studies. It also allows at least semiquantitative estimates of the parameters which enter many electronic properties. Finally, it provides a basis for experimentally determining a more precise band structure in terms of the observed deviations from single-OPW behavior. In this last regard, it is worth remarking that for most purposes one does *not* desire a precise description of the band structure. One more generally wishes to have a parameterized model of the structure which is simple enough to allow the calculation of a particular property, but reliable enough to include the important features of that effect.

In connection with the detailed studies of zinc and cadmium, rather explicit descriptions of the Fermi surfaces of these metals emerged. The surfaces resemble those shown in Fig. 3 for valence two. The surfaces in the combined first and second bands are distorted qualitatively in the manner indicated on the right-hand side of Fig. 4, but not to the extent that the lateral ridges on the surface meet the lateral edges of the zone. The ridges are everywhere rounded off somewhat such that the cross sections of the diagonal arms are reduced by a few percent. In zinc the cross section of the horizontal ring is narrowed down by a factor of ten midway between the lateral edges of the zone; in cadmium the ring is pinched off completely in these regions. In the combined third and fourth bands the needle-like segments along the lateral edges of the zone are narrowed down greatly in zinc and disappear in cadmium. In both cases the horizontal central disk is reduced appreciably in size, while the V-shaped segments at the lateral zone faces are rounded off, but presumably not greatly reduced in size.

In addition to the applications mentioned above, a study of the anomalous skin effect in aluminum was made, although there were no suitable data on single-crystal specimens to allow for comparison. The oscillatory magnetoacoustic effect was also discussed briefly in terms of the method. Finally, the generalization of the scheme to a study of alloys was outlined.

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Surface Magnetostatic Modes and Surface Spin Waves

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Examination of the spatial configuration of the magnetostatic modes of a ferromagnetic body shows that those modes whose frequency lies between $\omega=\gamma(B_iH_i)^{\frac{1}{2}}$ and $\omega=\gamma(H_i+2\pi M)$ are surface modes. It is also found that the complete spin-wave spectrum consists of a set of surface spin waves in addition to the spin-wave band usually considered. The magnetostatic mode spectrum thus merges smoothly into the spin-wave spectrum.

The characteristic equation for the surface modes on a plane surface at an arbitrary angle to the applied dc field is given. The properties of the surface modes on plane surfaces and on spheroidal bodies are discussed.

THE characteristic magnetostatic modes of a ferromagnetic slab, magnetized parallel to its surface, were recently examined by the authors.[1] It was found that the mode spectrum extends over the same frequency range as the magnetostatic mode spectrum of a spheroid,[2] namely from $\omega=\gamma H_i$ to $\omega=\gamma(H_i+2\pi M)$. It was also found that the spectrum of a slab clearly divides into two regions, one region extending from $\omega=\gamma H_i$ to $\omega=\gamma(B_iH_i)^{\frac{1}{2}}$ (coincident with the spin-wave band at long wavelengths[3]) in which the modes are spatially harmonic plane waves, and the other region from $\omega=\gamma(B_iH_i)^{\frac{1}{2}}$ to $\omega=\gamma(H_i+2\pi M)$ in which the modes are surface waves with exponentially decaying amplitude as one goes from the surface towards the interior of the ferromagnetic medium. The purpose of the present note is to show the general existence of surface modes in this frequency region immediately above the spin-wave band for uniformly magnetized samples of rather arbitrary shape and to discuss some of their properties. We find that not only may the magnetostatic mode spectrum in general be regarded as divided into volume and surface modes, but that the complete spin-wave spectrum similarly consists of a set of surface spin waves in addition to the usual spin-wave spectrum of an infinite medium.[3] Therefore, if one includes the surface spin waves, the magnetostatic mode spectrum merges smoothly into the total spin-wave spectrum.[4,5]

The general features of the surface modes may be derived by considering a semi-infinite ferromagnetic medium whose surface is at an angle, α, to the internal dc magnetic field, H_i, as shown in Fig. 1. (In the limit

[1] R. W. Damon and J. R. Eshbach, *Fifth Conference on Magnetism and Magnetic Materials, Detroit, Michigan, November 16–19, 1959* [J. Appl. Phys. (to be published)].
[2] L. R. Walker, Phys. Rev. **105**, 390 (1957).
[3] C. Herring and C. Kittel, Phys. Rev. **81**, 869 (1951).
[4] L. R. Walker, J. Appl. Phys. **29**, 318 (1958).
[5] C. R. Buffler, Suppl. J. Appl. Phys. **30**, 172S (1959); Robert L. White, Suppl. J. Appl. Phys. **30**, 182S (1959).

of large wave numbers this may be considered to be a small portion of the surface of a finite sample.)

