilon_n$, however, k_n is pure imaginary, and the acceptable radial function decays exponentially, $R_{nl}(r)\rightarrow e^{-|k_n|r}/r$. In other words, we are looking at a wave function in a region of negative kinetic energy, and it

[8] D. M. Chase, Phys. Rev. **106**, 516 (1957).

[9] For a more thorough discussion of the relation between direct interactions and the optical model, see H. Feshbach, Ann. Phys. **5**, 357 (1958).

[10] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949). See also reference 9.

displays the characteristic asymptotic behavior of a bound state.

Notice that precisely at threshold for the level ϵ_n, however, $k_n=0$, and the wave equation becomes the Laplace equation, for which the asymptotic behavior is

$$R_{nl}(r) \to r^{-(l+1)}. \qquad (7)$$

The kinetic energy is not negative in this case, but the angular momentum barrier attenuates all partial waves except the S wave more rapidly than r^{-1} in the external region, making them unobservable at large distances from the scatterer.

Let us now specialize to the case of a nonspherical nucleus, which is free to rotate, so that the φ_n's represent rotational states of the target nucleus. In the very low bombarding energy region, where direct excitation is impossible, the way in which the experimental parameters such as Γ_n/D and R' (see references 2 and 7 for their exact definitions) depend upon the nuclear deformation is rather complicated, and seems worth a brief description.

For $E<\epsilon_1$, direct excitation is impossible, and in the asymptotic region we then need consider only the $f_0(\mathbf{r})$ part of the wave function. Whether the potential is spherical or not, we can define phase shifts for the problem by expanding $f_0(\mathbf{r})$ in spherical harmonics in the external region, and if we assume the potential to have a finite range, say R, only those phase shifts for which $l<kR$ will contribute significantly to the cross sections. Since the "low-energy" parameters Γ_n/D and R' are measured by experiments done at energies low enough so that $kR<1$ (say 10 kev or less), only the S-wave phase shift will contribute significantly to the cross section. This is entirely due to the angular momentum barrier, in the usual way, and is strictly true at $k=k_0=0$, where it agrees with the assertion of Eq. (7), that only the S wave is detectable in the asymptotic region.

The nonspherical potential well problem is distinguished from the spherical well problem, however, by the fact that, even in this zero-energy limit, the neutron's wave function contains more than S waves near and inside the well. The noncentral interaction generates (even from an incoming S-wave) many higher partial waves in the internal region, which are damped out in the asymptotic region. The parameter governing this admixture of higher partial waves seems to be $(K\Delta R)$, where K is the internal neutron wave number and ΔR is a typical deformation length, such as the difference between major and minor axes for a spheroid. This parameter is by no means small in the larger deformed nuclei, and in the "near zone" of such a nucleus the neutron's wave function contains many higher partial waves. Although they are unimportant very far from the nucleus, and their phase shifts do not contribute directly to the cross sections, they interfere strongly with the S wave in the near zone, causing the S-wave phase shift to be different for a spheroid, e.g., from what it is for a sphere of the same volume. It is through this interference that the effect of the deformation appears in the observable parameters at low bombarding energy, and it is thus not surprising that the "extra" resonances in Γ_n/D, e.g., appear at the resonant energies which characterize the higher partial waves in scattering from a spherical well.

II. ROTATIONAL EXCITATIONS AND THE ADIABATIC APPROXIMATION[11]

It is especially simple to apply this type of direct interaction formalism to a rigid, axially-symmetric rotator, constrained not to rotate about its symmetry axis, for then the internal Hamiltonian is just $H_A = L^2/2\mathcal{I}$ ($\mathcal{I}$ being the moment of inertia and L^2 the usual angular momentum operator), and its eigenfunctions $\varphi_n(\xi)$ are the spherical harmonics $Y_{lm}(\Omega')$, where $\Omega'\equiv(\theta',\varphi')$ is the direction of the axis.

