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Magnetic-Field Dependence of the Ultrasonic Attentuation in Metals

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A self-consistent, semiclassical treatment is given for the attenuation of a sound wave by a free-electron gas in a positive background which supports the sound wave. Emphasis is placed upon the kinds of magnetic-field dependence which can be found under a wide range of values for the magnetic field, frequency, and mean free path. Applications of the general formalism include propagation parallel and perpendicular to the magnetic field. The special phenomena studied include geometric resonances, cyclotron resonances, and magneto-plasma resonance. A qualitative physical interpretation of the various effects found in the detailed calculations is also presented.

I. INTRODUCTION

THE first experimental evidence that electrons in metals could contribute significantly to the attenuation of megacycle sound waves was uncovered by Bömmel[1] and MacKinnon.[2] The importance of the electronic system was indicated by the observed change in attenuation upon crossing the superconducting transition. Various theoretical discussions of the contribution of the electrons to the attenuation in normal metals were put forward,[3–5] but the first complete theory was that developed by Pippard[6] for the free-electron gas. Pippard's theory has successfully accounted for the major experimental features of the attenuation.

By this time Bömmel[7] had found that the attenuation showed nonmonotonic dependence upon magnetic field in tin at helium temperatures. The fluctuations in attenuation appeared at magnetic fields inconsistent with either cyclotron resonance or deHaas-van Alphen oscillations. Pippard[8] proposed that such oscillations could arise from a matching of the diameters of electron orbits in a magnetic field and the wavelength of the incident sound wave, an explanation which subsequent work (including the present) has shown to be correct in its essentials. Nevertheless, a detailed quantitative theory of the dependence of the attenuation upon magnetic field has been much slower in coming forward than was the original zero-field theory of Pippard. During the period of theoretical silence a number of experiments were performed which amply confirmed Bömmel's original experiments and Pippard's suggestion that the oscillations provided a new tool for studying the Fermi surfaces of metals. The principal experimental investigations have been those of Morse and his collaborators on copper[9,10] and tin[11] and that of Reneker on bismuth.[12] The experiments of Morse *et al.*, were carried out with the magnetic field perpendicular to the direction of propagation of the sound wave; Steinberg[13] and Harrison[14] have discussed some aspects of the observed behavior. Kjeldaas,[15] on the other hand, has provided a theory of the attenuation for a magnetic field parallel to the direction of propagation, which is valid for currently employed experimental conditions.

More recently Rodriguez[16] attempted the solution of the problem of attenuation in a transverse magnetic field. His formulation of the problem was correct in most respects, but most of his results were invalidated by inadvisable physical and mathematical approximations. In particular, he found no oscillations in the attenuation of the type suggested by Pippard and observed experimentally. The absence of oscillations derives from the mathematical procedures employed by Rodriguez, but more fundamental difficulties were that his treatment of the electric fields was not self-consistent and that the collision-drag effect[17] was not included adequately. A more careful treatment by Kjeldaas and Holstein[18] has subsequently shown that salient features of the experiments are understandable in terms of the free-electron model. The work of

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[8] A. B. Pippard, Phil. Mag. **2**, 1147 (1957).
[9] R. W. Morse, H. V. Bohm, and J. D. Gavenda, Phys. Rev. **109**, 1394 (1958).
[10] R. W. Morse and J. D. Gavenda, Phys. Rev. Letters **2**, 250 (1959); J. D. Gavenda and R. W. Morse, Bull. Am. Phys. Soc. **3**, 167 (1959).
[11] R. W. Morse, H. V. Bohm, and J. D. Gavenda, Bull. Am. Phys. Soc. **3**, 44 (1958); T. Olson and R. W. Morse, Bull. Am. Phys. Soc. **3**, 167 (1959).
[12] D. H. Reneker, Phys. Rev. Letters **1**, 440 (1958); Phys. Rev. **115**, 303 (1959).
[13] M. S. Steinberg, Phys. Rev. **109**, 1486 (1958); **110**, 772 (1958).
[14] M. J. Harrison, Phys. Rev. Letters **1**, 441 (1958).
[15] T. Kjeldaas, Phys. Rev. **113**, 1473 (1959).
[16] S. Rodriguez, Phys. Rev. **112**, 80 (1958).
[17] T. Holstein, Phys. Rev. **113**, 479 (1959).
[18] T. Kjeldaas and T. Holstein, Phys. Rev. Letters **2**, 340 (1959).

Kjeldaas and Holstein relates primarily to the range of conditions embraced by the existing experiments of Morse and collaborators. The emphasis in their work has been on the numerical and analytical aspects of the theory. There remains the task of providing an overall theoretical survey of the dependence of the attenuation upon magnetic field under a wide range of experimental conditions. Such a survey is the subject of the present paper.

In the following, therefore, we give a self-consistent semiclassical treatment of a free-electron gas in a positive background supporting a sound wave. In Sec. II we develop a general formulation of the theory of the attenuation. Starting from the Boltzmann equation we develop a constitutive equation giving the response of an electron gas to the electric field, the collision drag, and the electron density gradient which accompany the sound wave. From the constitutive equation in conjunction with Maxwell's equations we derive a general formula for the attenuation. In our formulation, the conductivity tensor plays a central role and is studied in detail in Sec. III. In the subsequent sections we apply this formulation to cases of magnetic field perpendicular to and parallel to the direction of sound propagation. We consider wide variations in magnetic field, frequency of the sound wave, and electronic mean free path. In Sec. IV we consider the oscillations observed experimentally (which we term "geometric resonances").[19] In addition, we discuss the low-field limit, including a detailed quantitative theory of the cyclotron resonance first proposed by Mikoshiba.[20] Finally we discuss the behavior of the attenuation at very high fields. All calculations include high- and low-frequency limits. In Sec. V we extend the parallel-propagation case treated by Kjeldaas to higher fields and frequencies than considered by him. Our concluding section, Sec. VI, contains a qualitative physical interpretation of the various effects found in the detailed calculations.

II. FORMAL THEORY OF THE ATTENUATION

A. The Constitutive Equation

In place of a real metal, we consider a gas of N_0 electrons per unit volume moving through a uniform background of positive charge of the same density. The discreteness of the ion cores in real metals is unimportant when the sound wavelength greatly exceeds the interatomic separation. A sound wave of propagation vector $\mathbf{q}$ and frequency ω manifests itself as a velocity field $\mathbf{u}(\mathbf{r},t)\propto \exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$ in the positive background. In the present model, interactions between particles are replaced by interactions of individual particles with a self-consistent electromagnetic field derived from Maxwell's equations. The latter can be reduced to

$$\mathcal{E}_{\parallel}=(4\pi/i\omega)j_{\parallel},\quad \boldsymbol{\mathcal{E}}_{\perp}=\frac{(4\pi i/\omega)(v_s/c)^2\mathbf{j}_{\perp}}{1-(v_s/c)^2},\tag{2.1}$$

where $\boldsymbol{\mathcal{E}}$ is the electric field and $\mathbf{j}$ the total current density accompanying the sound wave, and both vary as $\exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$. The subscripts $\parallel$ and $\perp$ in (2.1) refer to components parallel and perpendicular to $\mathbf{q}$, respectively, and v_s is the sound velocity.

The total current density contains a contribution from the electrons, $\mathbf{j}_e$, and one from the positive background, $Ne\mathbf{u}$,

$$\mathbf{j}=\mathbf{j}_e+Ne\mathbf{u}.\tag{2.2}$$

The electronic current $\mathbf{j}_e$ excited by the sound wave obeys a constitutive equation analogous to Ohm's law, which has already been given correctly by Rodriguez.[16] For completeness and convenience, we rederive the constitutive equation here. Because the sound wavelength is much longer than the electron wavelengths and the atomic separations, all further developments can be carried out in macroscopic terms.

