Analysis of the Two-Mode-of-Fission Hypothesis*

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(Received March 16, 1959; revised manuscript received March 3, 1960)

Mass yields for gamma fission of U^{238}, proton fission of Th^{232}, alpha fission of U^{238}, and deuteron fission of normal uranium have been examined in terms of vector spaces for consistency with the hypothesis that for each target and projectile there are two and only two modes of fission. The first three cases are on the whole consistent with the hypothesis. The fourth is not consistent with the hypothesis; the measured yields include some yields from neutron fission. Some consequences of the hypothesis with the added requirement of non-negative yields and non-negative coordinates are derived.

INTRODUCTION

TURKEVICH and Niday[1] have suggested that there are two fundamentally different modes of fission with mass yield characteristics that explain the increase in proportion of symmetric fissions with increasing excitation energy. This idea may be reconciled with the large multiplicity of fission products by assuming that for a given target and projectile there are two and only two configurations for the nucleus that may result in fission. Once one of these configurations is attained, many results are possible but in different frequencies for the two configurations. In this form the hypothesis, which originally was concerned primarily with mass yields, extends readily to other things such as prompt gamma rays, neutrons, and the energies of fission fragments. The hypothesis would apply to spallation products if they result only from the two hypothetical configurations.

Using the methods of linear algebra, we compare this hypothesis with the mass yield data of Schmitt and Sugarman[2] for gamma-induced fission of U^{238}, the mass yield data of Tewes and James[3] for proton fission of Th^{232}, the excitation functions of Ritsema[4] for alpha-particle fission of U^{238}, and the yields of Sugihara, Drevinsky, Troianello, and Alexander[5] for deuteron fission of natural uranium.

In Sec. I we review a few elementary principles of linear algebra and we identify sets of fission yields (fission yield curves) with vectors. In Sec. II the method for a least squares test of the two-mode-of-fission hypothesis is developed and the significance of such tests is discussed. In Sec. III the results of the statistical tests for the cases under consideration are discussed in detail. In Sec. IV some ranges of possible shapes of fission yield curves for the two hypothetical modes of fission are derived from measured fission yields. In the final section the lack of invariance of the treatment to a more or less arbitrary choice of two mass numbers is considered.

I. VECTORS

The set of mass yields corresponding to a given way of inducing fission in a given nuclide may be arranged into an n-tuple or a vector[6] $(y_1,\cdots,y_n)$, where y_i is the fission yield of mass number A_i. Such a vector then represents a mass yield distribution or fission yield curve, that is, a curve where the y_i's are plotted against the A_i's. Other measured fission parameters that are linear in number of fissions could be included with the y's, but we will not do so.

The two hypothetical modes of fission are represented by two such vectors,[7] $\boldsymbol{\beta}_1=(y_{11},\cdots,y_{1n})$ and $\boldsymbol{\beta}_2=(y_{21},\cdots,y_{2n})$. The hypothesis of Turkevich and Niday implies that any fission yield curve is a linear combination of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. That is, if $\boldsymbol{\varepsilon}$ is a vector representing a set of actual fission yields, then

$$\begin{aligned}\boldsymbol{\varepsilon}&=c_1\boldsymbol{\beta}_1+c_2\boldsymbol{\beta}_2\\&=(c_1y_{11}+c_2y_{21},\cdots,c_1y_{1n}+c_2y_{2n}),\end{aligned}$$

for some two numbers c_1 and c_2. This conclusion follows from the hypothesis whether or not the spectrum of the particles including fission is monoenergetic. The two vectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, representing the two hypothetical modes of fission, are characteristic of a given target and projectile, and independent of projectile energy. $\boldsymbol{\beta}_2$ is chosen as the one whose relative amount increases with energy.

For most of our purposes it is desirable not to require that fission yield curves be adjusted so that the sum of yields is 200%. With this understanding, the vectors can represent the mass distribution in a mixture of fission products from any number of fissions. Addition of vectors corresponds to combining two mixtures of fission products, and multiplication by a number corresponds to taking an aliquot or to taking a larger or smaller amount of the same mixture of fission products.

* Work performed under the auspices of the U. S. Atomic Energy Commission.

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2 R. A. Schmitt and N. Sugarman, Phys. Rev. **95**, 1260 (1954).
3 H. A. Tewes and R. A. James, Phys. Rev. **88**, 860 (1952).
4 Susanne Elaine Ritsema, *University of California Radiation Laboratory Report* UCRL-3266 (Office of Technical Services, U. S. Department of Commerce, Washington, D. C., 1956).
5 T. T. Sugihara, P. J. Drevinsky, E. J. Troianello, and J. M. Alexander, Phys. Rev. **108**, 1264 (1957).
6 Robert R. Stoll, *Linear Algerbra and Matrix Theory* (McGraw-Hill Book Company, Inc., New York, 1952).
7 Greek letters are used for vectors and Roman letters are used for numbers.

The two vectors, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, representing the two hypothetical modes of fission, span a subspace included in the space of n-tuples. These two vectors must be linearly independent, otherwise there would be only one shape of fission yield curve, contrary to experience. Thus, the two-mode-of-fission hypothesis implies that the observed fission yield curves for a particular way of inducing fission in a particular nuclide will lie in a two-dimensional subspace. A two-dimensional subspace consists of all linear combinations of two linearly independent vectors.

For use in deriving the equations of condition in a least squares calculation to be described later, we now introduce two vectors, $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, and develop some of their properties. If all of a set, $\boldsymbol{\varepsilon}_1,\cdots,\boldsymbol{\varepsilon}_r$ of fission yield curves are of the form $\boldsymbol{\varepsilon}_i=c_{1i}\boldsymbol{\beta}_1+c_{2i}\boldsymbol{\beta}_2$, then any three or more of the $\boldsymbol{\varepsilon}$'s are linearly dependent. This means that each of them may be expressed as a linear combination of any two linearly independent linear combinations of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. The two vectors $\boldsymbol{\eta}_1=(1,0,a_3,\cdots,a_n)$ and $\boldsymbol{\eta}_2=(0,1,b_3,\cdots,b_n)$ are obviously linearly independent. The a's and b's may be chosen so that $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ lie in the two-dimensional subspace of fission yield curves. To make this choice we let $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ be two linearly independent fission yield curves, or any other pair of linearly independent linear combinations of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. Since $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are linearly independent, the two-by-n matrix with $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ as rows contains a nonzero second order determinant. We choose A_1 and A_2 (called base nuclides) to be mass numbers whose respective yields u, u' and w, w' form a nonzero determinant $uw'-u'w$. Then the two equations

$$\boldsymbol{\varepsilon}=u\boldsymbol{\eta}_1+w\boldsymbol{\eta}_2,$$

and

$$\boldsymbol{\varepsilon}'=u'\boldsymbol{\eta}_1+w'\boldsymbol{\eta}_2,$$

may be solved for $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. This procedure gives $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ as linear combinations of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$, which in turn are linear combinations of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. Therefore $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are in the subspace of fission yield curves. For any fission yield curve $\boldsymbol{\varepsilon}''$ there is a linear relation $c''\boldsymbol{\varepsilon}''+c_1\boldsymbol{\eta}_1+c_2\boldsymbol{\eta}_2=0$ since all three vectors are in the same two-dimensional subspace. c'' is not zero; if it were zero, then $c_1\boldsymbol{\eta}_1+c_2\boldsymbol{\eta}_2$ would be zero and $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ would not be linearly independent. Thus,

$$\boldsymbol{\varepsilon}''=(-c_1/c'')\boldsymbol{\eta}_1-(c_2/c'')\boldsymbol{\eta}_2,$$

and any fission yield curve is a linear combination of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$.