Following Walker's method for the magnetostatic mode problem, it is found that the characteristic potential function, ψ, may be written as a product, $\psi = X(x)Y(y)Z(z)$, and, as expected, the solutions are plane waves. $Y(y)$ and $Z(z)$ have the form $e^{ik_y y}$ and $e^{ik_z z}$, where k_y and k_z must be real and may take on continuous values, due to the infinite extent of the surface. Outside of the ferromagnetic medium $X^e(x) = \exp(k_x^e x)$ where $k_x^e = |(k_y^2+k_z^2)^{\frac{1}{2}}|$. Within the medium $X^i(x) = \exp(ik_x^i x)$ and for solutions with frequencies within the spin-wave band k_x^i is real and boundary conditions may be satisfied for a given spin wave, $e^{i\mathbf{k}\cdot\mathbf{r}}$, by adjustment of the spatial phase of the spin-wave configuration. For a given k_y and k_z we find that there is no more than one solution, in terms of k_x^i and ω, which lies outside of the spin-wave band. When it exists this solution lies in the frequency range $\gamma(B_iH_i)^{\frac{1}{2}} \leq \omega \leq \gamma(H_i+2\pi M)$ and the wave vector component k_x^i must have an imaginary part, thus assuring a surface wave character. For these solutions

$$k_x^i = k_{xr}^i + ik_{xi}^i,$$

where

$$k_{xr}^i = -\frac{\kappa \sin\alpha \cos\alpha}{1+\kappa\cos^2\alpha}k_z$$

$$k_{xi}^i = \left|\left[\frac{1+\kappa}{1+\kappa\cos^2\alpha}\left(k_y^2 + \frac{1}{1+\kappa\cos^2\alpha}k_z^2\right)\right]^{\frac{1}{2}}\right|.$$

The characteristic equation for the surface modes, as derived from the magnetostatic boundary conditions, is

$$(1+\kappa\cos^2\alpha)k_{xi}^i + (\nu\cos\alpha)k_y + k_x^e = 0, \qquad (1)$$

where

$$\kappa = \frac{\gamma^2 4\pi M H_i}{\gamma^2 H_i^2 - \omega^2} \quad \text{and} \quad \nu = \frac{\gamma 4\pi M\omega}{\gamma^2 H_i^2 - \omega^2}.$$

The first and third terms of Eq. (1) are always positive for $\omega > \gamma(B_iH_i)^{\frac{1}{2}}$, and, since ν is negative, solutions are only possible for positive k_y. Thus all the surface wave solutions must be traveling waves with respect to their y component.

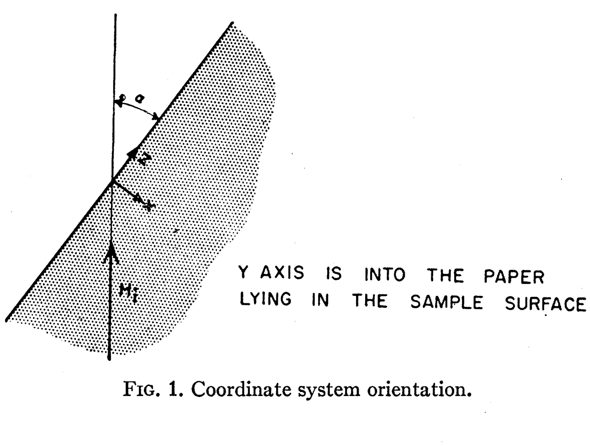

FIG. 1. Coordinate system orientation.