The electronic contribution is given by

$$\mathbf{j}_e(\mathbf{r},t)=-e\int \mathbf{v}f(\mathbf{r},\mathbf{v},t)d\mathbf{v},\tag{2.3}$$

where $f(\mathbf{v},\mathbf{r},t)$ is the distribution function in phase space (μ-space) for electrons of velocity $\mathbf{v}$ and position $\mathbf{r}$. When no sound wave is present, the distribution function reduces to the thermal equilibrium Fermi-Dirac function $f_0(\mathbf{v},E_F{}^0)$, $E_F{}^0$ being the Fermi energy, and does not depend explicitly on the static magnetic field $\mathbf{H}_0$ (Bohr-van Leewen theorem). When a sound wave is present, the distribution function is determined from Boltzmann's equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}}=\left.\frac{\partial f}{\partial t}\right)_{\text{coll.}}.\tag{2.4}$$

In (2.4), $\mathbf{F}$ is the Lorentz force,

$$\mathbf{F}=-e[\boldsymbol{\mathcal{E}}+(\mathbf{v}/c)\times\mathbf{H}],\tag{2.5}$$

where the magnetic field $\mathbf{H}$ includes a part $\mathbf{H}_1$ associated with the sound wave in addition to $\mathbf{H}_0$.

For the collision term on the right-hand side of (2.4) we make the relaxation time ansatz

$$\left.\frac{\partial f}{\partial t}\right)_{\text{coll.}}=-(f-f_s)/\tau.\tag{2.6}$$

The meaning of (2.6) is that $f(\mathbf{r},\mathbf{v})/\tau$ electrons are scattered out of a unit volume of phase space around $(\mathbf{r},\mathbf{v})$ in unit time and $f_s(\mathbf{v},\mathbf{r})/\tau$ are scattered in. Here τ is the relaxation time and f_s the distribution of electrons after scattering. When the impurities are at

[19] This work on geometric resonances was carried out independently of the work of Holstein and Kjeldaas; we are in complete agreement with their results wherever there is overlap.

[20] N. Mikoshiba, J. Phys. Soc. Japan 13, 759 (1958).

rest, the electrons scatter into an isotropic velocity distribution centered about the impurity velocity, namely zero; f_s is simply $f_0(\mathbf{v},E_F{}^0)$. When the impurities are moved by the sound wave, the electrons scatter into an isotropic velocity distribution centered about the impurity velocity $\mathbf{u}$. Further, the scattering is local and cannot change the electron density. We therefore have

$$f_s(\mathbf{r},\mathbf{v},t)=f_0(\mathbf{v}-\mathbf{u}(\mathbf{r},t),\, E_F(\mathbf{r},t)), \tag{2.7}$$

for the distribution after scattering, where the Fermi energy $E_F(\mathbf{r},t)$ is chosen to give the correct electron density $N(\mathbf{r},t)$.

The problem of deriving the constitutive equation for $\mathbf{j}_e$ is now completely specified. We must solve the Boltzmann Eq. (2.4) with the collision term given by (2.6) and (2.7), and then we must substitute the result into (2.3). The Boltzmann equation is readily solved by a method due to Chambers.[21] A particle contributes to the distribution function $f(\mathbf{r},\mathbf{v},t)$ only if it is at the point $(\mathbf{r},\mathbf{v})$ in phase space at time t. The particle will have been on the single trajectory T which passes through $(\mathbf{r},\mathbf{v})$ since the time t' at which it was scattered onto T at $(\mathbf{r}',\mathbf{v}')$, Fig. 1. The number of electrons scattered onto T in dt' is $f_s(\mathbf{r}',\mathbf{v}',t')dt'/\tau$, and the probability that an electron will not scatter again before reaching $(\mathbf{r},\mathbf{v})$ is $\exp[-(t-t')/\tau]$. The distribution function is therefore given by

$$f(\mathbf{r},\mathbf{v},t)=\int_{-\infty}^{t} f_s(\mathbf{r}',\mathbf{v}',t')e^{-(t-t')/\tau}dt'/\tau. \tag{2.8}$$

Expanding f to first order in $\mathbf{u}$ and quantities proportional to $\mathbf{u}$,

$$f=f_0(\mathbf{v},E_F{}^0)+f_1, \quad N=N_0+N_1 \tag{2.9}$$

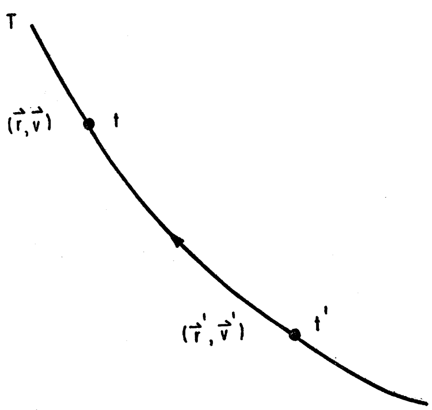

FIG. 1. The trajectory T of an electron in phase space under the combined influence of the static magnetic field $\mathbf{H}_0$ and the sound wave. An electron which contributes to the distribution function $f(\mathbf{r},\mathbf{v},t)$ must have been scattered onto T at some previous time t' at the corresponding point $(\mathbf{r}',\mathbf{v}')$ and have followed T without scattering until it arrives at the point $(\mathbf{r},\mathbf{v})$ at the time t.

[21] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952); **A238**, 344 (1957).

we obtain[22]

$$f_1=-\frac{\partial f_0}{\partial E}\int_{-\infty}^{t}[\mathbf{v}'\cdot(-e\boldsymbol{\mathcal{E}}'+m\mathbf{u}'/\tau) +(\tfrac{2}{3})E_F{}^0N_1'/N_0\tau]e^{-(t-t')/\tau}dt'. \tag{2.10}$$

In (2.10) $\mathbf{r}'$ and $\mathbf{v}'$ now lie on the trajectory T_0 followed in the absence of the sound wave. The electric field $\boldsymbol{\mathcal{E}}$ was introduced into (2.10) by expressing $f_0(\mathbf{v}',E_F{}^0)-f_0(\mathbf{v},E_F{}^0)$ in terms of the energy change along the trajectory and then integrating by parts. The quantities $\boldsymbol{\mathcal{E}}'$, $\mathbf{u}'$, and N_1' are all proportional to $\exp[i(\mathbf{q}\cdot\mathbf{r}'-\omega t')]$ so that (2.10) may be rewritten in terms of $\boldsymbol{\mathcal{E}}$, $\mathbf{u}$, and N_1,

$$f_1(\mathbf{r},\mathbf{v},t)=-\frac{\partial f_0}{\partial E}\left[\mathbf{J}(\mathbf{v})\cdot\left(\boldsymbol{\mathcal{E}}-\frac{m\mathbf{u}}{e\tau}\right) +\frac{2E_F{}^0}{3N_0\tau}K(\mathbf{v})N_1\right], \tag{2.11}$$

where

$$(\mathbf{J}(\mathbf{v}),K(\mathbf{v}))=\int_{-\infty}^{t}(-e\mathbf{v}',1) \times\exp\{i[\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})-\omega(t'-t)] -(t-t')/\tau\}dt'. \tag{2.12}$$

By substituting (2.10) into (2.3) we obtain the desired constitutive equation

$$\mathbf{j}_e=\boldsymbol{\sigma}\cdot(\boldsymbol{\mathcal{E}}-m\mathbf{u}/e\tau)-\mathbf{R}N_1ev_s. \tag{2.13}$$

In (2.11), $\boldsymbol{\sigma}$ is the magneto-conductivity tensor for frequency ω and wave number $\mathbf{q}$,

$$\boldsymbol{\sigma}=\int d\mathbf{v}\,(-e\mathbf{v})\mathbf{J}(\mathbf{v})(-\partial f_0/\partial E), \tag{2.14}$$

and as such enters all time- and position-dependent transport phenomena; e.g., the Azbel'-Kaner effect.[23] Collisions with moving impurities have the effect of adding an apparent drag $m\mathbf{u}/\tau$ in (2.13) to the forces acting on the electrons. The magnetic field $\mathbf{H}_1$ associated with the sound wave does not enter explicitly. The third term in (2.13) arises from diffusion of the nonuniformly distributed electrons, the vector $\mathbf{R}$ being given by

$$\mathbf{R}=\frac{2E_F{}^0}{3N_0\tau v_s}\int \mathbf{v}K(\mathbf{v})\left(\frac{\partial f_0}{\partial E}\right)d\mathbf{v}. \tag{2.15}$$

[22] One can see from (2.10) or (2.11) the close relation between Chamber's method[20] and the idea of a local Fermi surface employed by Pippard.[6] The change in the radius of the Fermi surface $\Delta v(\mathbf{r},t;\hat{v}_F)$ considered by Pippard is simply the integral in (2.10) divided by mv_F.