The two vectors, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, may be thought of as determining a plane through the origin in the n-dimensional space. If the two-mode-of-fission hypothesis is true an actual set of fission yields is represented by a point in the plane of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$; a measured set of fission yields is represented by a point near the plane of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. If only three yields were measured, the n-dimensional space would be three-dimensional and easily visualized.

We define an inner product, $(\boldsymbol{\zeta},\boldsymbol{\psi})$, of any two vectors $\boldsymbol{\zeta}=(p_1,\cdots,p_n)$ and $\boldsymbol{\psi}=(q_1,\cdots,q_n)$ as $\sum_{i=1}^n p_iq_i$. The definition of an inner product permits the definition of the length of $\boldsymbol{\zeta}$ as $(\boldsymbol{\zeta},\boldsymbol{\zeta})^{\frac{1}{2}}$ and the cosine of the angle between $\boldsymbol{\zeta}$ and $\boldsymbol{\psi}$ as $(\boldsymbol{\zeta},\boldsymbol{\psi})/(\boldsymbol{\zeta},\boldsymbol{\zeta})^{\frac{1}{2}}(\boldsymbol{\psi},\boldsymbol{\psi})^{\frac{1}{2}}$. These definitions change the vector space into a Euclidian space and make it possible to plot graphs representing two-dimensional spaces or subspaces. In such a plot a fission yield curve is represented by a point.

Other inner products could be defined, but to change the vector space into a Euclidian space it is sufficient that the inner product be real, symmetric, bilinear, and positive definite (e.g., see reference 6, pp. 213–214). Symmetric means $(\boldsymbol{\zeta},\boldsymbol{\psi})=(\boldsymbol{\psi},\boldsymbol{\zeta})$. Bilinear means that the inner product is linear in each argument, i.e., for any vectors $\boldsymbol{\zeta}_i$ and any numbers r_j, $(r_1\boldsymbol{\zeta}_1+r_2\boldsymbol{\zeta}_2,\boldsymbol{\zeta}_3)=r_1(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_3)+r_2(\boldsymbol{\zeta}_2,\boldsymbol{\zeta}_3)$ and $(\boldsymbol{\zeta}_4,r_5\boldsymbol{\zeta}_5+r_6\boldsymbol{\zeta}_6)=r_5(\boldsymbol{\zeta}_4,\boldsymbol{\zeta}_5)+r_6\times(\boldsymbol{\zeta}_4,\boldsymbol{\zeta}_6)$. Positive definite means $(\boldsymbol{\zeta},\boldsymbol{\zeta})\geqslant 0$ and $(\boldsymbol{\zeta},\boldsymbol{\zeta})=0$ only if $\boldsymbol{\zeta}=(0,\cdots,0)$. The inner product as defined in the previous paragraph obviously satisfies these three requirements. The inner product will not be used in the test of the hypothesis, but is useful in a discussion of the possible shapes of fission yield curves for the two hypothetical modes of fission.

II. METHOD FOR THE TEST OF THE HYPOTHESIS

It is the two-dimensional subspace implication of the hypothesis of Turkevich and Niday that can be compared with experimental mass yields. If all fission yield curves were known with zero error the hypothesis could be tested by solving for $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in terms of two fission yield curves and then testing the other fission yield curves for linear dependence on $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. Or more simply, any two linearly independent fission yield curves could be used instead of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. However, because of errors in fission yield measurements three or more observed fission yield curves do not lie exactly in a two-dimensional subspace even if the hypothesis is true. For this reason, instead of the above scheme we use a least squares calculation.

It was shown above that if a set of fission yield curves are in a two-dimensional subspace, then each fission yield curve of the set is a linear combination of two vectors $\boldsymbol{\eta}_1=(1,0,a_3,\cdots,a_n)$ and $\boldsymbol{\eta}_2=(0,1,b_3,\cdots,b_n)$. Such linear combinations may be written as

$$u_j\boldsymbol{\eta}_1+w_j\boldsymbol{\eta}_2=(u_j,\ w_j,\ a_3u_j+b_3w_j,\ \cdots,\ a_nu_j+b_nw_j)=(x_{j1},\cdots,x_{jn}),$$

where x_{ji} is the yield of mass number A_i in fission yield curve j. These equations mean that

$$x_{ji}=u_ja_i+w_jb_i,$$

so that the hypothesis may be tested by determining whether the data may be represented by such equations. This was done by the least squares method of Deming.[8]

[8] W. Edwards Deming, *Statistical Adjustment of Data* (John Wiley & Sons, Inc., New York, 1938), 1st ed.

TABLE I. Summary of the cases tested for consistency with the two-mode-of-fission hypothesis. $q(S,f)$ is the probability of obtaining a minimized sum of squares larger than S if the hypothesis is true.

Reaction	Number of yield curves	Energy range	Total No. of fission yields	Base nuclides	Number of degrees of freedom, f	S, minimized sum of squares	$q(S,f)$
Th^{232} p-f	7	6.7 Mev–21.1 Mev	82	115 140	42	46.64	0.29
U^{238} γ-f	7	7 Mev–300 Mev	87	111 140	32	74.43	0.00001
U^{238} γ-f	6	7 Mev–100 Mev	82	111 140	27	25.93	0.53
U^{238} α-f	6	22.6 Mev–45.4 Mev	53	97 115	20	3.26	>0.9995
Normal U d-f	3	5 Mev–13.6 Mev	45	97 115	13	25.5	0.021

There is a least squares calculation for each mass number except A_1 and A_2. In each least squares calculation there is an equation of condition, $x_{ji}=u_ja_i+w_jb_i$, for each energy j for which the yield, x_{ji}, of mass number A_i was measured. a_i and b_i are the parameters that are evaluated by the least squares calculation. Fission yields must have been measured for the base nuclides, A_1 and A_2, for each energy. For example, for gamma-induced fission of U^{238} A_1 and A_2 were chosen to be 111 and 140. The errors listed in the fission yield tables in references 2, 3, and 5 were interpreted as standard deviations for the least squares calculation. The probable errors given in reference 4 for formation cross sections were multiplied by 1.4826 to give standard deviations.

The numbers $a_1,\cdots,a_n$ ($a_1=1$, $a_2=0$) arranged into an n-tuple form $\boldsymbol{\eta}_1$ and the numbers $b_1,\cdots,b_n$ ($b_1=0$, $b_2=1$) form $\boldsymbol{\eta}_2$. A knowledge of $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ within experimental error results from the test of the hypothesis; it is not a prerequisite of the test.

The minimized sum of the squares of weighted residuals, S, is a measure of the degree of fit obtained from the least squares calculation; the larger S the poorer the fit. It is well known that if only one of u_j, w_j, and x_{ji} is subject to error, the sum of squares, S, is distributed like χ^2 with $f=m-2$ degrees of freedom, where m is the number of measured yields. Deming[8] states that this is still true if two of them are subject to error, and we assume that it is true if all three are subject to error. It is a property of the χ^2 distribution that if S_1 and S_2 are distributed like χ^2 with f_1 and f_2 degrees of freedom, then S_1+S_2 is distributed like χ^2 with f_1+f_2 degrees of freedom. Thus, the S for each mass number for a particular case of fission as well as the sum of the S's for all the mass numbers constitute χ^2 tests for the hypothesis.