That is, in this model the nonspherical nucleus is presumed to have a symmetry axis (the usual assumption of the collective model[12]), and only its rotational degree of freedom is to be considered in detail. Any other degrees of freedom are to be included in the complex-potential description. Further, only the excitation of rotational states of the $K=0$ band is considered, which restricts the model to the description of even-even nuclei. The Hamiltonian is thus taken to be

$$H=-\nabla^2/2m+V(\mathbf{r},\Omega')+L^2/2\mathcal{I}, \qquad (8)$$

and if (r,Ω) are the spherical coordinates of $\mathbf{r}$, the wave function ψ has the form $\psi(r,\Omega,\Omega')$. We shall be interested only in very low-energy scattering, and so neglect spin-orbit coupling.

The expansion of the wave function in the eigenstates $Y_{l'm'}(\Omega')$ of $H_A(\Omega')$ is

$$\psi(r,\Omega,\Omega')=\sum_{l'm'}\xi_{l'm'}(r,\Omega)Y_{l'm'}(\Omega'). \qquad (9)$$

If $f_{l'm'}(r,\Omega)$ is expanded in spherical harmonics of Ω, we shall have a double sum, over (l,l',m,m'), but in fact the form of the m and m' sums is determined by the rotational conditions which ψ must satisfy. That is, in the noncentral field represented by $V(\mathbf{r},\Omega')$, the neutron's angular momentum $\mathbf{l}^2$ will, of course, not be a good quantum number, but $\mathbf{J}^2=(\mathbf{l}+\mathbf{L})^2$ will. The eigenstates of $\mathbf{J}^2$ are just the $\mathcal{Y}_{ll'}{}^{JM}(\Omega,\Omega')$ which result from the vector addition of $Y_{lm}(\Omega)$ and $Y_{l'm'}(\Omega')$. ψ must be expandable in terms of these J^2 eigenfunctions, and so only those combinations of $Y_{lm}(\Omega)$ and $Y_{l'm'}(\Omega')$ can occur in (9) which do in fact possess these rotational properties.

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[11] Many of the conclusions of this section were implied in the comprehensive work of Chase et al., reference 6, whose primary concern was with numerical application of the formalism. Our aim in this section is to describe somewhat more explicitly than these authors did the relation between their work and that of Margolis and Troubetzkoy, reference 7, who used the adiabatic approximation.

[12] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 27, No. 16 (1953).

Since we wish to consider only the very low-energy region, we can restrict our considerations to incoming S waves.[13] But since the (even-even) nucleus is initially in its ground state, $L=0$, only the eigenstates with $J=0$ can occur in ψ. This restricts the l' value to $l'=l$, and we thus have ψ expanded in the simple form,

$$\psi(r,\Omega,\Omega')=\sum_l R_l(r)\mathcal{Y}_{ll}^{00}(\Omega,\Omega'). \tag{10}$$

Further,

$$\begin{aligned}\mathcal{Y}_{ll}^{00}(\Omega,\Omega')&=\sum_m C(l,l,0;m,-m,0)Y_{lm}(\Omega)Y_{l'-m}(\Omega')\\ &=(2l+1)^{-\frac{1}{2}}\sum_m(-1)^m Y_{lm}(\Omega)Y_{l'-m}(\Omega')\\ &=(2l+1)^{\frac{1}{2}}(4\pi)^{-1}P_l(\cos\theta),\end{aligned} \tag{11}$$

by the addition theorem, where θ is the angle between the directions Ω and Ω'. With a slight redefinition, then, we have for incident S waves the simple result,

$$\psi(r,\Omega,\Omega')=\sum_l R_l(r)P_l(\cos\theta). \tag{12}$$

ψ thus depends only on the angle between $\mathbf{r}$ and the nuclear symmetry axis, which we might have guessed from the beginning, for with both the incoming neutron and the target nucleus in S states, these are the only two physical directions in the problem. Note also from (10) that whenever the neutron acquires an angular momentum (l,m), the nucleus spins in the opposite direction at the same rate $(l,-m)$ in order to maintain the rotational invariance of ψ.