[23] G. Dresselhaus and D. Mattis, Phys. Rev. **111**, 403 (1958).

There is a relation between $\mathbf{J}$ and K obtained by integrating by parts on t' in Eq. (2.12),

$$i\mathbf{J}\cdot\mathbf{q}=-e\{1-[(1-i\omega\tau)/\tau]K\}, \qquad (2.16)$$

which leads directly to an interesting and useful relation between $\boldsymbol{\sigma}$ and $\mathbf{R}$,

$$i\boldsymbol{\sigma}\cdot\mathbf{q}=-[3N_0e^2v_s(1-i\omega\tau)/2E_F{}^0]\mathbf{R}. \qquad (2.17)$$

In order to develop the significance of Eq. (2.17), we consider the generalization of (2.13) to arbitrary position and time dependence. The current induced by the electric field is

$$j_e(\mathbf{r},t)=ecV\int d\mathbf{r}'\int_{\infty}^{t}dt'\,\boldsymbol{\Sigma}(\mathbf{r}-\mathbf{r}',\,t-t')\cdot\boldsymbol{\mathcal{E}}(\mathbf{r}',t'). \qquad (2.18)$$

Taking the Fourier transform of (2.18) leads back to the corresponding part of (2.13)

$$\mathbf{j}_{e1}(\mathbf{q},\omega)=\boldsymbol{\sigma}(\mathbf{q},\omega)\cdot\boldsymbol{\mathcal{E}}(\mathbf{q},\omega), \qquad (2.19)$$

but with $\boldsymbol{\sigma}(\mathbf{q},\omega)$ expressed as

$$\boldsymbol{\sigma}(\mathbf{q},\omega)=\int_0^{\infty}ds\,\boldsymbol{\Sigma}(\mathbf{q},s)e^{i\omega s}, \qquad (2.20)$$

where $s=t-t'$. Proceeding similarly with the diffusion current, we start with the general relation

$$\mathbf{j}_{e2}(\mathbf{r},t)=e\int d\mathbf{r}'\int_{-\infty}^{t}dt' \times\mathbf{P}(\mathbf{r}-\mathbf{r}',\,t-t')\cdot\boldsymbol{\nabla}'N_1(\mathbf{r}',t'). \qquad (2.21)$$

and obtain

$$\mathbf{j}_{e2}(\mathbf{q},\omega)=e\mathbf{D}(\mathbf{q},\omega)\cdot i\mathbf{q}N_1(\mathbf{q},\omega),$$
$$\mathbf{D}=\int_0^{\infty}ds\,\mathbf{P}(\mathbf{q},s)e^{i\omega s}. \qquad (2.22)$$

The diffusion coefficient $\mathbf{D}(\mathbf{q},\omega)$ is directly related to $\mathbf{R}$ [see Eq. (2.13)],

$$i\mathbf{D}(\mathbf{q},\omega)\cdot\mathbf{q}=-\mathbf{R}v_s. \qquad (2.23)$$

A mobility tensor can be defined as usual

$$\boldsymbol{\sigma}(\mathbf{q},\omega)=N_0e\boldsymbol{\mu}(\mathbf{q},\omega). \qquad (2.24)$$

Comparison of (2.17) and (2.23) shows that (2.17) is equivalent to a generalization of the ordinary Einstein relation to hold for nonlocal, time-dependent processes occurring in constant magnetic fields,[24]

$$\boldsymbol{\mu}(\mathbf{q},\omega)=\frac{e\mathbf{D}(\mathbf{q},\omega)}{2E_F{}^0/3}(1-i\omega\tau). \qquad (2.25)$$

Kubo's generalization of the Einstein relation[25] was limited to $\mathbf{q}=0$. It is curious that (2.25) differs from the usual Einstein relation only through the factor $(1-i\omega\tau)$ despite the finite wave number.

B. The Attenuation Coefficient

The sound wave feeds both kinetic and potential energy into the electron system as it propagates. The electrons dissipate this energy to the positive background through collisions. An individual collision is a local event so that only kinetic and not potential energy is changed after a collision. By straightforward manipulation based on the Boltzmann equation one can show that the average rate of loss of electronic kinetic energy via collisions is $\langle\mathbf{j}_e\cdot\boldsymbol{\mathcal{E}}\rangle_{\text{Av}}$ per unit volume. Not all of this kinetic energy is dissipated as heat, a part being coherently fed back into the sound wave. Because the average electron velocity $\langle\mathbf{v}\rangle$ before collision in general differs from that after collision $\mathbf{u}$, there is a net force exerted by the electrons on unit volume of the positive charge equal to $N_0m(\langle\mathbf{v}\rangle-\mathbf{u})/\tau$. Energy is fed back into the sound wave at an average rate $\langle\mathbf{u}\cdot N_0m(\langle\mathbf{v}\rangle-\mathbf{u})/\tau\rangle_{\text{Av}}$ and the net power dissipated per unit volume is

$$Q=\tfrac{1}{2}\,\text{Re}[\mathbf{j}_e{}^*\cdot\boldsymbol{\mathcal{E}}-\mathbf{u}^*\cdot N_0m(\langle\mathbf{v}\rangle-\mathbf{u})/\tau], \qquad (2.26)$$

where we have used complex quantities as in the previous section for convenience. The correction to the $\mathbf{j}_e{}^*\cdot\boldsymbol{\mathcal{E}}$ term is related to the collision drag effect considered by Holstein.[17] It is of importance for the attenuation primarily at high frequencies or magnetic fields.

Equation (2.1) can be conveniently rewritten as

$$\mathbf{j}=-\sigma_0\mathbf{B}\cdot\boldsymbol{\mathcal{E}}, \qquad (2.27)$$

where the tensor $\mathbf{B}$ has the principal components $i(-\gamma,\beta,\beta)$ relative to axes defined by $\mathbf{q}$ and two transverse directions; σ_0 is the dc conductivity; and

$$\beta=\omega c^2/4\pi\sigma_0v_s{}^2,\quad \gamma=\beta(v_s/c)^2, \qquad (2.28)$$

after the $(v_s/c)^2$ in the denominator of (2.1) is neglected. The above expression (2.26) for Q can be transformed to

$$Q=-\tfrac{1}{2}\,\text{Re}[N_0e\mathbf{u}^*\cdot(\mathbf{I}+\mathbf{B})\cdot\boldsymbol{\mathcal{E}}], \qquad (2.29)$$

with the aid of (2.2), (2.3), and (2.27). The electric field $\boldsymbol{\mathcal{E}}$ must be linearly related to $\mathbf{u}$,

$$\boldsymbol{\mathcal{E}}=\mathbf{W}\cdot N_0e\mathbf{u}/\sigma_0. \qquad (2.30)$$

After (2.30) is inserted for $\boldsymbol{\mathcal{E}}$, (2.29) becomes

$$Q=N_0(\tfrac{1}{2}m|\mathbf{u}|^2/\tau)\hat{u}\cdot\mathbf{S}\cdot\hat{u} \qquad (2.31)$$

where

$$\mathbf{S}=-\text{Re}[(\mathbf{I}+\mathbf{B})\cdot\mathbf{W}], \qquad (2.32)$$

and $\hat{u}$ is a unit vector in the direction of the

[24] The foregoing argument is incomplete. We can conclude from comparison of (2.17) and (2.23) only that

$$[\mathbf{D}-(\tfrac{2}{3})E_F{}^0\boldsymbol{\mu}/e(1-i\omega\tau)]\cdot\mathbf{q}=0,$$

or

$$\mathbf{D}=[(\tfrac{2}{3})E_F{}^0\boldsymbol{\mu}/e(1-i\omega\tau)]+\mathbf{D}_b,$$

where $\mathbf{D}_b\cdot\mathbf{q}$ vanishes. However, from (2.22) a term like $\mathbf{D}_b$ cannot give rise to a diffusion current and may be disregarded.