A function, q, of S and f may be defined as the integral from S to infinity of the χ^2 probability density function with f degrees of freedom. When testing a true hypothesis, $q(S,f)$ is the probability of obtaining a minimized sum of squares larger than S. If it is decided before a test to reject the hypothesis when q turns out less than some fixed value p, then p is the probability of rejecting a true hypothesis, i.e., of making an error of the first kind. The term p is called the level of significance of the test. The probability of accepting a false hypothesis (making an error of the second kind) is only rarely known. It is not known for the cases under consideration.

The definition of the random variable $q(S,f)$ is not common in chi-square testing but appears to be valid, and it simplifies the discussion.

III. RESULTS OF THE TEST

Table I gives a tabular summary of the cases tested. The sums of S's for individual mass numbers are given with the corresponding $q(S,f)$. For proton fission of Th^{232}, q is 0.29 and the hypothesis must be accepted unless one is demanding a better fit than may be expected in 29% of the cases when a true hypothesis is being tested. For gamma fission of U^{238} there are two entries. For the first, with yields up to 300 Mev, q is 0.00001. If the hypothesis is true for this case a very improbable thing has happened; the hypothesis must be rejected at any reasonable level of significance. The second entry for gamma fission of U^{238} is the result of omitting some 300-Mev yields, as explained below. The result is $q=0.53$ and the modified hypothesis must be accepted at any reasonable level of significance.

For deuteron fission of normal uranium the hypothesis is known to be false; Sugihara *et al.*[5] attribute most of the 5-Mev fissions to a neutron background. For this case q turned out to be 0.021 and the hypothesis may reasonably be rejected. Although it is not sufficient to make any probability statements, this case provides experience about errors of the second kind and offers a reason (how good a reason is not known) for accepting the hypothesis when q is reasonably large.

For alpha fission of U^{238}, q is beyond the range of readily available χ^2 tables but it is larger than 0.9995. That is, the fit is better than may be expected in almost all cases. Such a good fit tends to confirm the hypothesis but is very unlikely. It is attributed to the use of errors quoted with the data and intended to be errors of absolute cross sections. For the least squares calculation the unrecorded, but presumably smaller, errors of relative cross sections are the appropriate ones. For this reason the results cannot be used as a χ^2 test of the hypothesis.

The results of the least squares calculation for individual mass numbers for proton fission of thorium are given in Table II. Mass numbers A_1 and A_2 were

TABLE II. Proton fission of thorium. The over-all sum of squares of weighted residuals, omitting Pa^{232}, is 46.64. There are 42 degrees of freedom. Base nuclides are: $A_1=115$, $A_2=140$.

Mass	η_1	η_2	6.7-Mev fission yield Obs	6.7-Mev Calc	8.0-Mev fission yield Obs	8.0-Mev Calc	9.3-Mev fission yield Obs	9.3-Mev Calc	13.3-Mev fission yield Obs	13.3-Mev Calc	17.8-Mev fission yield Obs	17.8-Mev Calc	19.5-Mev fission yield Obs	19.5-Mev Calc	21.1-Mev fission yield Obs	21.1-Mev Calc	S	f	$q(S,f)$
77	0.0163	0.0031			0.026±0.011	0.0260	0.034±0.018	0.0272	0.032±0.011	0.0343	0.030±0.007	0.0411			0.052±0.006	0.0425	5.99	3	0.12
78	−0.0019	0.0110			0.061±0.039	0.0507	0.060±0.012	0.0524	0.047±0.005	0.0483	0.036±0.004	0.0479			0.064±0.006	0.0462	14.46	3	0.0031
82	−0.0058	0.0035							0.010		0.0062±0.0025				0.0057±0.0014				
83	−0.0003	0.3514			1.67±0.07	1.655	1.51±0.32	1.722	1.65±0.08	1.619	1.51±0.07	1.640			1.64±0.07	1.584	2.63	3	0.46
84	−0.0600	0.5748			2.90±0.50	2.683	2.53±0.69	2.773	2.44±0.53	2.577	2.66±0.56	2.579			2.49±0.50	2.488	0.32	3	0.96
89	−0.4093	1.2679	6.55±0.96	2.948	6.00±0.37	5.728	6.24±0.49	5.914	5.25±0.18	5.346	5.31±0.12	5.205	5.01±0.14	5.293	5.07±0.14	5.006	4.21	5	0.52
91	−0.0839	1.0424			5.48±0.53	4.883	5.15±0.52	5.047	4.58±0.21	4.703	4.57±0.28	4.721			4.61±0.14	4.555	1.07	3	0.78
95	−0.8599	1.3110	5.93±1.14	2.826	6.05±1.05	5.621	5.29±1.10	5.797	4.93±0.18	4.995	4.57±0.22	4.618	5.14±0.32	4.747	4.27±0.38	4.416	3.49	5	0.53
97	−0.7625	1.1224	3.97±0.84	2.406	4.08±0.61	4.794	4.07±0.24	4.943	4.36±0.11	4.244	4.08±0.15	3.907	4.32±0.20	4.021	3.63±0.09	3.735	7.00	5	0.22
115	1	0	0.51±0.09		0.69±0.03		0.73±0.04		1.22±0.04		1.75±0.11		1.66±0.10		1.74±0.07				
131	0.2501	0.4302			2.40±0.24	2.212	2.24±0.22	2.291	2.27±0.07	2.289	2.45±0.18	2.447			2.38±0.05	2.376	0.38	3	0.94
132	−0.5498	0.8423			3.03±0.90	3.613	3.56±0.72	3.726	3.48±0.35	3.212	2.38±0.34	2.972			3.02±0.24	2.842	3.82	3	0.28
139	0.0461	1.0264			5.95±0.68	4.897	4.75±0.41	5.063	4.72±0.18	4.788	4.63±0.22	4.874			4.96±0.29	4.709	2.78	3	0.43
140	0	1	2.49±1.16		4.74±0.53		4.90±0.61		4.61±0.11		4.67±0.23		4.71±0.22		4.51±0.07				
156	−0.0115	0.0095			0.055±0.026	0.0371	0.037±0.018	0.0382	0.029±0.004	0.0298	0.024±0.002	0.0243			0.023±0.001	0.0228	0.49	3	0.92
232	−33.0683	11.7725	147±4	12.5	116±17	33.0	34.3±5.3	33.6	5.08±0.75	13.9	2.64±0.40	2.89	0.88±0.14	0.555	0.81±0.12	4.44	30.98	5	<0.0005

TABLE III. Gamma fission of U^{238}. S's in the last column are sums of squares of weighted residuals. The over-all sum of squares of weighted residuals with 300-Mev yields for the 43-day isomer of Cd^{115} and for masses 112, 117, 132, and 143 omitted is 25.97. There are 27 degrees of freedom. $q(25.93, 27)=0.53$. Base nuclides are: $A_1=111$, $A_2=140$.