Since

$$L^2P_l(\cos\theta)=l(l+1)P_l(\cos\theta), \tag{13}$$

the substitution of the expansion (12) into the Schrödinger equation with the Hamiltonian (8) can be written as

$$[-\nabla^2+2mV(r,\theta)]\sum_l R_l(r)P_l(\cos\theta)=2m\sum_l\left(\frac{k^2}{2m}-\frac{l(l+1)}{2\mathcal{I}}\right)R_l(r)P_l(\cos\theta). \tag{14}$$

The notation is simplified slightly if we let R be the mean radius of the nucleus and define M, an effective rotating mass of the nucleus, by

$$\mathcal{I}\equiv MR^2. \tag{15}$$

Then if we further define an effective wave number k_l by

$$k_l^2=k^2-\frac{m}{M}\frac{l(l+1)}{R^2}, \tag{16}$$

we can re-write (14) as

$$[-\nabla^2+2mV(r,\theta)]\sum_l R_l(r)P_l(\cos\theta)=\sum_l k_l^2R_l(r)P_l(\cos\theta). \tag{17}$$

The equation is particularly interesting in this form, for it is very similar to the Schrödinger equation describing the scattering of a particle by nonspherical potential with a fixed orientation. Such an equation differs from (17) only in the replacement of k_l^2 by k^2, which is the same for all partial waves.

This is as it should be, for $k_l^2\to k^2$ is just the $M\to\infty$ limit, which is the adiabatic limit[8] of the rotator problem; it is thus explicitly clear how the rotating-nucleus model of Chase et al.[6] goes over, in this limit, into the static-nucleus model of Margolis and Troubetzkoy.[7] The investigation of this limit is very analogous to Born and Oppenheimer's[14] investigation of a similar limit in molecular wave functions. Because of the large discrepancy between the masses of the electrons and nuclei in a molecule, the nuclear motion (by momentum conservation) is very slow compared to the electron motion, suggesting that electronic wave functions might be obtained with considerable accuracy by considering the nuclei to be rigidly fixed in their average positions. This idea is given formal expression in the Born-Oppenheimer treatment by expanding the molecular wave function in powers of the $\frac{1}{4}$-root of (m/M), the ratio of the electronic to the nuclear mass.

The analogous procedure in the present problem is to employ an expansion of the wave function in some power of (m/M), the ratio of the neutron mass to the effective rotating mass of the nucleus [which ratio appears in k_l, Eq. (16)]. It is not difficult to see that this expansion has the form

$$\psi=\psi_0+(m/M)\psi_1+\cdots. \tag{18}$$

ψ_0 is then the wave function of the adiabatic limit, which as we noted above is just the wave function for the scattering of neutrons by a nucleus of fixed orientation, i.e., of infinite moment of inertia, all of whose rotational excitation energies are zero.

The solution of (17) even in this limit is not obtainable in simple analytic form, for the wave equation for a nonspherical potential well does not separate in any coordinate system.[7] The equation of ψ_1 is similar, with the addition of an inhomogeneous term, and is just as difficult to solve, so that we have found it impossible to give an analytic estimate of the error made in neglecting ψ_1, for any given (m/M).

On physical grounds, one might argue as follows that the neglect of ψ_1 should be not a serious one for heavy nuclei. If we evaluate $\mathcal{I}$, the moment of inertia, from the rotational spectrum of the target nucleus by setting $E_l=l(l+1)/2\mathcal{I}$, then for a nucleus near the end of the periodic system, we find $m/M\sim 1/35$, which is a reasonably small number. Physically, this implies that the nucleus will turn through only a very small angle during the time a neutron takes to traverse the nucleus at its *internal* velocity. For if v is this velocity and R mean radius of the nucleus, the maximum angular momentum of the neutron inside the nucleus is about mvR; then

[13] We shall follow the procedure customary in optical-model calculations, and assume that the "effective neutron" being scattered is a spinless particle. This presumably takes care of the fact that the model is to be applied only to *averages* of cross sections over many resonances with different spins.

[14] M. Born and J. R. Oppenheimer, Ann. Physik **84**, 457 (1927).

since we are assuming the total angular momentum to be zero, the angular momentum of the nucleus is

$$L=\mathcal{I}\omega=MR^2\omega\lesssim mvR,$$

or

$$R\omega/v\lesssim m/M\sim 1/35.$$

But this is approximately the ratio of the traversal time of the neutron to the period of nuclear rotation, and suggests that, as in the molecular case, one could obtain a fairly accurate neutron scattering wave function by considering the nucleus to be fixed rigidly in a given orientation.

