mental points are not in agreement with the shape factor predicted for a once-forbidden unique transition. It should be pointed out that small changes in the end-point energy used in the calculation of the experimental shape factor will not alter the above conclusion.

CONCLUSIONS

The gamma-ray measurements of the present work are in essential agreement with that of other investigators.[6,7] The single gamma-ray energy was measured to be 1.208±0.010 Mev and is to be compared with 1.22±0.01 Mev[6] and 1.190±0.005 Mev[7]; with no evidence for any other gamma transitions.

The beta spectrum measured in coincidence with the 1.208-Mev gamma is in agreement with that expected for a single beta-group having a maximum electron energy of 0.319±0.010 Mev. The maximum electron energy as determined by other experimenters is reported to be 0.33±0.01 Mev[6] and 0.36±0.02 Mev.[7] The present measurement supports the conclusion that the 0.319-Mev transition is not a once-forbidden unique transition. Within the statistical accuracy of the experiment, the shape of this weak beta group corresponds to a statistical or allowed shape. This conclusion together with a comparative half-life of 8.8 suggests that the transition is a nonunique once-forbidden transition which would support the proposed alternative assignments of $\frac{1}{2}+$ or $\frac{3}{2}+$ of Way et al.,[2] for the first excited state of Y^{91}. These measurements yield a Y^{91}-Zr^{91} mass difference of 1.527±0.014 Mev in agreement with the measured beta-ray end point of 1.537±0.007 Mev by Langer and Price.[3]

ACKNOWLEDGMENT

The authors wish to thank Professor R. W. King for helpful discussions related to this investigation.

PHYSICAL REVIEW VOLUME 118, NUMBER 5 JUNE 1, 1960

Low-Energy Photodisintegration of H^3 and He^3 and Neutron-Deuteron Scattering*

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(Received December 11, 1959; revised manuscript received January 4, 1960)

The cross sections for electric dipole photodisintegration of H^3 and He^3 at low energies are expressed in terms of the effective range parameters of the doublet *n-d* scattering matrix. Agreement with the experimental results is possible for either set of *n-d* scattering lengths.

I. INTRODUCTION

DIRECT elastic scattering experiments do not define the neutron-deuteron doublet and quartet scattering lengths uniquely, but lead to the two alternative sets[1]

$$\text{(A)}\quad {}^2a=0.8\pm0.3,\quad {}^4a=6.2\pm0.1,\qquad \text{(B)}\quad {}^2a=8.3\pm0.2,\quad {}^4a=2.4\pm0.2 \text{ fermis},$$

and theoretical arguments have been advanced for both sets (A)[2–4] and (B).[5–8] It is therefore of interest to point out that the low-energy H^3 and He^3 photodisintegration cross sections can be expressed in terms of the *n-d* doublet effective range. This in turn is determined, if we neglect higher terms in the effective range expansion, by the doublet scattering length and the binding energy of the last neutron in H^3. Then the large difference between the doublet scattering lengths of sets (A) and (B) leads to very large differences in the predictions for the photodisintegration cross sections. These differences do not disappear on consideration of higher terms in the expansions, but do so if, as is not unlikely, $k\cot\delta$ has a pole on the imaginary axis between zero and the triton bound state. The existence of such a pole is necessary to give agreement between the observed $d(p,\gamma)He^3$ cross section and calculations pointing to set (A) as the correct set of scattering lengths.

II. BOUND STATES AND THE EFFECTIVE RANGE EXPANSION

For any system the scattering matrix $\mathbf{S}$ referring to open channels only is, if the representation is suitably chosen, unitary and symmetric; then $\mathbf{S}$ can be diagonalized:

$$\mathbf{S}=\mathbf{T}^{-1}\exp(2i\boldsymbol{\delta})\mathbf{T}, \qquad (1)$$

* This work is supported in part by funds provided by the U. S. Atomic Energy Commission, the Office of Naval Research, and the Air Force Office of Scientific Research.