Mass number	η_1	η_2	7-Mev fission yields Obs	7-Mev Calc	10-Mev fission yields Obs	10-Mev Calc	16-Mev fission yields Obs	16-Mev Calc	21-Mev fission yields Obs	21-Mev Calc	48-Mev fission yields Obs	48-Mev Calc	100-Mev fission yields Obs	100-Mev Calc	300-Mev fission yields Obs	300-Mev Calc	S	f	$q(S,f)$
83	0 1504	0.0924									0.59±0.06	0.578	0.62±0.07	0.643	0.73±0.08	0.726	0.14	1	0.71
84	0.1152	0.1802									1.03±0.17	0.990	1.04±0.15	1.073	1.09±0.18	1.081	0.10	1	0.76
89	0.3067	0.4924							2.6±0.1	2.55	2.8±0.1	2.70	2.8±0.1	2.92	3.0±0.1	2.94	1.95	2	0.38
97	−0.6398	1.2236							5.7±0.2	5.72	5.8±0.2	5.63	5.8±0.2	5.83				1	0.56
99	0.2621	1.2511	6.6	7.27	6.6	7.15	6.6	6.33	6.6	6.24	6.6	6.46	6.6	6.90	6.6	6.50			
103	0.3869	0.5396							3.0±0.3	2.81	2.9±0.2	3.00	3.2±0.2	3.25	3.4±0.3	3.32	0.66	2	0.72
106	0.7569	0.3273							2.1±0.3	1.93	2.0±0.2	2.22	2.6±0.2	2.51	3.0±0.2	2.99	1.49	2	0.48
111	1	0	0.046±0.004		0.065±0.007		0.30±0.01		0.43±0.01		0.77±0.02		1.02±0.02		1.88±0.06				
112	0.6233	0.0003	0.031±0.005	0.0303	0.047±0.005	0.0421	0.160±0.01	0.1884	0.26±0.02	0.269	0.52±0.03	0.481	0.71±0.04	0.637	1.14±0.08	1.173	11.15	5	
	(0.6279)[a]	(0.0002)[a]		(0.0300)[a]		(0.0420)[a]		(0.1894)[a]		(0.271)[a]		(0.485)[a]		(0.641)[a]			(10.97)[a]	(4)[a]	(0.027)[a]
113	0.5549	0.0361									0.60±0.3	0.608	0.77±0.04	0.757	1.21±0.06	1.216	0.31	1	0.58
115/54	0.6226	−0.0024			0.030±0.004	0.0269	0.16±0.01	0.175	0.25±0.01	0.256	0.47±0.02	0.468	0.67±0.03	0.622	1.15±0.05	1.159	4.30	4	0.37
115/43	0.0587	−0.0013					0.013±0.001	0.0113	0.018±0.001	0.0191	0.041±0.002	0.0389	0.048±0.002	0.0532	0.20±0.02	0.104	24.34	3	
	(0.0514)[a]	(−0.0006)[a]						(0.0124)[a]		(0.0191)[a]		(0.0366)[a]		(0.0492)[a]			(1.85)[a]	(2)	(0.40)[a]
117	0.6414	−0.0019			0.027±0.007	0.0309					0.50±0.02	0.484	0.69±0.02	0.644	1.04±0.04	1.197	13.65	2	
	(0.6844)	(−0.0031)[a]				(0.0267)[a]						(0.511)[a]		(0.715)[a]			(1.40)[a]	(1)[a]	(0.24)[a]
127	0.5621	0.0947							0.70±0.02	0.706	0.93±0.02	0.906	1.07±0.03	1.075	1.49±0.09	1.511	0.54	2	0.47
131	0.4034	0.7845							4.1±0.2	4.02	4.3±0.1	4.23	4.4±0.2	4.57	4.6±0.2	4.52	0.92	2	0.63
132	−0.4761	1.0207							5.0±0.1	4.80	4.9±0.1	4.74	4.6±0.1	4.92	4.3±0.1	4.00	9.10	2	
	(−1.4344)[a]	(1.1510)[a]								(5.02)[a]		(4.65)[a]		(4.63)[a]			(1.09)[a]	(1)[a]	(0.30)[a]
139	1.0991	0.7507									4.6±0.1	4.60	5.1±0.1	5.10				0	
140	0	1	5.8±0.3		5.7±0.2		5.0±0.1		4.9±0.1		5.0±0.3		5.3±0.1		4.8±0.2				
143	−0.2367	0.8044							4.0±0.1	3.84	3.8±0.3	3.84	3.8±0.1	4.02	3.6±0.1	3.42	5.58	2	
	(−0.9478)[a]	(0.8998)[a]								(4.00)[a]		(3.77)[a]		(3.80)[a]			(0.007)[a]	(1)[a]	(0.94)[a]
144	−1.4418	0.9020							3.8±0.2	3.80	3.4±0.2	3.40						0	

[a] 300-Mev yields omitted.

TABLE IV. Alpha fission of U^{238}. The over-all sum of squares is 3.26. There are 20 degrees of freedom. $q(3.26, 20)>0.9995$. Base nuclides are: $A_1=97$, $A_2=115$.

			22.6-Mev fission yield		27.1-Mev fission yield		33.8-Mev fission yield		38.6-Mev fission yield		40.1-Mev fission yield		43.9-Mev fission yield		45.4-Mev fission yield				
Mass	η_1	η_2	Obs	Calc	Obs	Calc	Obs	Calc	Obs	Calc	Obs	Calc	Obs	Calc	Obs	Calc	S	f	$q(S,f)$
95	0.6521	0.0523	4.7	5.35	29	24.3	28		38	37.7	35	38.3	41	37.6	36	36.5	0.386	4	0.98
97	1	0	8.0		36		41		54		54		53		52				
103	0.7885	0.0932	6.5	6.55					47	47.1	44	48.2	51	47.2	47	45.6	0.095	3	0.99
105	0.7658	0.1267	7.0	6.46					36	47.4	53	49.0	55	47.9	48	46.0	0.703	3	0.87
115	0	1	2.6		15.4				48		60		58		49				
139	0.8509	−0.1327	6.5	6.46					36	39.6			37	37.4	42	37.7	0.097	2	0.95
140	0.7463	−0.0781	5.8	5.77			29		35	36.5	35	35.6	36	35.0	36	35.0	0.019	3	0.999
143	0.6200	0.0961			23	23.8			44	38.1	49	39.2	30	38.4			0.835	2	0.66
147	0.4685	−0.0649			15	15.9			27	22.2	19	21.4					0.268	1	0.61
156	0.0344	0.0351			1.8	1.78			3.4	3.54	4.1	3.97					0.022	1	0.89
157	0.0427	0.0002			1.5	1.54			2.5	2.32	2.2	2.32					0.053	1	0.82
237	−1.4728	3.5739			3	2.0			100	92.0			110	129.2	150	98.5	0.580	2	0.75
239	−0.2135	0.9368	0.29	0.728	12	6.7			25	33.4					44	34.8	0.092	2	0.96
240	−0.0549	0.1935	0.027	0.257	1.3	1.00			6.8	6.32					5.9	6.63	0.113	2	0.95

chosen to be 115 and 140. Entries are arranged in order of increasing mass number; entries for A_1 and A_2 are no longer first and second. S and $q(S,f)$ are given in the last two columns. The results indicate acceptance of the hypothesis except for mass number 78 and possibly for mass number 77. For mass number 77, $q(S,f)$ is 0.12, so that a level of significance that would result in rejection can be expected to result in errors of the first kind (wrong rejections) in about twelve percent of the cases tested. For mass 78, $q(S,f)$ is 0.0031 and it may be rejected with very little risk of an error of the first kind. However, we are reluctant to reject the hypothesis on the basis of one very poor fit and one mediocre fit when everything else fits so well. In this connection it should be pointed out that a necessary part of the hypothesis being tested is that no mistakes have been made in the measurements or calculations of the yields. The results for the p-n product Pa^{232} do not agree with the hypothesis. If Pa^{232} results only from the two hypothetical configurations represented by β_1 and β_2 the probability of obtaining a fit this bad or worse is less than 0.0005. Since it may be reasonably supposed that not all of the Pa^{232} results from the two hypothetical modes of fission, this case may be regarded as further experience about the probability of errors of the second kind. For proton fission of thorium we regard the data as being in agreement with the hypothesis.