This expectation has been confirmed by the work of Chase et al.,[6] who compared the results of the adiabatic-limit of Margolis and Troubetzkoy[7] with a calculation which included at least the most important part of ψ_1 (and higher terms). They found the adiabatic approximation to be valid, at $k=0$, to within something like one percent. One might in general expect that another necessary condition for the validity of such an approximation would be the requirement that the neutron *outside* the nucleus should also move fast relative to the nuclear rotations. In general, this is probably true, but if the neutron is asymptotically in an S state, it is insensitive to the orientation of the nucleus, and so at $k=0$ it is not unreasonable that the adiabatic approximation should be a good one for heavy nuclei.[15]

III. THE EFFECT OF "PEAR-SHAPED" DEFORMATIONS IN THE ADIABATIC APPROXIMATION

The above calculations were all done for spheroidal nuclei. That is, the target nucleus was assumed to have an axis of symmetry, and its section in a plane through this axis was taken to be[16]

$$R(\theta)=R_0[1+a_2P_2(\cos\theta)]. \tag{19}$$

This is a parity-conserving interaction. If the initial state contains only incoming S waves, this interaction will couple it, in the internal region of the nucleus, with higher partial waves of the same parity.

Among the observable parameters of the low-energy scattering, the strength function $\bar{\Gamma}_n/D$, is most sensitive to the appearance of these higher partial waves in the wave function. $\bar{\Gamma}_n/D$, for a spherical-well optical model at low energies, has a characteristic resonant behavior as a function of R, with resonances appearing at the usual S-wave positions. When a spheroidal deformation is introduced, further resonances appear in $\bar{\Gamma}_n/D$, even at $k=0$, which are associated with the $l=2$, $l=4$, etc., partial waves.

There is a set of these "extra" resonances for each value of the principal quantum number n, i.e., for each S-wave resonance, and as n increases, the positions of the resonances of each such set cluster more and more tightly about the corresponding S-wave resonance position. (This corresponds to $j_l(x)$ approaching more and more closely to its asymptotic sinusoidal behavior as $x\rightarrow\infty$.)

If now an odd-l deformation were introduced into (19), odd partial waves would also appear in the neutron's wave function, and additional resonances in $\bar{\Gamma}_n/D$ would appear at the $l=1$, $l=3$, etc., resonant values of R. These are characteristically located *between* the S-wave positions, so they should show up quite distinctly.

The investigation of the effects of such "pear shaped" deformations on neutron scattering is of interest for two reasons. In the first place, the Brookhaven group[17] has noticed a set of experimental $\bar{\Gamma}_n/D$ values in the neighborhood $A\sim 230$–240 which are anomalously high compared with those found for $A\sim 100$, which are located symmetrically on the other side of the S-wave resonance at $A=150$. This is interesting, for a P wave, if present in the neutron's wave function, should be in resonance at about $A\sim 230$.

Secondly, Stephens, Asaro, and Perlman,[18] in investigating the low-lying rotational levels of even-even nuclei in this region, find that they do not follow the usual $0+$, $2+$, $4+$, $\cdots$ sequence, but contain 1- and 3-states as well. It has been suggested that these levels can be understood as part of a rotation -inversion spectrum, of the type found in the NH_3 molecule. That is, if the nucleus does have a pear-like shape, it should be possible for the large and small ends to exchange positions by an inversion process, involving a passage through a potential barrier. There should then be a series of vibrational-type levels corresponding to this inversion, which are of alternating even and odd parity and will be grouped in even-odd pairs if the potential barrier is high. One could then have an even-l rotational band built on the even (lower) member of such a pair, and an odd-l band built on the other, thus reproducing spectra of the type observed among these heavy nuclei. An estimate of the frequency of the inversion period is given by the splitting between an even-odd pair.[19] Since these and other properties of such a spectrum have already been discussed by Lee and Inglis[20] and by Alder et al.,[21] we shall not consider the details further here.

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[15] There is also another factor assisting the adiabatic approximation in this low-energy region. This is the fact that k_0 [Eq. (16)], the wave number associated with the S-wave part of the wave function, is always equal to k, i.e., it does not change as we deviate from the adiabatic limit by increasing (m/M). But for reasonable deformations the S-wave remains the dominant part of the wave function, and the fact that its wave number is independent of (m/M) should give the whole wave function an additional stability about the value $m/M=0$.