[25] R. Kubo, Can. J. Phys. **34**, 1274 (1956); J. Phys. Soc. Japan **12**, 570 (1957).

polarization. The three independent directions of polarization $\hat{u}_i$, $i=1,2,3$, form the coordinate axes in which **B** has already been expressed and in which **W** and **S** can also be. We choose $\hat{u}_1$ along **q** and $\hat{u}_3$ to lie in the plane of **q** and $\mathbf{H}_0$, Fig. 2. For a particular polarization the power dissipated Q_i is

$$Q_i = N_0 \frac{m|\mathbf{u}|^2}{2\tau} S_{ii}, \tag{2.33}$$

so that our problem amounts to calculating a diagonal element of the tensor **S**. One would have expected the factor $N_0(\frac{1}{2}m|\mathbf{u}|^2/\tau)$ to establish the scale of Q from simple dimensional arguments alone, which suggests that S_{ii} does not ordinarily differ from unity by more than a few orders of magnitude.

The quantity of interest experimentally is the attenuation coefficient α, which gives the exponential decay of sound intensity with distance. α is the power density dissipated per unit energy flux, or

$$\alpha = Q/\tfrac{1}{2}M|\mathbf{u}|^2 v_s, \tag{2.34}$$

where M is the atomic mass of the metal being represented by our simple model. From (2.33), we have

$$\alpha_i = \frac{mv_F}{Mv_s}\frac{S_{ii}}{l} \tag{2.35}$$

for α, where $l = v_F\tau$ is the mean free path of an electron and v_F the Fermi velocity. Since α, by its nature, is the reciprocal of the mean free path L of the sound wave, the relation between L and l implied by (2.35) is of particular interest. For copper, for example, Mv_s/mv_F is about 100 so that

$$L_i \approx 100l/S_{ii} \quad \text{for} \quad \text{Cu}.$$

Ease of measurement and the general order of magnitude of background attenuation require that L be of the order of 1 cm or perhaps somewhat less. Thus, with S_{ii} of order unity, the mean free path l must be of order 0.1 mm or longer for readily observable attenuation by electrons. There is the additional point that only with long mean free paths can information about the details of the electronic structure be obtained rather than macroscopic properties. These dictate the requirements of pure materials and low temperatures; e.g., for very pure Cu, τ might be as long as 10^{-10} sec at helium temperatures and hence $L_i \simeq 4/S_{ii}$. There are two limitations on the maximum value of S_{ii} which can be tolerated. First, L_i should not be very much smaller than the thickness of the thinnest specimen which can be conveniently studied, say $\sim$0.1 mm, and hence for Cu, S_{ii} should not be larger than 10^3. Second, if L_i becomes comparable to the wavelength, $\sim$0.05 mm for 60 Mc/sec, then a correction to the frequency or velocity of sound becomes necessary because of the Kramers-Kronig relations, which imposes essentially the same upper bound on S_{ii}.

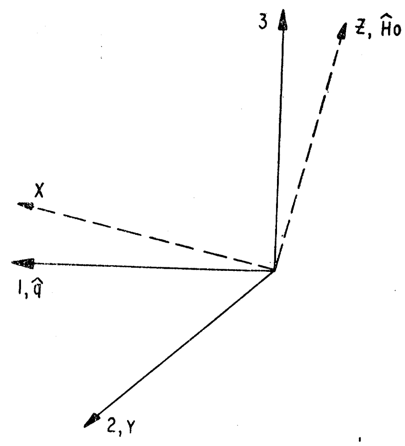

FIG. 2. The coordinate systems utilized for the expression of quantities like **B**, **S**, and $\boldsymbol{\sigma}$. The system (1,2,3) fixed to the direction of propagation **q** is most convenient for the expression of **B** and **S**. The system (x,y,z) fixed to the direction of the field $\mathbf{H}_0$ is most convenient for the expression of the conductivity tensor $\boldsymbol{\sigma}$.

We turn now to the derivation of the tensor **W** and the explicit form of **S**. The equation of continuity relates N_1 to $j_{e\parallel}$;

$$j_{e\parallel} = -N_1 e v_s. \tag{2.36}$$

The third term in (2.14) can therefore be written as

$$\mathbf{R}\dot{j}_{e\parallel} = \mathbf{R}\cdot\mathbf{j}_e, \tag{2.37}$$

where the tensor **R** has the components

$$R_{ij} = R_i \delta_{1j}, \tag{2.38}$$

and from (2.17)

$$R_{i1} = \frac{-i\omega\tau v_F^2}{3\sigma_0(1-i\omega\tau)v_s^2}\sigma_{i1} = -\Delta\sigma_{i1}. \tag{2.39}$$

We can now simplify the constitutive equation (2.13) to

$$\mathbf{j}_e = \sigma_0 \boldsymbol{\sigma}' \cdot (\boldsymbol{\mathcal{E}} - m\mathbf{u}/e\tau) \tag{2.40}$$

by use of (2.37), where

$$\boldsymbol{\sigma}' = [\mathbf{I} - \mathbf{R}]^{-1} \cdot \boldsymbol{\sigma}/\sigma_0 \tag{2.41}$$

is an effective conductivity which includes diffusion and is measured in units of σ_0. Substitution of (2.27) and (2.41) into (2.2) leads to an expression for **W**,

$$\mathbf{W} = -[\boldsymbol{\sigma} + \mathbf{B}]^{-1} \cdot [\mathbf{I} - \boldsymbol{\sigma}'] \tag{2.42}$$

We are thus led to the sought-after expression for **S** by inserting (2.42) into (2.32),

$$\mathbf{S} = \mathrm{Re}\{[\mathbf{I} + \mathbf{B}] \cdot [\boldsymbol{\sigma}' + \mathbf{B}]^{-1} \cdot [\mathbf{I} + \mathbf{B}]\} - \mathbf{I}, \tag{2.43}$$

or, in components

$$S_{ii} = \mathrm{Re}\{(1 + B_{ii})^2 [(\boldsymbol{\sigma}' + \mathbf{B})^{-1}]_{ii}\} - 1. \tag{2.44}$$

Equation (2.44) together with Eq. (2.35) completes the formal solution of the ultrasonic attenuation problem. Our results are exact within the limitations of semiclassical transport theory and the relaxation time ansatz.

Our goal is to use (2.44) as a basis for exploring the possible kinds of dependence on frequency and magnetic field entering the attenuation through $\boldsymbol{\sigma}'$ and $\mathbf{B}$. The derivation and analysis of convenient expressions for the conductivity components are carried out in the next section, and the classification of the field and frequency dependences entering $\mathbf{S}$ is deferred to Secs. 4 and 5. However, it is possible to delimit here the dependences entering $\mathbf{S}$ primarily through $\mathbf{B}$. By combining (2.27) and (2.30) we obtain

$$\mathbf{j}=\mathbf{B}\cdot\mathbf{W}\cdot N_0 e\mathbf{u} \tag{2.45}$$

for the relation between the total current and the current of positive charge. The latter is screened by the electron current when $\mathbf{j}$ is much smaller than $N_0 e\mathbf{u}$. From (2.42) the conditions for screening are that

$$\text{(a)}\ |B_{ii}|\ll 1, \quad \text{(b)}\ |B_{ii}|\ll|\sigma_{ii}'|.$$

If either or both of these is violated, the screening is incomplete. The meaning of condition (a) is that screening occurs when the electric fields generated by the positive ion motion are strong enough to force the electrons to follow. This is not possible at frequencies sufficiently high that $|B_{ii}|\gg 1$, the high-frequency limit. On the other hand, (b) means that screening occurs when the conductivity is large; i.e., when a large electron current results from a given electric field. It is shown in the following that $\boldsymbol{\sigma}$ and hence $\boldsymbol{\sigma}'$ becomes small at high fields; that is, $|B_{ii}|\gg|\sigma_{ii}'|$ in the high field limit, where screening is incomplete.