The detailed results for gamma fission of U^{238} are given in Table III. No S is given for mass 99 since no errors were quoted with the data. With all 300-Mev yields included, the hypothesis is to be rejected as indicated earlier. It is reasonable to examine the possibility that the large deviations are caused by a third type of fission at 300 Mev. This could be done by repeating all of the calculations with the 300-Mev yields omitted. However, it is sufficient to repeat the calculations for those mass numbers with unusually large S's since if the others agree with the hypothesis up to 300 Mev, then they must agree up to 100 Mev. Mass numbers 112, 115 (43-day isomer of Cd^{115}), 117, 132, and 143 have unusually large S's. The calculations were repeated for them omitting the 300-Mev yields. These results are given in parenthesis in Table III. The agreement with the hypothesis is now satisfactory except for mass number 112. The hypothesis could be rejected on the basis of the result for this mass number at a level of significance of 0.027. However, we are again reluctant to reject the hypothesis on the basis of the results of one mass number.

The results of the least squares calculation for individual mass numbers are given in Table IV for alpha fission of U^{238}. A_1 and A_2 were chosen to be mass numbers 97 and 115 (both isomers of Cd^{115}). Yields for 33.8 Mev were omitted in the least squares calculations because no yields were given for mass numbers 97 and 115. As mentioned earlier, the results do not constitute a χ^2 test because of the errors used. It should be noted that the results for mass numbers 237, 239, and 240 agree about as well as the fission products. This is experience about errors of the second kind, but in the wrong direction.

The detailed results for deuteron fission of normal uranium are given in Table V. Mass numbers 97 and 115 (54-hour isomer of Cd^{115}) were chosen for A_1 and A_2. As already mentioned, the hypothesis is to be rejected for this case.

IV. RESULTS OF THE HYPOTHESIS: SHAPES OF FISSION YIELD CURVES FOR THE TWO HYPOTHETICAL MODES OF FISSION

For this section the hypothesis is regarded as being true, and some consequences are examined. The test of the hypothesis is independent of a knowledge of β_1 and β_2, the vectors associated with the two modes of fission. The least squares procedure does not determine β_1 and β_2 but it does set limits on them by determining, within experimental error, a two-dimensional subspace containing them. Further limits may be set on them

TABLE V. Deuteron fission of normal uranium. The over-all sum of squares of weighted residuals is 25.5. There are 13 degrees of freedom. $q(25.5, 13)$ is 0.021. Base nuclides are: $A_1=97$ and $A_2=115$ (54-hour isomer of Cd^{115}).

Mass	η_1	η_2	5-Mev fission yield Obs	Calc	10-Mev fission yield Obs	Calc	13.6-Mev fission yield Obs	Calc	S	$q(S,f)$
91	0.6017	−0.8050	143±7	136.3	2.3±0.1	2.54	12.2±0.8	11.12	4.17	0.043
95	0.8717	−1.0803	205±15	197.9	3.5±0.2	3.74	16.8±0.7	16.47	1.14	0.29
97	1	0	233±16		5.2±0.2		24.1±0.2			
115/54	0	1	4.8±0.5		0.73±0.03		4.2±0.2			
115/43	0.0002	0.0663	0.4±0.2	0.37	0.046±0.004	0.0495	0.30±0.02	0.283	1.12	0.29
131	0.5113	2.0252	128±1	128.9	4.2±0.1	4.14	19±2	20.8	0.97	0.33
132	0.7225	0.7580	172±3	172.0	4.2±0.2	4.31	20.6±1.1	20.60	0.00	
140	0.7612	−0.3578	181±9	175.6	3.4±0.2	3.70	17.5±0.9	16.84	2.01	0.17
141	0.6622	−0.5648	160±16	151.6	2.9±0.1	3.03	15.6±1.6	13.59	2.35	0.13
143	0.5141	−0.0730	120±7	119.4	2.6±0.1	2.62	12.1±0.3	12.08	0.03	0.87
144	0.8122	−1.5025	183±4	182.0	3.0±0.3	3.13	14±2	13.3	0.27	0.60
147	0.3670	−0.0937	89±4	85.1	1.6±0.1	1.84	8.7±0.3	8.45	4.59	0.03
153	0.0471	0.1524	12.1±0.1	11.7	0.33±0.01	0.356	2.0±0.1	1.78	8.21	0.0046
156	0.0109	0.1022	3.0±0.5	3.04	0.132±0.002	0.1315	0.66±0.08	0.693	0.18	0.67
157	0.0055	0.0659	1.6±0.1	1.6	0.076±0.002	0.0766	0.45±0.06	0.409	0.49	0.49

by the two requirements: (1) coordinates with respect to β_1 and β_2 of observed fission yield curves cannot be negative; (2) β_1 and β_2 may have no negative yields.

To apply condition 1 a graph may be plotted of a two-dimensional subspace of fission yield curves. In such a graph the inner product defined earlier determines angles and distances. In these graphs a point represents a fission yield curve. All of the points on a line through the origin represent the same shape of fission yield curve, and any shape that may occur is represented by a point on the unit circle. In these graphs the yields u and w of mass numbers A_1 and A_2 could be used as coordinates with respect to η_1 and η_2. However, in order to avoid excessive dependence on two yields, the data for each fission yield curve were

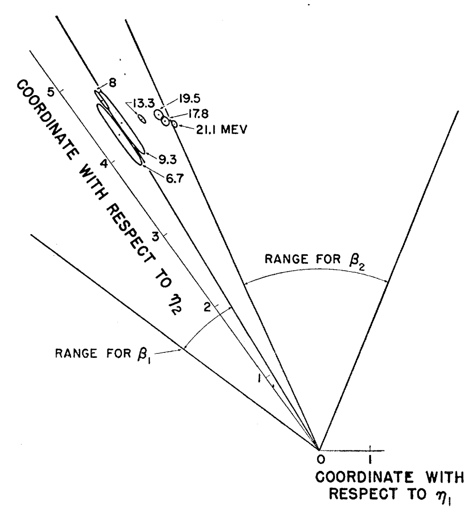

FIG. 1. Points representing fission yield curves for proton fission of Th^{232}. Ellipses indicate uncertainty of position of the points. Regions that may contain points representing the two hypothetical fission modes are determined by the requirements of no negative yields and no negative coordinates. The angle between coordinate axes is not 90° because $(\eta_1,\eta_2)\neq 0$. The scale is not the same on the two axes since $(\eta_1,\eta_1)\neq(\eta_2,\eta_2)$. Points are all about the same distance from the origin because yields are adjusted to make their sum 200%.

fitted to η_1 and η_2 by least squares. It was assumed for this purpose that η_1 and η_2 are accurately known. Errors of coordinates with respect to η_1 and η_2 are not independent but are related by a moment matrix occurring in the least squares calculation. Errors are illustrated on the graphs by ellipses obtained by equating to unity the quadratic form of the matrix of the least squares normal equations. This results in an ellipse homothetic with an ellipse of concentration[9] and half its size. The size was chosen to make the length of a semiaxis correspond to one standard deviation.

Condition 1, that observed fission yield curves cannot have negative coordinates with respect to β_1 and β_2, implies that β_1 and β_2 may not be in the region of observed fission yield curves. Since the location of fission yield curves is uncertain, it seems reasonable to use limits of one standard deviation. For this reason the limits imposed on β_1 and β_2 by condition 1 were set by drawing tangents to the ellipses at each edge of the region of observed fission yield curves. The resulting angles were measured from η_1 with a protractor. These limits, being one-standard-deviation limits, have about the same significance as saying that a measured quantity is less than the measured value plus one standard deviation.