[16] Our definition of the quadrupole deformation parameter a_2 agrees with the definition used in reference 7, and the parameter β of Chase et al., reference 6, is $\beta=1.58a_2$.

[17] R. B. Schwarz, V. E. Pilcher, D. J. Hughes, and R. L. Zimmerman, Bull. Am. Phys. Soc. **1**, 347 (1956).

[18] F. S. Stephens, Jr., F. Asaro, and I. Perlman, Phys. Rev. **100**, 1543 (1955).

[19] F. Hund, Z. Physik **43**, 820 (1927).

[20] K. Lee and D. R. Inglis, Phys. Rev. **108**, 774 (1957).

[21] K. Alder, A. Bohr, T. Huss, B. Mottelson, and A. Winther, Revs. Modern Phys. **28**, 432 (1956).

The inclusion of $P_1(\cos\theta)=\cos\theta$ in the shape expansion (19) corresponds merely to a shift of the center of mass of the nucleus, so the lowest non-trivial odd-order deformation is the octupole deformation, $P_3(\cos\theta)$. In fact, it is appropriate to include both $l=3$ and $l=1$ harmonics, choosing the coefficient of P_1 in such a way as to keep the center of mass of the nucleus at the origin of the coordinate system; this condition gives $a_1=-27a_2a_3/(35+30a_2)$. Including a quadrupole deformation as well, the nuclear shape is then

$$R(\theta)=R_0[1+a_1P_1(\cos\theta)+a_2P_2(\cos\theta)+a_3P_3(\cos\theta)]. \quad (20)$$

Knowing from the numerical results of Chase et al., that the adiabatic approximation is a good one for the rotational degree of freedom, we shall consider the target nucleus to be fixed in orientation. If the rotation-inversion picture of the spectrum outlined above is the correct one, it might be expected that the inversion degree of freedom should also be included explicitly in the model Hamiltonian. However, we may use the following estimate to indicate that this degree of freedom can also safely be treated in the adiabatic approximation. The classical inversion period, as we noted above, is given by the splitting between an even-odd pair of vibrational levels. Assuming the validity of the inversion-rotation picture of the spectrum, this splitting may be estimated from the known levels as $T_i\sim\hbar/\Delta E$, with $\Delta E\sim E_1-(E_0+E_2)/2$. Since Stephens et al., find $\Delta E\sim 250$ kev, this gives $T_i\sim 1.6\times 10^{-20}$ sec as the inversion period. But the traversal time of a neutron inside the nucleus (assuming as in the rotational case that, for incident S waves, this is the relevant time) is more than 100 times smaller than T_i (for a 40 Mev well). We therefore expect the adiabatic approximation to be extremely good for the inversion process as well as for the rotations, and we shall apply it to both.

In this static limit, the most convenient method for obtaining a numerical solution of the Schrödinger equation seems to be that employed by Margolis and Troubetzkoy[7] for a spheroidal deformation, and it is this method which we have used. The formalism needed when a nuclear octupole moment is included differs from theirs only in the replacement of their sums over even partial waves by sums over all partial waves. Since the difference is so slight, we shall not describe the method further here, but refer the interested reader to their paper for the necessary details. It is unfortunate that this method of solution is applicable only to a square-well potential. This means that we shall not, in this paper, be able to investigate the simultaneous effects of an octupole deformation and a diffuse nuclear surface.

The parameters which enter the calculation are R, the effective radius of the well; V_0, the depth of the real part; ζV_0, the depth of the imaginary part; and the deformation parameters a_2 and a_3. With five parameters available, there is no question of our ability to fit the small group of experimental points around $A\sim 235$ mentioned earlier.[17] In fact, they can be fit by many different sets of values for these parameters, so that little meaning can be attributed to a chance fit unless at least most of the parameters can be determined in advance by other means.