For transverse waves, $i=2$ or 3, $|B_{ii}|$ is simply the β of Eq. (2.28) and equals $\frac{1}{2}\pi^2$ times the square of the ratio of the classical skin depth to the phonon wavelength. In copper, such as that used by Morse *et al.*,[9,10] β is 10^{-4} at frequencies of order tens of megacycles. Thus condition (2) would be satisfied up to frequencies of order 10^5 megacycles, or, in high resistivity metals, to frequencies correspondingly lower. Inasmuch as ultrasonic frequencies of order 10^4 megacycles are now attainable,[26] we shall treat both the low-frequency limit, $\beta\ll 1$, and the high-frequency limit, $\beta\gg 1$, in Secs. 4 and 5. For longitudinal waves, however, $|B_{11}|=\gamma$ is practically always negligible so that (2.44) may be simplified to

$$S_{11}=\mathrm{Re}[(\boldsymbol{\sigma}'+\mathbf{B})^{-1}]_{11}-1. \tag{2.46}$$

Similarly, the high-field limits $\beta\gg\sigma_{22}'$, σ_{33}' are attainable for the transverse components, whereas that for the longitudinal component $\beta(v_s/c)^2\gg\sigma_{11}'$ is both beyond reach and in the quantum limit. Nevertheless, we include the high-field limit because of the rather unexpected nature of the results.

[26] K. N. Baranskii, Doklady Akad. Nauk S.S.S.R. **114**, 517 (1957) [transalation: Soviet Phys.-Doklady **2**, 237 (1957)]; H. E. Bömmel and K. Dransfeld, Phys. Rev. Letters **1**, 234 (1958); **2**, 298 (1959); **3**, 83 (1959). E. H. Jacobsen, Phys. Rev. Letters **2**, 249 (1959).

III. CONDUCTIVITY TENSOR

The magnetic field dependence of the conductivity tensor is implicit in the integral expressions (2.13) and (2.15). The present task is to exhibit this dependence explicitly in a convenient form by evaluation of the integrals. We first note that for arbitrary $g(\mathbf{v})$

$$\int g(\mathbf{v})\left(-\frac{\partial f_0}{\partial E}\right)d\mathbf{v}=\frac{3N_0}{8\pi E_F{}^0}\int d\Omega\, g(\mathbf{v}_F), \tag{3.1}$$

so that $\mathbf{J}$ and K need only be evaluated for $v=v_F$. We choose a coordinate system having z along $\mathbf{H}$, x in the plane of $\mathbf{q}$ and $\mathbf{H}$ and y coinciding with our previous axis 2, Fig. 2. The relation between $(\mathbf{r},\mathbf{v})$ and $(\mathbf{r}',\mathbf{v}')$ is

$$\begin{aligned}
&v_x'=v_\perp\cos[\omega_c(t'-t)+\phi],\\
&x'=x+(v_\perp/\omega_c)\{\sin[\omega_c(t'-t)+\phi]-\sin\phi\},\\
&v_y'=v_\perp\sin[\omega_c(t'-t)+\phi],\\
&y'=y-(v_\perp/\omega_c)\{\cos[\omega_c(t'-t)+\phi]-\cos\phi\},\\
&v_z'=v_z=v\cos\theta,\quad z'=z+v_z(t'-t),\\
&v_\perp=v\sin\theta,
\end{aligned} \tag{3.2}$$

in this system, where

$$\omega_c=eH/mc \tag{3.3}$$

is the cyclotron frequency and θ and ϕ are the polar angles of $\mathbf{v}$. Substituting this expression of the kinetics into (2.13), we obtain

$$(\mathbf{J}(\mathbf{v}_F),K(\mathbf{v}_F))$$
$$=\exp[-iX\sin\theta\sin\phi]\begin{pmatrix}(iev_F/X)\partial/\partial\phi\\ iev_F\partial/\partial X,\ 1\\ -ev_F\cos\theta\end{pmatrix}\mathcal{J}(X,\theta,\phi), \tag{3.4}$$

where

$$\mathcal{J}(X,\theta,\phi)=\int_0^\infty ds\,\exp\{-i[X\sin\theta\sin(\omega_c s-\phi) +q_z v_F\cos\theta\, s-\omega s]-s/\tau\} \tag{3.5}$$

after writing X for $q_x v_F/\omega_c$ and replacing $t-t'$ by s. Note that $R=v_F/\omega_c$ is the orbital radius of an electron moving perpendicular to $\mathbf{H}_0$ with the Fermi velocity. The integral over s in (3.5) is readily performed with the aid of[27]

$$e^{iz\sin\psi}=\sum_{n=-\infty}^{\infty}J_n(z)e^{in\psi}; \tag{3.6}$$

$$\mathcal{J}=\tau\sum_{n=-\infty}^{\infty}\frac{J_n(X\sin\theta)e^{in\phi}}{1+i(n\omega_c+q_z v_F\cos\theta-\omega)\tau}. \tag{3.7}$$

[27] Eredelyi, Magnus, Oberhettinger, and Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, New York, 1953), Vol. 2, p. 7.

The results obtained thus far permit us to write the conductivity tensor $\boldsymbol{\sigma}$ in the form

$$\boldsymbol{\sigma}=\frac{3\sigma_0}{4\pi}\sum_n\int d\Omega\left[\begin{Bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{Bmatrix}e^{-iX\sin\theta\sin\phi}\right.$$
$$\left.\times\begin{Bmatrix}(-i/X)\partial/\partial\phi\\ -i\partial/\partial X\\ \cos\theta\end{Bmatrix}\frac{J_n(X\sin\theta)e^{in\phi}}{1+i(n\omega_c-\omega+q_zv_F\cos\theta)\tau}\right]. \quad (3.8)$$

The integration over ϕ can be carried out by repeating the procedures used to obtain (3.4) and (3.8), and we arrive at the following useful expansion of the conductivity tensor

$$\boldsymbol{\sigma}=3\sigma_0\sum_{n=-\infty}^{\infty}\frac{1}{2}\int_0^{\pi}\left[\begin{bmatrix}n/X\\ i\partial/\partial X\\ \cos\theta\end{bmatrix}J_n(X\sin\theta)\right]$$
$$\times\left[\begin{bmatrix}n/X\\ -i\partial/\partial X\\ \cos\theta\end{bmatrix}J_n(X\sin\theta)\right]$$
$$\times\frac{\sin\theta d\theta}{1+i(n\omega_c-\omega+q_zv_F\cos\theta)\tau}. \quad (3.9)$$

An important property of the conductivity tensor is displayed by (3.9).

$$\sigma_{ij}(\mathbf{H})=(-1)^a\sigma_{ji}(\mathbf{H}), \quad (3.10)$$

where "a" is the number of times y appears among the indices i and j. This interchange relation implies that the components of $\boldsymbol{\sigma}(\mathbf{H})$ satisfy the same Onsager reciprocity relations for finite $\mathbf{q}$ and ω as they do for vanishing $\mathbf{q}$ and ω. The system as a whole is invariant to reflection in the x-z plane and simultaneous change in sign of $\mathbf{H}$, with the consequence that[28]

$$\sigma_{ij}(\mathbf{H})=(-1)^a\sigma_{ij}(-\mathbf{H}). \quad (3.11)$$

Combining (3.10) and (3.11) leads to the Onsager relation

$$\sigma_{ij}(\mathbf{H})=\sigma_{ji}(-\mathbf{H}), \quad (3.12)$$

as stated. Equation (3.12) applies to $\Sigma_{ij}(\mathbf{r}-\mathbf{r}',t-t')$ as well, so that we have deduced by detailed calculation that the Onsager relations apply to nonlocal, time-dependent phenomena. Kubo's derivation of the Onsager relations[25] was limited to the case $\mathbf{q}=0$.