In applying condition 2 that β_1 and β_2 may have no negative yields it seems reasonable to call a yield negative when the calculated yield plus one standard deviation is zero or negative. For this purpose curves of calculated yield as a function of angle from η_1, with error bands one standard deviation above and below (reference 8, p. 168), were prepared. These curves were used to set lower limits of shapes of β_1 and upper limits of shapes of β_2. Limits set in this way have about the same significance as the other limits set on β_1 and β_2 since these too are one-standard-deviation limits.

The graph of the two-dimensional subspace for

[9] Harald Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, New Jersey, 1946).

proton fission of Th^{232} is given in Fig. 1. Regions that may contain β_1 and β_2 are indicated. The range of angles for β_1 is from 142°30′ (condition 2 and mass 77) to 120°48′ (condition 1 with the 6.7-Mev fission yield curve). These angles, as well as those that follow, are measured from η_1. The range of angles for β_2, the high-energy fission mode, is from 67°53′ (condition 2, mass 156) to 113°28′ (condition 1, 21.1-Mev fission yield curve). Calculated yields for the limits of β_2 are shown in Figs. 2 and 3. Uncertainties in η_1 and η_2 are illustrated by error bars one standard deviation above and below. These were determined from the error bands used in connection with condition 2.

The graph of the two-dimensional subspace for gamma fission of U^{238} is shown in Fig. 4. For this case the

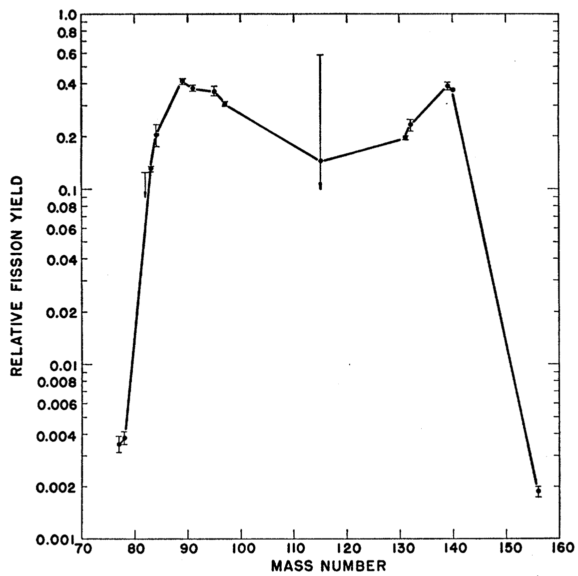

FIG. 2. Proton fission of Th^{232}. These are calculated yields for the low-energy limit of β_2, the high-energy fission mode. These yields were determined by the condition that observed fission yield curves may not have negative coordinates with respect to β_1 and β_2.

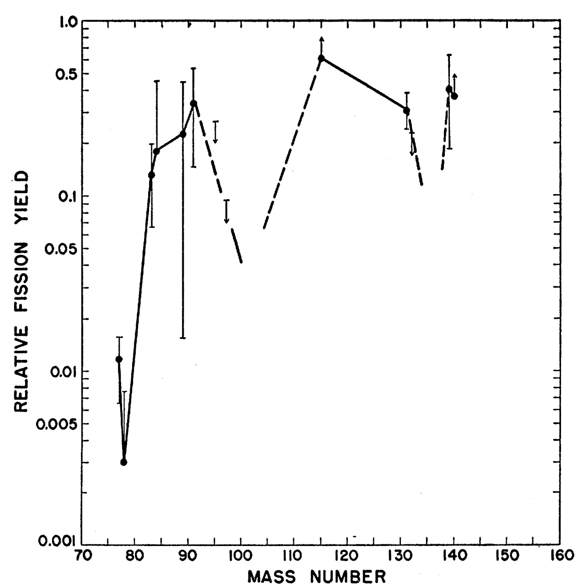

FIG. 3. Proton fission of Th^{232}. These are calculated yields for the high-energy limit of β_2, the high-energy fission mode. This limit for β_2 is determined by the condition that β_1 may not have negative yields. This condition was applied to mass 156.

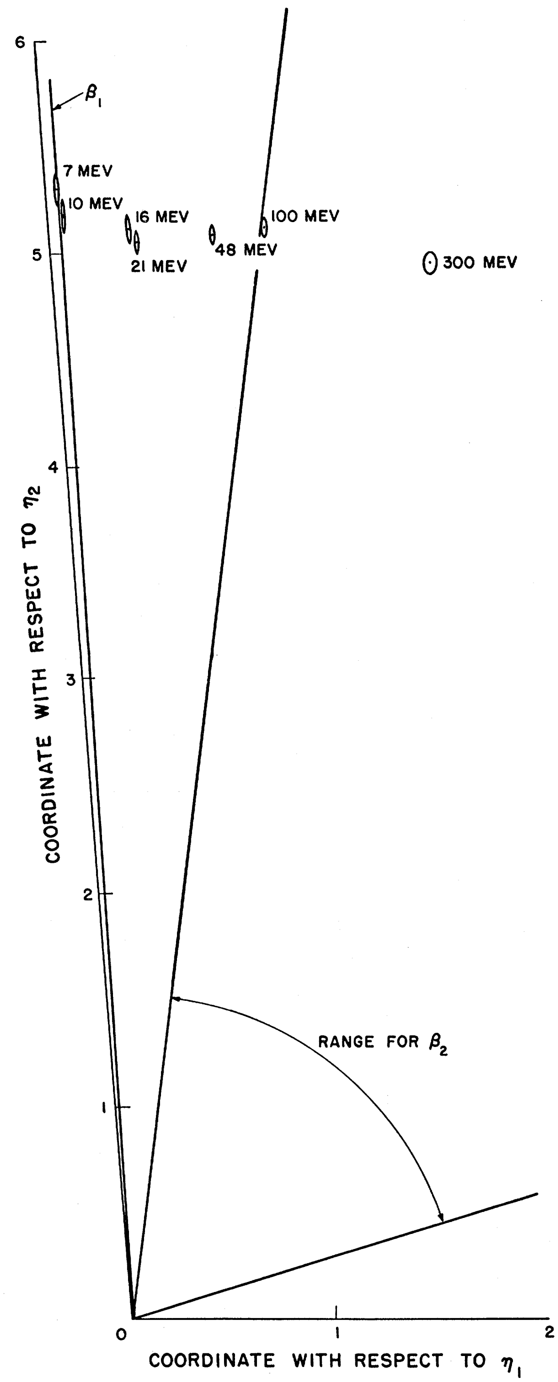

FIG. 4. Points representing fission yield curves for gamma fission of U^{238}. All of the comments for Fig. 1 are applicable.