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where $\mathbf{T}$ is a real orthogonal matrix and $\boldsymbol{\delta}$ is real and diagonal. The eigenstates Ψ_α of the scattering are then given asymptotically, in the author's notation[9] by

$$\Psi_\alpha^\infty = j_\alpha \Sigma_\gamma m_\gamma^{\frac{1}{2}} k_\gamma^{-\frac{1}{2}} T_{\alpha\gamma} \times[\cot\delta_\alpha F_\gamma(k_\gamma r_\gamma)+G_\gamma(k_\gamma r_\gamma)] r_\gamma^{-1}\phi_\gamma \mathcal{Y}_\gamma, \quad (2)$$

where j_α is a normalizing constant and the $T_{\alpha\gamma}$ give the relative amplitudes of the various channels at infinity. F_γ and G_γ are the usual regular and irregular solutions of the Schrödinger equation in the external region of channel γ, and ϕ_γ, $\mathcal{Y}_\gamma$ are, respectively, the product of the intrinsic wave functions of the final nuclei, and the usual spin-angle function.

We may continue Eqs. (1) and (2) analytically into regions for which some of the channel energies are negative; or we can include these explicitly from the start, in which case the unitary property of $\mathbf{S}$ is generalized to[10] $\mathbf{S}(k)[\mathbf{S}(k^*)]^{*T}=\mathbf{1}$.

Further

$$\mathbf{S}(-k)\mathbf{S}(k)=\mathbf{1}. \quad (3)$$

$\mathbf{S}$ is still symmetric and (1) is retained, with $\boldsymbol{\delta}$ and $\mathbf{T}$ still diagonal and orthogonal, respectively, but no longer real; they are just the analytic continuations of the corresponding matrices in the positive energy region, regarded as functions of the channel wave numbers, into the complex plane. The continuation is determined by the boundary conditions to be that into the upper half plane.[11]

Now Ψ represents a bound state when in every channel there exist only exponentially decaying waves; from (2) this is given by the condition

$$\cot\delta_\alpha = i, \quad (4)$$

together with the requirement that the channel energy be negative for all channels connected to α by nonzero $T_{\alpha\gamma}$.

Thus we have a bound state of an arbitrary system whenever one of the characteristic phase shifts passed through $i\infty$; this is a straightforward generalization of the one channel case. Moreover, the $T_{\alpha\gamma}$ give the asymptotic amplitudes of the various channels in the bound system.

If we now restrict ourselves to the case when α is an S-wave neutral channel, we can expand $\cot\delta_\alpha$ about zero energy in that channel as follows[9]:

$$k\cot\delta_\alpha = -1/a_0+\tfrac{1}{2}r_0k^2+Tk^4+\cdots, \quad (5a)$$

or about the bound-state energy using

$$\cot\delta_\alpha = i+\tfrac{1}{2}r_0'k'^2+T'k'^4+\cdots, \quad (5b)$$

where k' is measured from a zero at the bound-state energy. Expansion (5b) is not valid for energies such that k is real; but (5a) can be used for k imaginary. Writing $k'^2=k^2+\gamma^2$ and assuming the radius of convergence of (5a) is large enough, we can equate coefficients of the two series to find

$$\begin{aligned} i&=i/(\gamma a_0)+ir_0\gamma/2-i\gamma^3T+\cdots,\\ \tfrac{1}{2}r_0'&=i/(2\gamma^3a_0)-ir_0/4\gamma+3i\gamma T/2. \end{aligned} \quad (6)$$

Moreover, we can evaluate r_0' explicitly in terms of the asymptotic form of the bound state. If this has the form in the external region

$$\Psi^\infty = A\{[F_\alpha(i\gamma r)-iG_\alpha(i\gamma r)]\phi_\alpha\mathcal{Y}_\alpha/r_\alpha + \sum_{\lambda\neq\alpha} B_\lambda[F_\lambda(i\gamma r)-iG_\lambda(i\gamma r)]\phi_\lambda\mathcal{Y}_\lambda/r_\lambda\},$$

where F_α, G_α are the S-state channel wave functions and A the S-state amplitude, we find by evaluating the relevant equation (21a) of reference 9:

$$r_0'/2=-iT_{\alpha\alpha}^2/\gamma A^2, \quad (6a)$$

where $T_{\alpha\alpha}$ is evaluated at $(i\gamma)$. Then (6) becomes

$$\gamma = 1/a_0+\tfrac{1}{2}r_0\gamma^2-T\gamma^4, \quad (6b)$$

$$A^2=2\gamma T_{\alpha\alpha}^2/(\gamma r_0-1-4\gamma^3T). \quad (6c)$$

In particular, (6b) and (6c) are valid for the triton, with $i\gamma$ the wave number of the last neutron and a_0, r_0, T the $d(n,n)d$ doublet S-wave characteristic phase-shift parameters. The corresponding relations for He^3 are more complicated but we shall not need them; we shall refer the properties of He^3 back to the n-d parameters rather than to the p-d parameters.