Solving Eq. (1) for ω leads to a very simple expression for the frequencies of the allowed surface modes in terms of the applied field, the saturation magnetization, the angle α (previously defined) and an angle β, which is the angle between the surface propagation vector k_s (where $k_s^2 = k_y^2 + k_z^2$) and the positive y direction (i.e., $\tan\beta = k_z/k_y$). The result is

$$\frac{\omega}{\gamma} = \frac{H_i}{2}\frac{1}{\cos\alpha\cos\beta} + \frac{B_i}{2}\cos\alpha\cos\beta, \qquad (2)$$

where solutions are only permitted for

$$1 \geq \cos\alpha\cos\beta \geq (H_i/B_i)^{\frac{1}{2}}. \qquad (3)$$

For $\cos\alpha\cos\beta = 1$ ($\alpha=0$, $\beta=0$) the frequency is $\omega = \gamma(H_i + 2\pi M)$. Thus, the only mode existing at this frequency would be a purely transverse (y directed) wave on a sample magnetized parallel to its surface. Modes propagating at the limiting angles (either in α or β) have frequency $\omega = \gamma(B_iH_i)^{\frac{1}{2}}$. An increase in the dc field not only compresses the frequency range of the surface modes but limits their existence to more nearly transverse modes on surfaces more nearly parallel to the internal dc field.

In finite samples such as spheres and ellipsoids we would expect, from the above, that surface modes will exist in the frequency range $\gamma(B_iH_i)^{\frac{1}{2}} \leq \omega \leq \gamma(H_i+2\pi M)$ on those parts of the surface most nearly parallel to the internal field and that they will propagate in nearly transverse directions. Indeed in the case of spheriods[2] the argument of the characteristic function for the coordinate normal to the sample surface changes from real to imaginary as one goes from modes in the spin-wave band to those above it. This assures a surface mode character to the modes above $\gamma(B_iH_i)^{\frac{1}{2}}$ which becomes most pronounced when the principal mode number, n, is large. In each set of modes of a given (n,m) there is only one mode, the $(n,m,0)$ mode in Walker's notation, that has the possibility of lying above $\gamma(B_iH_i)^{\frac{1}{2}}$. Examination of the characteristic function of the $(n,m,0)$ modes lying above $\gamma(B_iH_i)^{\frac{1}{2}}$ shows that they have significant amplitude only on that part of the surface that is nearly parallel to the internal field. From Walker's Eq. (21) in reference 2 it may be shown that the $(n,m,0)$ mode of a sphere lies above $\gamma(B_iH_i)^{\frac{1}{2}}$ if

$$\frac{m}{n+1+(0,1)} > \left(\frac{H_i}{B_i}\right)^{\frac{1}{2}}. \qquad (4)$$

In the sphere we may identify n with the total surface wave number and m with its transverse component, and thus $m/n \to \cos\beta$. For $n \gg 1$ Eq. (4) then corresponds to Eq. (3) above. For small n the deviation of $m/[n+1+(0,1)]$ from m/n takes account of the penetration of the surface waves to depths comparable with sample

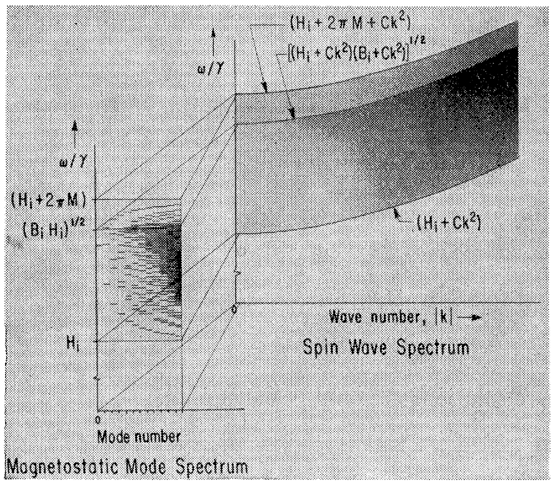

FIG. 2. A schematic representation of the magnetic mode spectrum of a finite ferromagnetic body.

dimensions, as well as the curvature encountered, in a similar way that $\cos\alpha$ accounts for the effect of a tipped surface in Eq. (3). Also we note that $(n,m,0)$ modes only exist for m positive, corresponding to our finding that k_y must be positive. Thus we conclude that in spheroids the $(n,m,0)$ modes are surface modes, when they lie above $\gamma(B_iH_i)^{\frac{1}{2}}$, with properties analogous to the surface modes of a plane surface.