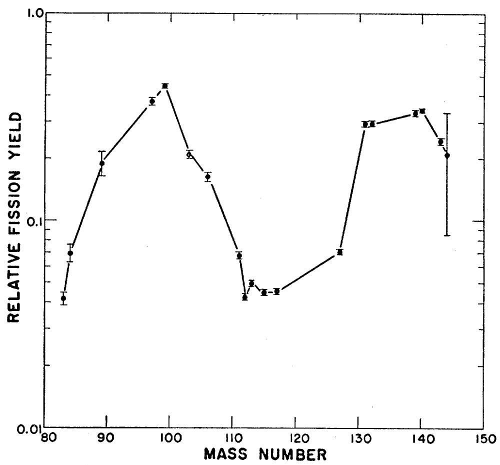

FIG. 5. Gamma fission of U^{238}. These are calculated yields for the low-energy limit of the high-energy fission mode, β_2. These yields were determined by the requirement that observed fission yield curves may not have negative coordinates with respect to β_1 and β_2.

hypothesis is consistent with the data only up to 100 Mev. However, least squares calculations omitting the 300-Mev points are available for only a few mass numbers, as explained earlier. For this reason the graph is drawn from the η's resulting from the inclusion of the 300-Mev yields. The 100-Mev point rather than the 300-Mev point is used for condition 1 that β_1 and β_2 cannot be in the region of observed fission yield curves. This puts a limit on β_2 at 83°03′. The other limit on β_2 is at 47°58′ where the yield plus one standard deviation of mass 132 is zero. For this mass number, calculations without the 300-Mev yield are available and are used. Calculated fission yields for the limits of β_2 are given in Figs. 5 and 6. Uncertainties in η_1 and η_2 are again indicated by error bars. The one-standard-

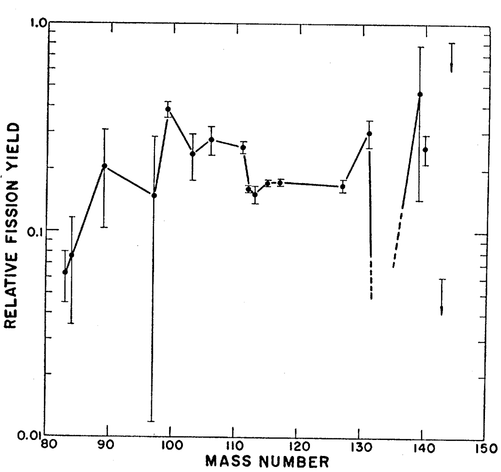

FIG. 6. Gamma fission of U^{238}. These are calculated yields for the high-energy limit of β_2, the high-energy fission mode. These yields were determined by the condition that β_2 may not have negative yields. This condition was applied to mass 132.

deviation limits of β_1 are at 93°57′ (54-hour isomer of Cd^{115} and condition 1) and at 93°33′ (7-Mev fission yield curve and condition 2). Calculated fission yields for 93°57′ are shown in Fig. 7. Yields for 93°33′ are similar.

The graph of the two-dimensional subspace for alpha fission of U^{238} is shown in Fig. 8. For this case since the appropriate errors are not known from a better source they were determined by external consistency (reference 8, p. 28). The limits of β_1 are from −3°29′ (condition 2, mass 115) to 9°40′ (condition 1, 22.6-Mev fission yield curve). One limit on β_2 was set at 103°48′ where the yield plus one standard deviation of mass 97 is zero. The other limit was set at 27°45′ by a line from the

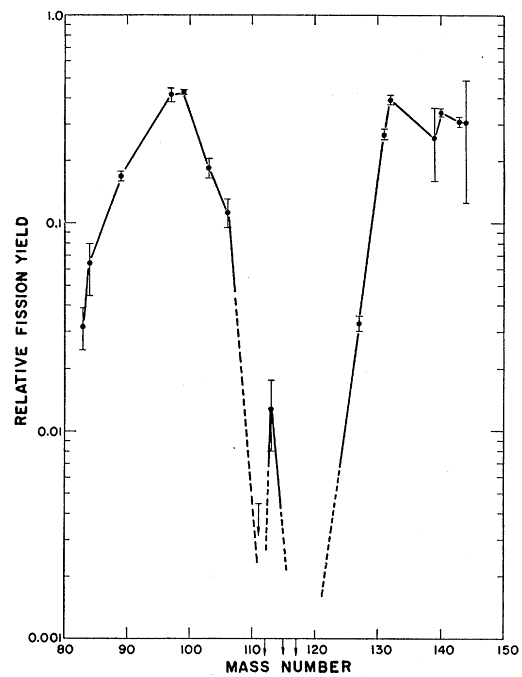

FIG. 7. Gamma fission of U^{238}. These are calculated yields for the low-energy limit of β_1, the low-energy fission mode. Calculated yields for the high-energy limit of β_1 are very nearly the same as these yields.

origin tangent to the curve for 40.1-Mev fission. Although 40.1-Mev is not the highest energy of alpha-particles used to induce fission, the point representing its fission yield curve is at the edge of the region of observed yields. Since the only estimate of errors is from external consistency we hesitate to attribute any significance to the apparent inversion in order of the points representing the 40.1-Mev, 43.9-Mev, and 45.4-Mev points in Fig. 8. Calculated yields for 103°48′, the upper limit for β_2, are shown in Fig. 9. Error bands are such that the lower limits of all except two mass numbers are negative.

Calculated yields for the two limits of β_2, the high-energy fission mode, for proton fission of Th^{232} are

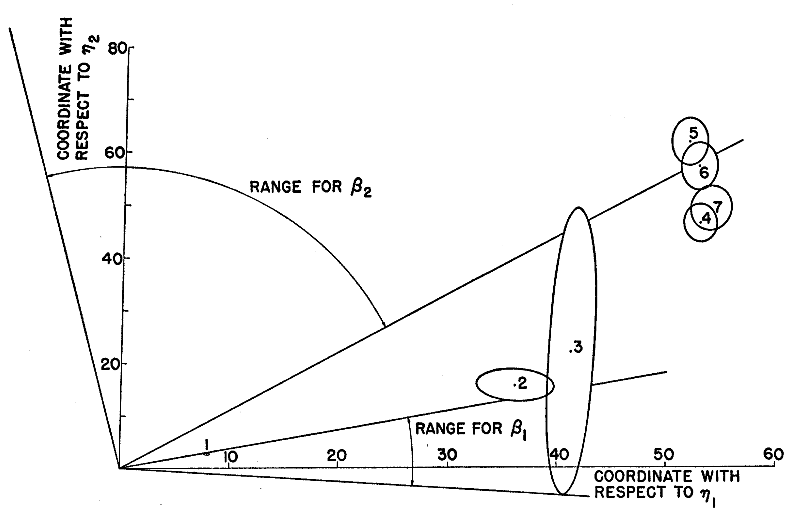

FIG. 8. Points representing sets of formation cross sections for alpha fission of U^{238}. 1, 22.6 Mev; 2, 27.1 Mev; 3, 33.8 Mev; 4, 38.6 Mev; 5, 40.1 Mev; 6, 43.9 Mev; and 7, 45.4 Mev. Errors, represented by ellipses, were determined by external consistency. Distance from the origin increases with energy because cross sections increase with energy.

shown in Figs. 2 and 3. The range of shapes for β_2 do not include a single narrow peak at near symmetric masses as is sometimes assumed (see for instance Schmitt and Sugarman[2]). β_2 may possibly be a standard two-peaked curve (Fig. 2) or a triple-peaked curve (Figs. 2 and 3) such as Fairhall and Jensen[10] observed for 11-Mev proton fission of Ra^{226}. If the latter is the case our treatment does not allow the interpretation that the central peak is due to one mode of fission and that the two other peaks are due to another mode. The two limits for β_2 for gamma fission of U^{238} are shown in Figs. 5 and 6. Again the standard two-peaked fission yield curve is a possibility. The yields for the other limit of β_2 in Fig. 6 are not very similar to any of the familiar fission yield curves, nor are those for the high-energy limit of β_2 for alpha fission of U^{238} in Fig. 9.