r_0 can be eliminated from (6c) using (6b) to give

$$|A| = \left|\left(\frac{2\gamma T_{\alpha\alpha}^2}{2/(a_0\gamma)-1+2\gamma^3T}\right)^{\frac{1}{2}}\right|.$$

III. PHOTODISINTEGRATION OF H^3 AND He^3

At the energies we are considering (neutron or proton final energies of $\lesssim\frac{1}{2}$ Mev) the only non-negligible transitions are magnetic and electric dipole. Both H^3 and He^3 are predominantly symmetric S state [$T_{\alpha\alpha}$ in (6c) ~ 1] and we shall neglect transitions from other than this state. For a totally space-symmetric S state H^3 or He^3 nucleus the magnetic dipole matrix elements vanish; this seems to have been noticed first by Verde[12] and accounts for the almost complete cancellation found by Massey and Burhop.[13] We shall therefore neglect magnetic dipole transitions; an effective range expansion can be given for them, but the approximations we shall make are worse than for electric dipole transitions, chiefly because the inner regions contribute more. The small magnetic cross sections observed experimentally[14] are due to nonsym-

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[11] R. G. Newton, Ann. Phys. **4**, 29 (1958).
[12] M. Verde, Helv. Phys. Acta **23**, 453 (1950).
[13] E. H. S. Burhop and H. S. W. Massey, Proc. Roy. Soc. (London) **A192**, 156 (1947).
[14] G. M. Griffiths (private communication).

metric S states and states of higher angular momentum, and can be subtracted before comparison is made.

We shall also neglect the very small spin-current contributions to the electric dipole transitions. The final states are the continuum doublet P states, and the photodisintegration cross section is then for unpolarized incident light:

$$\sigma_{\text{dis}} = \frac{\pi e^2(k^2+\gamma^2)}{\hbar c k}|\mathfrak{M}|^2, \quad (7)$$

where k is the wave number of the emitted particle and γ the wave number of the last bound particle in the ground state; the energy required to remove this particle is E_d, where

$$\gamma^2 = (4M/3\hbar^2)E_d, \quad (8)$$

with M the nucleon mass. The matrix element $\mathfrak{M}$ is

$$\begin{aligned} \mathfrak{M}(\mathrm{H}^3) &= \langle\phi_i|z_p|\phi_f\rangle, \\ \mathfrak{M}(\mathrm{He}^3) &= \langle\phi_i|z_{1p}+z_{2p}|\phi_f\rangle, \end{aligned} \quad (9)$$

where z_p is the z component of $\mathbf{r}_p$, the position vector of the proton relative to the c.m. We neglect recoil effects. ϕ_f is the continuum P state normalized to unit amplitude at infinity. We shall neglect the effects of the nuclear force in this state and approximate ϕ_f by the free-state P wave function. We consider first the triton matrix element. As in the two-body case, we argue that contributions to $\mathfrak{M}$ come predominantly from large distances, for which we can write $r_p=\frac{1}{3}r$ where $\mathbf{r}$ is the channel coordinate, and the only part of the H^3 wave function which contributes to $\mathfrak{M}$ is $\phi_i = \phi(\text{deuteron})\times Ae^{-\gamma r}\mathcal{Y}_0/r$ where A is given by (6c). We also write $\phi_f=\phi(\text{deuteron})\times[\sin kr/r-\cos kr]\mathcal{Y}_1/r$ and obtain the cross section for disintegration of the triton as (we have put $T_{\alpha\alpha}{}^2=1$)

$$\sigma_{\text{dis}}(\mathrm{H}^3) = \frac{8\pi e^2k^3\gamma}{9\hbar c(k^2+\gamma^2)^3}\times\frac{1}{|1-\gamma r_0+4\gamma^3T|}. \quad (10)$$