We shall assume that R, V_0, and ζ can be determined by fitting the optical model to low-energy scattering data for nuclei which are spherical. To determine an effective radius R for a nonspherical nucleus, we shall make the usual assumption that all nuclear matter has the same density, so R is taken as the radius of a sphere whose volume equals the volume enclosed by the surface (20). This R is not quite equal to the R_0 of (20), but is related to it by

$$R^3=R_0^3[1+\tfrac{3}{5}a_2^2+(3/7)a_3^2], \quad (21)$$

correct through second-order quantities in the deformations. We shall assume that $R=r_0A^{\frac{1}{3}}$ to relate R, and thus R_0, to A; r_0 is determined by fitting the model to spherical nuclei.

There seems to be no reason for thinking that V_0 should depend on the shape of the nuclear surface, so we shall use the V_0 which fits spherical nuclei best. Finally there is ζ, the parameter determining the depth of the imaginary potential. This, we feel, should also have the value determined by the fit to the scattering from spherical nuclei. This is because, although a nonspherical nucleus does possess rotational degrees of freedom which a spherical one does not, we have taken account of them explicitly, so that ζV represents only inelastic processes *other* than rotational excitations, just as in the spherical case.

Since we are employing a square-well potential, the parameters r_0, V_0, and ζ which we use should be determined by fitting the scattering from spherical nuclei

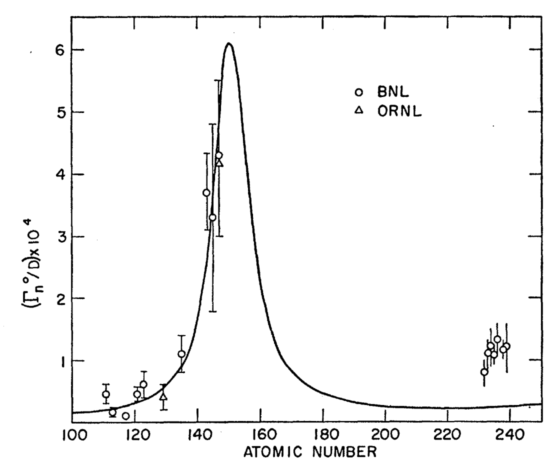

FIG. 1. Low-energy strength functions for near-spherical nuclei The curve is an optical model calculation for spherical nuclei with $r_0=1.45f$, $V_0=42$ Mev, and $\zeta=0.03$. The data for the nonspherical nuclei at $A\sim 235$ are included to emphasize their deviation from this curve.

with a square-well potential. This was just the model originally employed by Feshbach, Porter, and Weisskopf,[2] and in fact still provides as good a fit to the low-energy scattering by *spherical* nuclei as any of the more recent modifications employing diffuse-edge wells.[22] In Fig. 1 we have reproduced a plot of the strength function calculated from such a model, using the best choice of parameters of reference 2: $r_0=1.45$ fermis, $V_0=42$ Mev, and $\zeta=0.03$. The data plotted in Fig. 1 refer only to spherical or nearly-spherical nuclei[23] (except the points around $A=235$, in which we shall be particularly interested later), and they are seen to be quite reasonably reproduced by this choice of parameters. The fit to these data determines (V_0R^2), this being the quantity which determines the resonance positions at zero energy. We shall, below, wish to determine another resonance position very accurately, at $A\sim 235$. To do this we shall use the value of (V_0R^2) determined from Fig. 1 for spherical nuclei in some convenient region, say at $A=A_0$, and then assume that $R=r_0A^{\frac{1}{3}}$, so that (V_0R^2) at $A=A'$ is given by $(V_0R^2)'=(V_0R^2)_0(A'/A_0)^{\frac{2}{3}}$. To avoid extrapolating too far, we have chosen A_0 as large as possible, i.e., about 140. We estimate that the points in this region determine the position of the steeply-rising section of the curve to within about 5 units in A.

We shall, then, employ the values of reference 2 for r_0, V_0 and ξ. The quadrupole deformation parameter a_2 has been determined for two or three nuclei near the end of the periodic system by Coulomb excitation. Heydenberg and Temmer[24] use $r_0=1.2$ to interpret their measured $E2$ transition probability in terms of a quadrupole deformation parameter. Since it is the charge distribution which they are measuring, this use of the r_0 determined by electron scattering[25] seems correct, and gives $a_2\sim 0.2$, with an accuracy of perhaps 20%. This work has more recently been repeated by Alder et al.,[21] who find $a_2\sim 0.15$, with about the same accuracy claimed. This leaves a_3 as our only free parameter, which we can choose appropriately to fit the Brookhaven data on $\bar{\Gamma}_n/D$.