Calculated yields for the low-energy fission mode for gamma fission of U^{238} are shown in Fig. 7. The high yield of mass 113 relative to neighboring masses may be interpreted as evidence for a peak at the bottom of the valley. It should be mentioned that this calculated yield is a long extrapolation from the data; 48 Mev is the lowest energy for which Schmitt and Sugarman measured the yield of mass 113. Such a peak for 31-Mev bremsstrahlung fission of U^{238} was mentioned by Fairhall and Jensen as a private communication by Pappas. If there is such a peak at the bottom of the valley it cannot, according to our viewpoint, be attributed to a symmetric high-energy fission mode. We regard it as a feature of the fission yield curve for the low-energy fission mode, β_1.

V. EFFECT OF THE CHOICE OF BASE NUCLIDES ON THE SUBSPACE OBTAINED FROM THE LEAST SQUARES CALCULATION

For noncollinear points in a plane it is well known that the two regression lines do not coincide.[9] We expect something similar in our case where the choice of a pair of base nuclides from many possible pairs corresponds to the choice between the regression of y on x and the regression of x on y. In order to investigate this possibility, we chose another pair of base nuclides for gamma fission of U^{238}; they are the 54-hour isomer of Cd^{115} for A_1 and mass 99 for A_2. The standard deviation of the yield of mass 99 was assumed to be 0.1 or about 1.5% of the yield. The new choice of base nuclides results in the vectors ξ_1 and ξ_2 of Table VI. The subspace of ξ_1 and ξ_2 is to be compared with the subspace of η_1 and η_2 which resulted from the choice

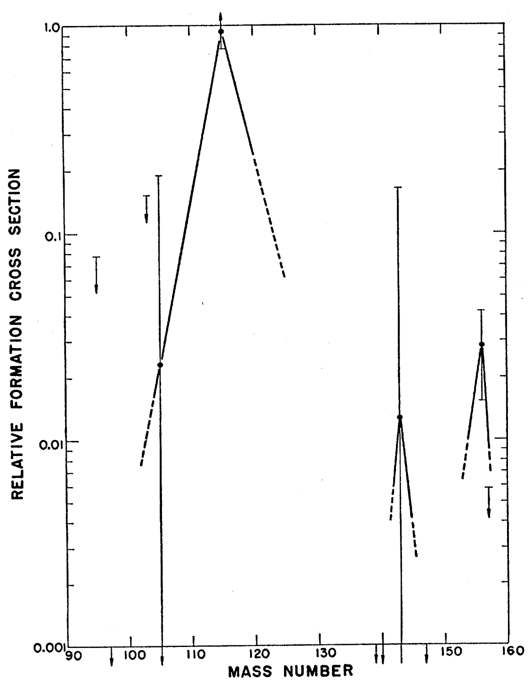

FIG. 9. Alpha fission of U^{238}. These are calculated relative formation cross sections for the high-energy limit of β_2, the high-energy fission mode.

[10] R. C. Jensen and A. W. Fairhall, Phys. Rev. **109**, 942 (1958).

TABLE VI. Gamma fission of U^{238}. Comparison of subspaces obtained using yields from different pairs of mass numbers.

Mass number	ξ_1	$d_1\eta_1+d_2\eta_2$	$\xi_1-d_1\eta_1-d_2\eta_2$	σ_1	ξ_2	$e_1\eta_1+e_2\eta_2$	$\xi_2-e_1\eta_1-e_2\eta_2$	σ_2
83	0.0742	0.0742	0.0	0.0206	0.2060	0.2103	−0.0043	0.1889
84	0.1490	0.1443	0.0047	0.0426	0.0913	0.1243	−0.0330	0.3565
89	0.3850	0.3942	−0.0092	0.0176	0.4079	0.3266	0.0813	0.1630
97	0.8567	0.9753	−0.1186	0.0549	0.2417	−1.4382	1.6799	0.7343
99	1	1	0		0	0	0	
103	0.4158	0.4321	−0.0163	0.0447	0.5773	0.4395	0.1378	0.4420
106	0.2483	0.2637	−0.0154	0.0402	1.2037	1.1047	0.0990	0.3591
111	0.0035	0.0031	0.0004	0.0013	1.5947	1.6049	−0.0102	0.0443
112	0.0022	0.0022	0.0000	0.0010	1.0061	1.0002	0.0059	0.0431
113	0.0269	0.0305	−0.0036	0.0119	0.8939	0.8785	0.0154	0.0380
115/54	0	0	0		1	1	0	
115/43	−0.0006	−0.0008	0.0002	0.0004	0.0957	0.0946	0.0011	0.0030
117	−0.0001	0.0005	−0.0006	0.0013	0.9967	1.0299	−0.0332	0.0342
127	0.0735	0.0773	−0.0038	0.0057	0.8979	0.8703	0.0276	0.0819
131	0.6115	0.6278	−0.0163	0.0294	0.5140	0.3838	0.1302	0.3034
132	0.7902	0.8137	−0.0235	0.0211	−0.8115	−1.1072	0.2957	0.1976
139	0.5189	0.6030	−0.0841	0.0790	2.5000	1.5116	0.9884	0.9153
140	0.7802	0.7987	−0.0185	0.0159	−0.1287	−0.3362	0.2075	0.2032
143	0.6207	0.6417	−0.0210	0.0214	−0.4367	−0.6502	0.2135	0.1843
144	0.6446	0.7160	−0.0714	0.0775	−1.8182	−2.6171	0.7989	1.3634

111 and 140 for base nuclides. For this comparison, ξ_1 and ξ_2 were fitted to η_1 and η_2 by least squares to determine how near ξ_1 and ξ_2 are to the subspace spanned by η_1 and η_2. In Table VI, $d_1\eta_1+d_2\eta_2$ and $e_1\eta_1+e_2\eta_2$ are the resulting linear combinations of η_1 and η_2 which are nearest in the sense of least squares to ξ_1 and ξ_2. The least squares calculations which gave ξ_1 and ξ_2 also furnish a standard deviation for each entry of ξ_1 and ξ_2. These are denoted by column headings σ_1 and σ_2 in Table VI.

For most cases, entries in ξ_1 and ξ_2 differ from the best fit with η_1 and η_2 by less than a standard deviation of the entry in ξ_1 and ξ_2. For mass 97, the differences for both ξ_1 and ξ_2 are more than two standard deviations. However, for mass 97 the entry in η_1 has a standard deviation of 97%. That is, the subspace is not very well determined with respect to mass 97. The agreement between the two subspaces is regarded as good.

VI. SUMMARY

The hypothesis of Turkevich and Niday that there are only two different modes of fission has been examined in terms of linear algebra. A chi-square test applied to several sets of fission-yield data indicates that the hypothesis is not to be rejected for these cases at any reasonable level of significance. The application of two conditions of positiveness to the data provides limits on the possible shapes of fission yield curves associated with the two hypothetical modes of fission. Some of the possible shapes between the limits are similar enough to the triple-peaked curve for fission of radium by protons to warrant comment. If these possible shapes are interpreted as being triple-peaked they cannot, according to our treatment, be regarded as a superposition of a symmetric high-energy and an asymmetric low-energy curve. Rather, the whole triple-peaked curve must be regarded as a single fission yield curve associated with one mode of fission.

ACKNOWLEDGMENTS

We wish to express our appreciation to Dr. A. Turkevich for many helpful discussions. We are indebted to Dr. G. A. Cowan, Dr. R. W. Spence, and Dr. J. D. Knight for their interest and valuable suggestions. The numerical work was done by Mrs. Teresa Kelley, J. J. Meaders, R. W. Williams, and R. F. Andrews.