Inspection of (3.9) or the earlier Eq. (2.13) shows that $\boldsymbol{\sigma}=\boldsymbol{\sigma}_1-i\boldsymbol{\sigma}_2$ has no poles in the upper half of the complex frequency plane and vanishes at infinite frequency, from which it follows that the Kramers-Kronig relations

$$\boldsymbol{\sigma}_{1,2}=\mp\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{-\boldsymbol{\sigma}_{2,1}(\omega')d\omega'}{\omega'-\omega} \quad (3.13)$$

hold for finite $\mathbf{q}$ and $\mathbf{H}$. Simple dispersion relations for the $\mathbf{H}$ dependence of $\boldsymbol{\sigma}$ appear to hold only for $\mathbf{q}=0$ but ω still arbitrary, when they are identical to those found by McClure[29] for $\mathbf{q}$ and ω both zero.

IV. PROPAGATION PERPENDICULAR TO FIELD

It is convenient at this point to specialize to particular directions of propagation in order to simplify the pertinent expressions. $\boldsymbol{\sigma}$ may be written explicitly for $\mathbf{q}\perp\mathbf{H}$. In this case our two coordinate systems coincide with $1\leftrightarrow x$, $2\leftrightarrow y$, and $3\leftrightarrow z$.

$$\begin{aligned}
\sigma_{11}&=\frac{3\sigma_0}{q^2l^2}(1-i\omega\tau)\left[1-\sum_{n=-\infty}^{\infty}\frac{(1-i\omega\tau)g_n(X)}{1+i(n\omega_c-\omega)\tau}\right],\\
\sigma_{22}&=3\sigma_0\sum_{n=-\infty}^{\infty}\frac{s_n(X)}{1+i(n\omega_c-\omega)\tau},\\
\sigma_{12}&=-\sigma_{21}=\frac{3\sigma_0}{2ql}\sum_{n=-\infty}^{\infty}\frac{(1-i\omega\tau)g_n'(X)}{1+i(n\omega_c-\omega\tau)},\\
\sigma_{33}&=3\sigma_0\sum_{n=-\infty}^{\infty}\frac{r_n(X)}{1+i(n\omega_c-\omega)\tau},\\
\sigma_{13}&=\sigma_{31}=\sigma_{23}=\sigma_{32}=0.
\end{aligned} \quad (4.1)$$

Here

$$\begin{aligned}
g_n(X)&=\int_0^{\pi/2}J_n^2(X\sin\theta)\sin\theta d\theta;\\
g_n'(X)&=(d/dX)g_n(X);\\
s_n(X)&=\int_0^{\pi/2}[J_n'(X\sin\theta)]^2\sin^3\theta d\theta;\\
r_n(X)&=\int_0^{\pi/2}J_n^2(X\sin\theta)\cos^2\theta\sin\theta d\theta.
\end{aligned} \quad (4.2)$$

The expressions in the first column have been rewritten from the form:

$$\sigma_{11}=\frac{3\sigma_0}{X^2}\sum_{n=-\infty}^{\infty}\frac{n^2g_n(X)}{1+i(n\omega_c-\omega)\tau},$$
$$\sigma_{12}=-\frac{3\sigma_0 i}{2X}\sum_{n=-\infty}^{\infty}\frac{ng_n'(X)}{1+i(n\omega_c-\omega)\tau}.$$

which is obtained directly from (3.9). This was done by noting that $\sum_{n=-\infty}^{\infty}g_n(X)=1$, which follows from the relation $\sum_{n=-\infty}^{\infty}J_n^2(X)=1$. We also make use of the fact that $g_n(X)=g_{-n}(X)$ and that $X\omega_c\tau=ql$. The evaluation of the functions g_n, g_n', s_n, and r_n is discussed in Appendix I.

[28] Detailed examination of (3.9) leads to the same conclusion as this symmetry argument.

[29] J. W. McClure, Phys. Rev. **112**, 715 (1958).

We may now write the three diagonal components of S [Eq. (2.44)], in terms of the nonvanishing components of $\boldsymbol{\sigma}'$:

$$S_{11}=\mathrm{Re}\left[\frac{\sigma_{22}'+i\beta}{\sigma_{11}'\sigma_{22}'+(\sigma_{12}')^2+i\beta\sigma_{11}'}\right]-1, \quad (4.3)$$

$$S_{22}=\mathrm{Re}\left[\frac{(1+i\beta)^2}{\sigma_{22}'+i\beta+(\sigma_{12}')^2/\sigma_{11}'}\right]-1, \quad (4.4)$$

$$S_{33}=\mathrm{Re}\left[\frac{(1+i\beta)^2}{\sigma_{33}'+i\beta}\right]-1. \quad (4.5)$$

In deriving (4.3) and (4.4) we have used the fact that $|\sigma_{11}'|\gg\gamma$ for all attainable fields and frequencies.

We now need only evaluate the tensor $\boldsymbol{\sigma}'$ for any case of interest and substitute into the above expressions.

A. Geometric Resonances

We expect geometric resonances when the phonon wavelength is of the order of the classical orbit radius; i.e., when X is of order unity. In this case ω_c is much greater than ω (by a factor of order v_F/v_s), and if in addition $|\omega_c\tau/(1-i\omega\tau)|^2\gg1$, we need keep only the $n=0$ terms in the summations appearing in (4.1). The resulting conductivity tensors become:

$$\sigma_{11}'=\frac{-3i\omega\tau(1-i\omega\tau)[1-g_0(X)]}{q^2l^2[1-i\omega\tau-g_0(X)]};$$

$$\sigma_{12}'=-\sigma_{21}'=\frac{-3i\omega\tau g_0'(X)}{2ql[1-i\omega\tau-g_0(X)]};$$

$$\sigma_{22}'=\frac{3}{1-i\omega\tau}\left(s_0+\frac{[g_0'(X)/2]^2}{1-i\omega\tau-g_0(X)}\right); \quad (4.6)$$

$$\sigma_{33}'=\frac{3r_0(X)}{1-i\omega\tau}.$$

The appearance of the Bessel function in the integrals g_0, r_0, and s_0 leads to oscillatory dependence of the components of $\boldsymbol{\sigma}'$ on $\mathbf{H}$. If the condition $|\omega_c\tau/(1-i\omega\tau)|^2\gg1$ is not well satisfied, terms with $|n|$ higher than zero enter. That these terms of higher n tend to wash out the oscillations can be seen from consideration of the relation $\sum_{n=-\infty}^{+\infty}J_n^2(z)=1$ and the slow variation of the frequency denominator with field in the range where $\omega_c\gg\omega$.

With Eq. (4.6) substituted into (4.3), (4.4), and (4.5) we obtain the attenuation when $|\beta(\sigma_0/\sigma_{22})|\ll1$.

$$S_{11}=\frac{q^2l^2}{3(1+\omega^2\tau^2)}\left(\frac{1}{1-g_0+(g_0'/2)^2/s_0}-1\right). \quad (4.7)$$

$$S_{22}=\left(\frac{1}{3[s_0+(g_0'/2)^2/(1-g_0)]}-1\right). \quad (4.8)$$

$$S_{33}=[(1/3r_0)-1]. \quad (4.9)$$

For S_{11} an additional term was obtained which was smaller by a factor of order $1/q^2l^2=1/(X^2\omega_c^2\tau^2)$. For our approximation $X\sim1$ and $\omega_c\tau\gg1$, so this is dropped.

The expression for S_{33} agrees with Eq. (50) of Rodriguez[16] when $ql\gg1$. Rodriguez has plotted this attenuation for several values of ql but did not display the ripples which are apparent in our plot for large ql.