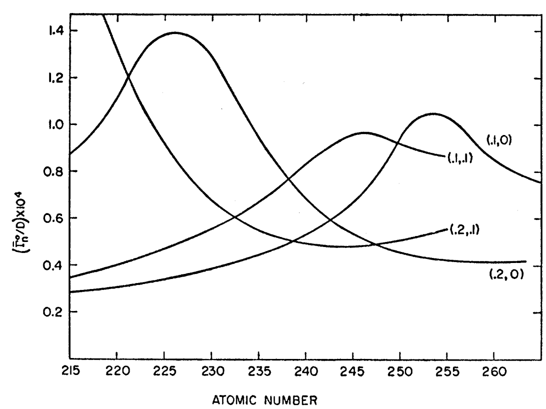

FIG. 2. Low-energy strength functions for heavy nuclei, calculated for several combinations of quadrupole and octupole deformation parameters. The curves are labelled by these parameters, as (a_2,a_3). A square-well potential was used, with $r_0=1.45$f, $V_0=42$ Mev, and $\zeta=0.03$.

Figure 2 gives the results of the calculation of the strength function for $A>215$, for four different choices of the deformation parameters, (a_2,a_3).[26] For a_2 different from zero, $\bar{\Gamma}_n/D$ has a resonance in this region which brings its values well above those of the spherical model ($\bar{\Gamma}_n/D\sim 0.25\times 10^{-4}$) shown in Fig. 1. It is rather surprising to find that this resonance is present for a *pure* quadrupole deformation, i.e., even when $a_3=0$, for this is a region very far from the S-wave resonant positions. This resonance, which was also seen by Chase et al.,[6] can be identified as due to the $l=4$ partial wave from the fact that it drifts rapidly to the right as a_2 decreases, and seems to disappear (as $a_2\rightarrow 0$) at about $A=285$, the $l=4$ resonant position.

As we mentioned earlier, the P wave is expected to show a resonance in this region if $a_3\neq 0$. However, the fact that there is also an $l=4$ resonance present means that we can expect an interference between them, with the resulting curve depending strongly on a_2 as well as on a_3. The effect of a_3 is felt most strongly in the region $A\sim 220$–230, as expected, and increasing a_3 at constant a_2 is seen to have the effect mainly of shifting the resonance to the left—an effect difficult to distinguish from increasing a_2 at constant a_3.

In Fig. 3 we have plotted curves with $a_2=0.15$, corresponding to the measurements of Alder et al., for $a_3=0$ and $a_3=0.10$.[27] We have also included the experimental points mentioned earlier,[16] and find that they are actually fit best by $a_2=0.15$, $a_3=0$. The choice $a_3=0.10$ appears to be definitely too large, and by interpolation it would seem difficult to fit these data with a value of a_3 greater than about 0.03, if the value $a_2=0.15$ is adhered to. (Using the Heydenburg-Temmer value $a_2=0.20$ *decreases* the upper limit on a_3, and in fact it makes it difficult to fit the data even for $a_3=0$).

The limit $a_3\leq 0.03$ should be taken with a grain of salt, however, for it depends sensitively on the values assumed for the other parameters of the problem. In

[22] See e.g., V. F. Weisskopf, Revs. Modern Phys. **29**, 174 (1957).

[23] The deformation of these nuclei are estimated from their static quadrupole moments, as given by C. H. Townes, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1958), Vol. 38, Part 1, p. 442.

[24] N. P. Heydenburg and G. M. Temmer, *Annual Review of Nuclear Science*, (Annual Reviews, Inc., Palo Alto, 1956), Vol. 6, p. 77.

[25] R. Hofstadter, Revs. Modern Phys. **28** (3), 214 (1956).