We can also evaluate $\mathfrak{M}(\mathrm{He}^3)$ approximately in terms of the same parameters r_0 and T. We have, to the same accuracy as for H^3, the same matrix element with $r/3$ replaced by r, and the wave functions replaced by the equivalent Coulomb functions. Distinguishing quantities referring to He^3 by a prime, we have again a relation for the asymptotic amplitude A' of the ground-state function in terms of the p-d nuclear doublet effective range expanded about the He^3 binding energy:

$$\tfrac{1}{2}r_0' = \int C_0{}^2(i\gamma')\{[\phi^2(\mathrm{He}^3)/A'^2] - [iF_0'(i\gamma' r)+G_0'(i\gamma' r)]^2\phi^2(\text{deuteron})\}d\tau, \quad (11)$$

where

$$C_0{}^2(k) = (2\pi/kD)[\exp(-2\pi/kD)-1]^{-1}; \quad D=3\hbar^2/2Me^2.$$

Contributions to this integral come only from the internal region; and in this region we expect the separate terms to be close to their neutral equivalents. Moreover, we have very closely, in this region,

$$\phi(\mathrm{H}^3)=\phi(\mathrm{He}^3); \text{ so we have } A'^2 = C_0{}^2(i\gamma')A^2. \quad (12)$$

Over the range of energy and radial distance in question, the final state wave function is given to within a few percent by

$$F_1(kr,1/kD) = \frac{C_0(k)kD}{(1+k^2D^2)^{\frac{1}{2}}}F_1(kr,0). \quad (12\text{a})$$

Using this we obtain finally

$$\sigma_{\text{dis}}(\mathrm{He}^3) = \frac{8\pi e^2k^3\gamma}{\hbar c(k^2+\gamma'^2)^3}\,\frac{C_0{}^2(k)k^2D}{2^{\frac{1}{2}}(1+k^2D^2)\gamma'[1-\cos(2\pi/\gamma' D)]^{\frac{1}{2}}} \times\frac{1}{|1-\gamma r_0+4\gamma^3T|}. \quad (13)$$

The distinction between γ and γ' is only about 7%.

Capture Cross Sections

The cross sections for proton or neutron capture by deuterium can be written down at once from (10) and (13). They are, for capture from an unpolarized beam:

$$\sigma_{\text{cap}}(n) = \frac{\pi e^2\hbar k\gamma}{3m^2c^3(k^2+\gamma^2)}\,\frac{1}{|1-\gamma r_0+4\gamma^3T|}, \quad (14\text{a})$$

$$\sigma_{\text{cap}}(p) = \frac{3\sqrt{2}\pi^2e^2\hbar k^3\gamma C_0{}^2(k)D}{m^2c^3\gamma'(k^2+\gamma'^2)(1+k^2D^2)[1-\cos(2\pi/\gamma' D)]^{\frac{1}{2}}} \times\frac{1}{|1-\gamma r_0+4\gamma^3T|}. \quad (14\text{b})$$

IV. COMPARISON WITH EXPERIMENT

An assumed scattering length and binding energy of the three-particle system define the disintegration cross sections uniquely if we neglect T. This is certainly valid in the corresponding two-particle problem, but less so here, due to the greater binding energies involved. T can be estimated in various ways. An exact calculation of the H^3 binding energy in the approximation of reference 8, compared with the calculated value of a_0 and r_0 given there, gave $T=+2$ (fermis)^3. The phase-shift plot of de Borde and Massey[15] and the "experimental" phase-shift analysis of Christian and Gammel[2] both give T close to zero, while the theoretical curve of reference 2 gives $T=-5$ in these units. None of these sets of phase shifts extrapolate back to a binding energy near that of the last neutron in the triton (6.266 Mev), although the last three are well fitted

[15] A. H. de Borde and H. S. W. Massey, Proc. Phys. Soc. (London) **A68**, 769 (1955).

by an expansion stopping at this point at energies up to more than 8 Mev in the c.m. system, and attempts to improve the fit by keeping a_0 and r_0 fixed and varying T lead in each case to a large positive value for T. Figure 1 gives the values of the quantity $|2/(a_0\gamma)-1+2\gamma^3T|$, which determines the photo cross sections, for the sets (A) and (B) of scattering lengths and various values of T.