These three results are plotted in Figs. 3 and 4. Note that the abscissa is proportional to the reciprocal of the magnetic field. We have plotted only the field-dependent factor, which is contained in brackets in the above expressions. These are independent of parameters associated with the material measured provided always that $\omega_c\tau\gg1$, $ql\gg1$, and $\beta\ll1$. Note the strong oscillations in S_{11} and S_{22}. The maxima and minima in S_{11} occur when g_0' vanishes. Positions of the first few extrema in S_{11} are given in Table I. Both S_{22} and S_{33} drop to zero at fields sufficiently high that $qR=X\ll1$ but not so high that the condition $\beta\sigma_0/|\sigma_{22}|\ll1$ is violated. For this same range of fields the field-dependent factor in S_{11} saturates at the value 1/5, a result which obtains even when ql is not large. Combining this result with Pippard's result for the

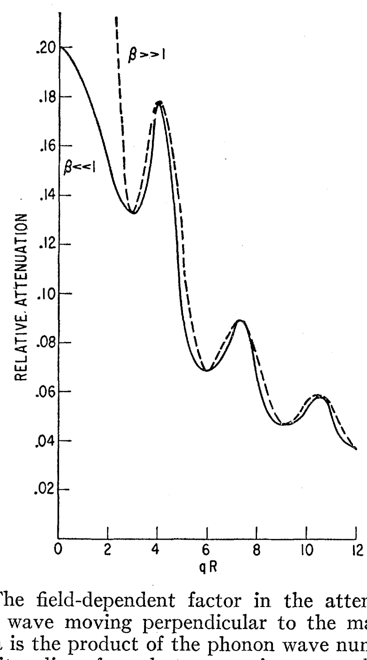

FIG. 3. The field-dependent factor in the attenuation of a longitudinal wave moving perpendicular to the magnetic field. The abscissa is the product of the phonon wave number and the classical orbit radius of an electron moving perpendicular to the magnetic field at the Fermi velocity. qR is inversely proportional to the field. The attenuation is obtained by multiplying the relative attenuation by $(Nm/\rho v_s\tau)[q^2l^2/3(1+\omega^2\tau^2)]$. The curve $\beta\ll1$ applies when the classical skin depth is much less than the wavelength; the curve $\beta\gg1$, when it is much greater. In both cases the cyclotron frequency is taken to be much greater than the electron scattering frequency.

zero field attenuation,[6] we see that the saturation value is larger than the zero field value when $ql>5\pi/2$ and smaller when $ql<5\pi/2$.

The parameter $\beta\sigma_0/|\sigma_{22}|$ can become greater than unity at fields in the geometric resonance range if the frequency is sufficiently high. The condition for this is that β itself be large, $\beta\gg1$. Under these circumstances substitution of the $\boldsymbol{\sigma}'$ of (4.6) into (4.3) to (4.5) yields

$$S_{11}=\frac{q^2l^2}{3(1+\omega^2\tau^2)}\left[\frac{g_0}{1-g_0}\right]. \tag{4.10}$$

$$S_{22}=1-\frac{3}{1+\omega^2\tau^2}\left(s_0+\frac{(g_0'/2)^2}{1-g_0}\right)\left(1+\frac{2\omega\tau}{\beta}\right). \tag{4.11}$$

$$S_{33}=1-\frac{3r_0}{1+\omega^2\tau^2}\left(1+\frac{2\omega\tau}{\beta}\right). \tag{4.12}$$

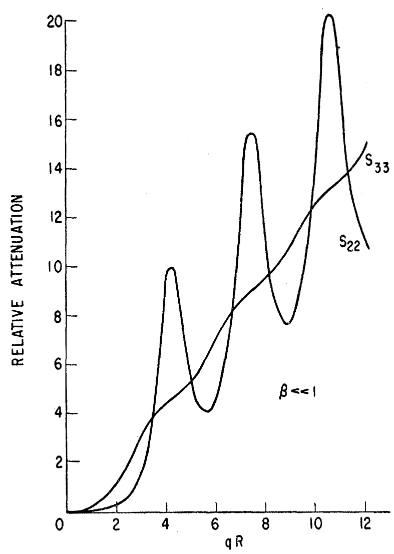

FIG. 4. The field-dependent factor in the attenuation of a transverse wave propagating perpendicular to the field, when the classical skin depth is much less than the wavelength and the cyclotron frequency is much greater than the electron scattering frequency. The abscissa is the product of phonon wave number and classical orbit radius. S_{22} corresponds to polarization perpendicular to the field; S_{33} corresponds to polarization parallel to the field. The attenuation is obtained by multiplying S_{ii} by $Nm/\rho v_s\tau$.

S_{11} again shows oscillations, the extrema of which coincide with those of the low-frequency case as displayed in Fig. 3. If $\omega\tau$ can remain small while β becomes large, S_{22} and S_{33} show oscillatory behavior as displayed in Fig. 5. Otherwise, the field dependent terms become small, and S_{22} and S_{33} equal one. Provided τ is less than 5×10^{-12} for a metal like copper, the conditions for oscillation can be met. When τ has the value 5×10^{-12}, the conditions are satisfied for $\omega=2\times10^{11}$. As τ decreases, the value of ω required decreases in proportion. To bring ω down to values such that X is

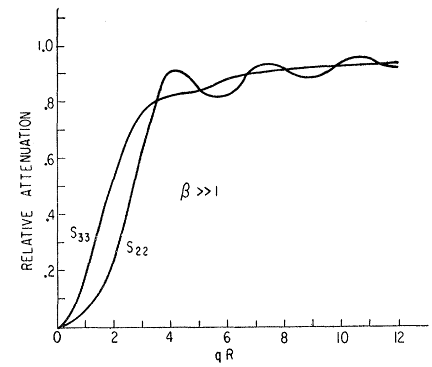

FIG. 5. The field-dependent factor in the attenuation of transverse waves propagating perpendicular to the magnetic field when the classical skin depth is much larger than the wavelength, but the frequency of the wave is much less than the electron scattering frequency. The cyclotron frequency is taken to be much greater than the electron scattering frequency. S_{22} corresponds to waves polarized perpendicular to the field; S_{33}, to waves polarized parallel to it. The attenuation is obtained by multiplying by $Nm/\rho v_s\tau$.

of order unity for attainable magnetic fields (10^5 gauss), τ would have to be lowered to about 10^{-13}.

B. Cyclotron Resonance and Low-Field Limit

We expect cyclotron resonance effects when the phonon frequency is of the order of the cyclotron frequency.[20] In these circumstances the frequency denominators in the conductivity tensors (4.1) can become small, and the possibility of oscillatory behavior arises. Under this condition X will be very large, since $X=(v_F/v_S)(\omega/\omega_c)$. Thus it will be convenient to take the asymptotic form for the tensors in (4.1). In Appendix II it is shown that $g_n(X)\cong1/(2X)+O(X^{-\frac{3}{2}})$, $r_n(X)\cong s_n(X)\cong1/(4X)+O(X^{-\frac{3}{2}})$, and $g_n'(X)\cong O(X^{-\frac{3}{2}})$. These expressions are valid only for $X>n$; when n exceeds X, the $g_n(X)$ become small. Hence, if we take the asymptotic form for $g_n(X)$, etc., in evaluating (16), we make an error of the form of the final term in the following equation:

$$\sum_{n=-\infty}^{\infty}\frac{g_n(X)}{1+i(n\omega_c-\omega)\tau}=\frac{1}{2X}\sum_{n=-\infty}^{\infty}\frac{1}{1+i(n\omega_c-\omega)\tau}-O\left[\frac{1}{2X}\sum_{n=X}^{\infty}\frac{2(1-i\omega\tau)}{(1-i\omega\tau)^2+n^2\omega_c^2\tau^2}\right].$$

TABLE I. Magnitude of X at Extrema of S_{11}.