[26] The sign of a_2 is significant, distinguishing between prolate and oblate quadrupole deformations; a_2 is positive for the few cases measured among the very heavy elements. The sign of a_3 on the other hand is meaningless. Reversing the sign of a_3 inverts or interchanges the large and small ends of the nucleus. But for an axially-symmetric body, the same effect can be achieved by a 180° rotation, and since the incoming S waves are insensitive to the orientation of the nucleus, this operation leaves the results of the calculation unchanged.

[27] For comparison, curves were also calculated using $\zeta=0.04$ rather than 0.03. This broadened the curves slightly, but did not shift them sideways.

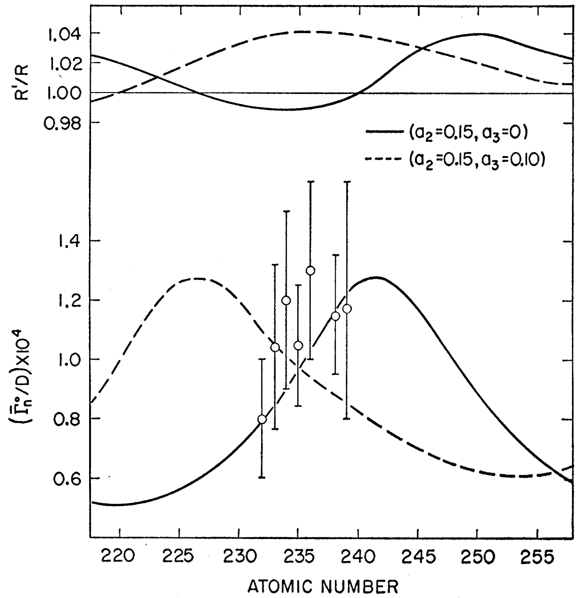

FIG. 3. Comparison of experimental strength function data with curves calculated for nuclei having quadrupole and octupole deformations. The square-well parameters are $r_0=1.45$f, $V_0=42$ Mev, $\zeta=0.03$. Calculated curves of R'/R are also included.

particular, if $(V_0r_0^2)$ is changed, the whole curve can be shifted to the right or left, and a different value of a_3 will then be needed to bring it back to the experimental points. As we mentioned earlier, $(V_0r_0^2)$ is determined principally by the points near $A=140$ of Fig. 1. Shifting the curve of Fig. 1 sideways by more than about 5 units in A would spoil the fit to these points, and this corresponds to a shift at $A=240$ of about 8 units. In other words, the positions of the resonances of Fig. 3 are really only determined to within about 8 units in A, which means that the upper limit on a_3 is known only with an error of something like 0.05.

We have also included in Fig. 3 the calculated values of R'/R (whose definition is given, e.g., in reference 2). For a spherical nucleus, it would be very nearly 1.0 throughout this region, and our choice of parameters is seen to keep it quite close to this value. R' has been measured for only two of these heavy nuclei,[28] and the experimental values fit neither the spherical-nucleus curve nor either of our curves for nonspherical nuclei.

It is interesting to note that Fröman[29] has recently estimated the value of a_3 for nuclei in this region by an analysis of their α decays. For the daughter nuclei of various isotopes of Th and U, he finds $a_3\sim 0.006$–0.02, and for the daughter of Cm^{242}, $a_3\sim 0.001$. These values are quite consistent with our "best estimate" of an upper limit on a_3.

In summary, a square-well potential model with the parameters $r_0=1.45\,f$, $V_0=42$ Mev, and $\xi=0.03$ is capable of fitting the zero-energy strength function data for the spherical-nuclei quite well, and can also fit the data for $A\sim 230$ with a quadrupole deformation of the size inferred from Coulomb excitation measurements. A nucleus with a small octupole deformation can also fit these latter points; with the above choice of parameters, and $a_2=0.15$ from Coulomb excitation measurements, we estimate that the most likely upper limit on a_3 is $a_3\leq 0.03$. This limit depends sensitively on the values assumed for the other parameters, and by stretching the fit to the spherical nuclei, it could be varied by perhaps as much as 0.05.

ACKNOWLEDGMENT

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[28] K. K. Seth, D. J. Hughes, R. L. Zimmerman, and R. C. Garth, Phys. Rev. **110**, 692 (1958).

[29] O. Fröman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Skrifter **1**, No. 3 (1957).