There are no experimental results using neutrons, but Griffiths[14] has measured the total cross section and angular distribution of the $d(p,\gamma)He^3$ reaction from 300-kev to 1-Mev laboratory proton energy. The angular distributions indicate that even at 300 kev the magnetic dipole transitions are only about 12% of the total cross section. Subtracting this contribution, the electric dipole cross sections are given by Griffiths as

$$\sigma(300\text{ kev})=(0.8\pm0.1)\times10^{-30}\text{ cm}^2,\quad \sigma(1\text{ Mev})=(3.1\pm0.3)\times10^{-30}\text{ cm}^2. \tag{15}$$

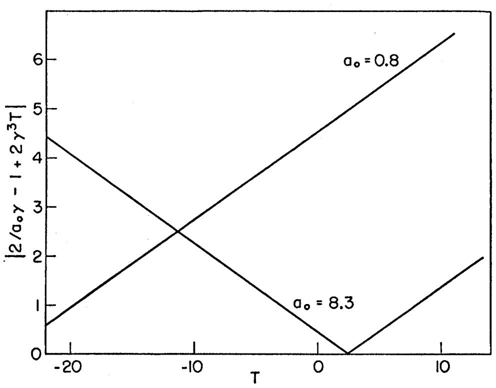

FIG. 1. The variation of $|1-\gamma r_0+4\gamma^3T|$ with T for assumed n-d doublet scattering lengths; the binding energy of H^3 is fitted.

Substitution of the 300-kev cross section into (14b) gives

$$|2/(a_0\gamma)-1+2\gamma^3T|=0.9. \tag{16}$$

First neglecting T, Fig. 2 gives a plot of the regions in the a_0-r_0 plane which are consistent with the observed possible scattering lengths, binding energy of the triton, and this result. There is a region of near overlap for these around $a_0=10$, but not near $a_0=1$; we would thus conclude that only the set (B) of scattering lengths is consistent with these data; and this conclusion is even strengthened by considering T. Equation (16) is satisfied by the following values of a_0 and T, together with the r_0 which they imply. Of these solutions, only $T=-2.4$ is a physically plausible one. The large negative values of T and r_0 associated with a scattering length of 0.8 fermi are physically very implausible, and disagree with the experimental phase-shift analysis of reference 2, which assumed $a_0=0.8$ and gave $r_0=+10$. The solution $a_0=8.3$, $T=7.6$ does not appear to be ruled out entirely; as observed above, accepting theoretical predictions of a_0 and r_0 leads to values of T of this magnitude, but this is not a realistic procedure, and such large values of T (positive or negative) are implausible.

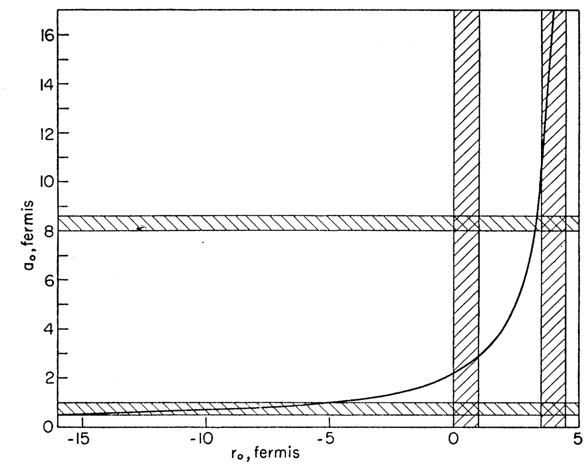

FIG. 2. The a_0-r_0 plane for $T=0$. The shaded regions are allowed by scattering and proton capture experiments; the smooth curve corresponds to the observed H^3 binding energy.

These results would then appear to rule out a scattering length near $a_0=0.8$ f. This disagrees with some recent calculations by Spruch and Rosenberg,[16] who have calculated the doublet and quartet scattering lengths assuming central forces and using a variational principle which gives an upper bound to the scattering length. For both states they find a result close to the set (A).