From the above, and the previous studies, it is apparent that the surface modes are merely part of the total set of solutions, the vast majority of which are modes within the spin-wave band. In fact, as one might expect, it can be shown that the number of surface modes increases as $|\mathbf{k}|$ while the number of volume type modes increases as $|\mathbf{k}|^2$. At large $|\mathbf{k}|$ therefore the number and frequency distribution of the volume modes is little affected by those that have become surface modes, and the volume mode spectrum merges into the spin-wave spectrum of an infinite medium[4] regardless of sample shape. The surface mode spectrum at large $|\mathbf{k}|$, on the other hand, is intimately related to sample shape, sample orientation in the dc field and strength of the dc field. The frequency density of the surface modes of a plane surface for a given k_s has a pole at both $\omega=\gamma(B_iH_i)^{\frac{1}{2}}$ and at $\omega=\gamma[(H_i/2\cos\alpha)+\frac{1}{2}B_i\cos\alpha]$, as shown in reference 1. In a finite sample whose surface has a distribution in α we would expect the pole at $\gamma(B_iH_i)^{\frac{1}{2}}$ to remain but the other pole may not.

Some observations may be made about the case where $\mathbf{k}$ is large enough that exchange energy becomes important. Initially, as $\mathbf{k}$ increases, the exchange energy may be considered as a perturbation on the problem already solved. In this case, following Kittel and Herring,[6] the result is that the frequency of a given surface wave increases in the same way as if an additional field $\Delta H_i=Ck^2$ were present, where C is a measure of the exchange interaction and $k^2=k_s^2+|k_x^i|^2$. Thus, if this value of k is used for the surface waves, the complete spin-wave spectrum is contained in the frequency range $\gamma(H_i+Ck^2)\leq\omega\leq\gamma(H_i+2\pi M+Ck^2)$ and the spin-wave spectrum may be plotted as in Fig. 2. For the surface wave at the upper limit $\alpha=0$ and $\beta=0$; thus $k_z=0$, $k_x^i=ik_y$, and $k^2=2k_s^2$. For other waves, however, k may be very large even for small k_s, so that exchange may no longer be considered a perturbation. In particular, for large $|k_x^i|$ additional boundary conditions must be considered (see Ament and Rado and Kittel[7]) and the problem becomes considerably more complex.

Finally, it is felt that the recognition of the existence of these surface modes may be of some importance in the explanation of the ferromagnetic resonance line-width, especially since surface roughness is known to play an important role in certain experiments.[5,8]

[6] C. Kittel and Conyers Herring, Phys. Rev. **77**, 725 (1950).
[7] W. S. Ament and G. T. Rado, Phys. Rev. **97**, 1558 (1955); C. Kittel, Phys. Rev. **110**, 1295 (1958).
[8] R. C. LeCraw, E. G. Spencer, and C. S. Porter, Phys. Rev. **110**, 1311 (1958).

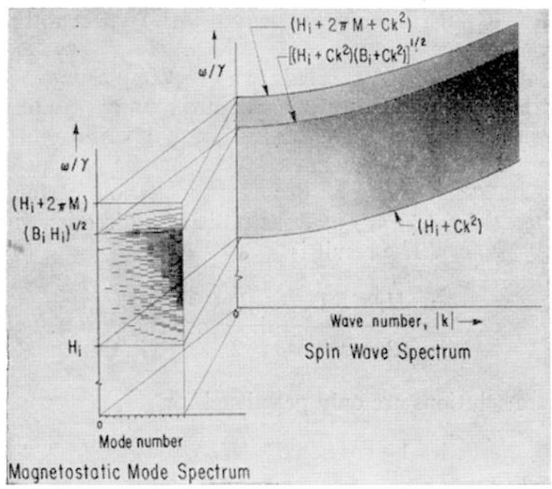

FIG. 2. A schematic representation of the magnetic mode spectrum of a finite ferromagnetic body.