Gammel and Baker[17] however have pointed out that $k\cot\delta$ may have a pole on the imaginary axis between 0 and $i\gamma$. In this case the expansion (5a) may be replaced by

$$k\cot\delta=\frac{-1/a_0+\frac{1}{2}r_0k^2+Tk^4+\cdots}{1+\Gamma^2k^2}, \tag{17}$$

where the pole is at $k=i/\Gamma$. The analysis then goes through as before, with the replacement of $|2/(a_0\gamma)-1+2\gamma^3T|$ by

$$\left|\frac{2/(a_0\gamma)-1+2\gamma^3T-\Gamma^2\gamma^2}{1-\Gamma^2\gamma^2}\right|.$$

TABLE I. Possible sets of doublet scattering parameters.

a_0, fermis	r_0, fermis	T, (fermis)^3
0.8	−16.2	−20.4
0.8	−20.2	−30.3
8.3	5.9	7.6
8.3	2.3	−2.4

[16] L. Spruch and L. Rosenberg (private communication); Nuclear Phys. (to be published).

[17] J. L. Gammel and G. A. Baker, Jr. (private communication).

The experiments do not then determine Γ well, but are consistent with solutions having $a_0=0.8$, $\Gamma^2\gamma^2\approx 3$. These give an effective range $=r_0+2\Gamma^2/a_0$ of about $+10$ fermis, which would agree with the estimates of reference 2.

The reasons for suspecting the existence of such a pole are the following: There is one in the two-body problem if the zero energy wave function has a node sufficiently close to the origin—for a square well potential, at a radius less than the potential range. A similar criterion for the three-body problem is not known; however, to the extent that the resonating group formalism neglecting deuteron distortion is valid, we might expect a similar result to hold if there is a node in the neutron radial wave function. The calculation of Christian and Gammel,[2] which gave such a node rather close to the origin, would then imply the existence of a pole in $k \cot\delta$ in the relevant region. Since the calculation also gave a small doublet scattering length, this argument is at least consistent, but it cannot be said to be conclusive. In particular, the "effective potential" defined by the resonating group formalism is energy dependent, so that even in this approximation the analogy with the two-body problem is not complete. Nevertheless, it would appear very likely, in view of the present result and the evidence[16] that set (A) is the correct set of scattering lengths, that there is such a pole. It would be of great interest to find conditions on the two-body potentials for the occurrence of such poles in the general case; this is a similar problem to that of finding criteria for the existence of many-body bound states, which is also not solved.

The 1-Mev results give $|1-\gamma r_0+4\gamma^3 T|\approx 1.6$. These are in reasonable agreement with the 300-kev results; the P phase shifts are already large at 1 Mev, so that our approximation is expected to be poor.

V. DISCUSSION

The formulas we have derived are not expected to be very accurate. Consistent neglect of other than S states may lead to errors in the electric dipole cross section of a few percent, as may the neglect of terms higher than k^4 in the expansion (5). Contributions from the internal region of the triton are overestimated by our approximation; these are in any case very small over the energy range we consider. Finally, the treatment of the Coulomb effects is somewhat crude, and may lead to uncertainties of up to 20%, due partly to equating the n-d and p-d effective ranges and partly to the treatment of the Coulomb barrier.[18] Thus the final formulas are accurate to perhaps 10% for neutrons and 30% for protons, over the energy range where the nuclear force may be neglected in the P state. Within this accuracy the experimental data agree well with a scattering length of 8.3 f, but are consistent also with $a_0=0.8$ f, if, as is plausible in this case, $k \cot\delta$ has a pole between zero energy and the triton bound state. Moreover, assuming $a_0=8.3$ leads to an effective range of 2.3 fermis, in excellent agreement with the calculations of reference 8, which gave a doublet effective range$=2.28$ f and the same scattering length; while assuming $a_0=0.8$ f gives $r_0\sim 10$ f, which is in agreement with the estimates of reference 2. It is thus not possible to distinguish between the two sets of scattering lengths in this way.

ACKNOWLEDGMENTS

I would like to thank Dr. Griffiths and Professor Spruch for sending me their results prior to publication, and Dr. Gammel for his interesting comments on the possibility of a singularity in $k \cot\delta$. Thanks are also due to Dr. Blin-Stoyle for his helpful interest during the whole of this work, to the Shell Petroleum Company for the award of a Shell Scholarship, and to Professor Weisskopf for extending the hospitality of the Massachusetts Institute of Technology, where this work was completed.

[18] These estimates of the error involved are borne out by numerical calculations at various energies carried out on the Mercury computer at Oxford. They also check that the contribution from the internal region is only a few percent. I am indebted to G. Thomas for carrying out these checks.