Maxima	Minima
0	2.94
4.04	6.04
7.27	9.16
10.45	12.28
13.61	15.41
16.77	18.55
19.92	21.68
23.07	24.82

The last term may be estimated by replacing the summation by an integration over n and the term is found to be of the order of $1/q^2l^2$ whereas the first is of order $1/ql$. We are interested in the case $\omega\tau$ large; hence, we may take ql to be very large and retain only the first term which may be evaluated directly, noting that[30]

$$\sum_{n=-\infty}^{\infty}\frac{1}{1+i(n\omega_c-\omega)\tau}=\frac{z}{1-i\omega\tau}\left[\frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+n^2\pi^2}\right]=\frac{z\coth z}{1-i\omega\tau},$$

where

$$z=(1-i\omega\tau)\pi/\omega_c\tau=(1-i\omega\tau)\pi X/ql. \quad (4.13)$$

Then, using (2.41), we obtain the limiting expressions for the $\boldsymbol{\sigma}'$.

$$\begin{aligned}\sigma_{11}'&=\frac{3}{q^2l^2}\left[\frac{\pi\omega^2\tau^2\coth z}{2ql}-i\omega\tau\right];\\ \sigma_{12}'&=-\sigma_{21}'=O[X^{-\frac{3}{2}}(ql)^{-1}];\\ \sigma_{22}'&=\sigma_{33}'=3\pi/(4ql)\coth z.\end{aligned} \quad (4.14)$$

In the expressions (4.3) and (4.4) for S_{11} and S_{22}, $\sigma_{12}'\sigma_{21}'$ will always be negligible in comparison to the terms we are keeping in $\sigma_{11}'\sigma_{22}'$. Noting also that the expressions for the transverse waves become identical, Eqs. (4.3), (4.4), and (4.5) for $ql\gg1$ become

$$S_{11}=\mathrm{Re}[(1/\sigma_{11}')-1], \quad (4.15)$$

$$S_{22}=S_{33}=\mathrm{Re}\left[\frac{(1+i\beta)^2}{\sigma_{33}'+i\beta}-1\right]. \quad (4.16)$$

We note that as the field goes to zero, $z=(1-i\omega\tau)\pi/\omega_c\tau$ becomes infinite and $\coth z$ and $\tanh z$ approach 1. The above expressions then become the same as those derived for zero-field by Pippard[6] for the case $ql\gg1$.

For the magnitudes of field of interest in cyclotron resonance, $\coth z$ will be of order unity. Thus we may drop terms of order $\omega\tau(\coth z)/ql$ and terms of order $1/(ql\coth z)$ in comparison to terms of order unity. Then (4.15) becomes

$$S_{11}=\frac{\pi ql}{6}\,\mathrm{Re}\coth z. \quad (4.17)$$

This expression is displayed in Fig. 6 for $\omega\tau=10$. As $\omega\tau$ is decreased, the oscillations are damped out for lower ω/ω_c.

Similarly, (4.16) becomes simply $S_{22}=S_{33}=1$ unless β is sufficiently small that terms of order $\coth z/ql\beta$ and $\coth z/ql\beta^2$ become appreciable. In this case, the last

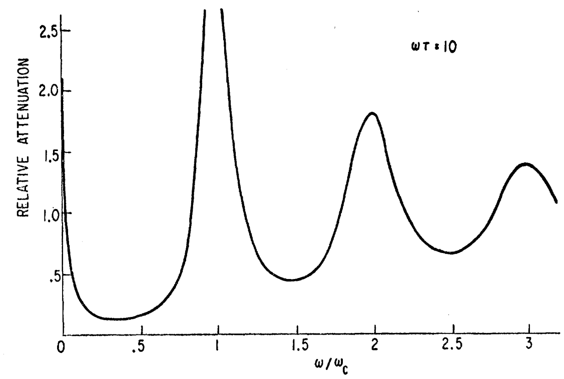

FIG. 6. The ratio of the attenuation of longitudinal waves as a function of magnetic field to that at zero field as a function of the ratio of phonon frequency to cyclotron frequency. ω/ω_c is inversely proportional to the magnetic field and is numerically equal to $qR(v_s/v_F)$. The product of the phonon frequency and the electron scattering time, $\omega\tau$, is taken equal to ten.

of these will be dominant and (4.16) becomes

$$S_{22}=S_{33}=1+(3\pi/ql\beta^2)\,\mathrm{Re}\coth z, \quad (4.18)$$

showing resonances similar to those for longitudinal waves. Such a case would be obtainable in copper if the scattering time were sufficiently long that the cyclotron resonance were observable.

C. High-Field Limit

The conductivity components (4.1) have the limiting behavior

$$\begin{aligned}\sigma_{11}&=\frac{\sigma_0(1-i\omega\tau)}{(\omega_c\tau)^2},\\ \sigma_{22}&=\frac{\sigma_0}{(\omega_c\tau)^2}\left[\frac{2}{5}\frac{(ql)^2}{1-i\omega\tau}+(1-i\omega\tau)\right],\\ \sigma_{21}&=-\sigma_{12}=\sigma_0/\omega_c\tau,\quad \sigma_{33}=\sigma_0/(1-i\omega\tau),\end{aligned} \quad (4.19)$$

leading to

$$\begin{aligned}\sigma_{11}'&=\frac{1-i\omega\tau}{(\omega_c\tau)^2+i\frac{1}{3}(ql)^2/\omega\tau},\\ \sigma_{22}'&=(\sigma_{22}/\sigma_0)+i\tfrac{1}{3}(ql)^2/\omega\tau(1-i\omega\tau)\\ &\quad\times[(\omega_c\tau)^2+i\tfrac{1}{3}(ql)^2/\omega\tau]^{-1},\\ \sigma_{12}'&=-\sigma_{21}'=\frac{-\omega_c\tau}{(\omega_c\tau)^2+i\frac{1}{3}(ql)^2/\omega\tau},\\ \sigma_{33}'&=1/(1-i\omega\tau),\end{aligned} \quad (4.20)$$

at fields high enough that $\omega_c\tau\gtrsim10ql$. Substitution of (4.20) into (4.4) and (4.5) yields

$$S_{22}=S_{33}=\frac{(\omega\tau)^2}{1+(\omega\tau+1/\beta)^2}, \quad (4.21)$$

[30] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1950), fourth edition, p. 136.

for the high-field limits of the transverse cases, a more precise result than that obtained above when terms of the order of (4.21) were omitted. The attenuation of longitudinal waves at very high fields is somewhat more complex. In the low-frequency limit $\beta \ll 1$, S_{11} first saturates at $(1/15)(ql)^2/[1+(\omega\tau)^2]$ provided that $(\omega_c\tau)^2 \gg \frac{1}{3}(ql)^2/\omega\tau$. Then, as $(\omega_c\tau)$ becomes comparable to $1/\beta$ a term $\beta^2(\omega_c\tau)^2$ appears in S_{11}. In the high-frequency limit $\beta \gg 1$, on the other hand, no saturation occurs, and one passes from geometric resonance directly to an H^2 dependence: $S_{11}=(\omega_c\tau)^2/[1+(\omega\tau)^2]$. In both cases, the attenuation continues to increase as H^2 for all attainable fields. However, if one imagined fields sufficiently large that $\gamma(\omega_c\tau)^2$ is of order unity, the H^2 dependence would pass into

$$S_{11}=\frac{(\omega_c\tau)^2}{1+[(\beta^{-1}+\omega\tau)+\gamma(\omega_c\tau)^2]^2}. \tag{4.22}$$

Equation (4.22) shows that there is an enormous broad maximum in the attenuation when

$$\omega_c=\omega_p[1+(\omega_p/cq)^2]^{\frac{1}{2}}, \tag{4.23}$$

of width in ω_c comparable to the value at maximum. Here ω_p is the plasma frequency, $\omega_p^2=4\pi N_0 e^2/m$. The maximum value of S_{11} is $\frac{1}{4}(\omega_p/\omega)^2/[1+(\omega_p/cq)^2]$ and is of order 10^{10} when $1/\beta \gg \omega\tau$. Finally, at fields beyond the maximum, S_{11} saturates again at unity.

The high-field limits for the transverse waves occur at fields both experimentally attainable and at which the present theory applies. For the longitudinal waves, however, the high-field maximum just described occurs at unattainably high fields for ordinary metals, which fields are also well beyond the limits of validity of the semiclassical transport theory used here. A completely quantum mechanical treatment of the magneto-plasma resonance is required. Further, the attenuation is so enormous that from the Kramers-Kronig dispersion relations it follows that the elastic constants must depend strongly on magnetic field when $\beta(\omega_c\tau) \gg 1$. The amplitude of the maximum may decrease by orders of magnitude when this effect on the sound velocity is included. Our calculation, therefore, is quite unrealistic, and our purpose in reporting it is solely to call attention to the possibility of such effects in metals. Moreover, in semimetals with low effective masses and low carrier densities, such behavior may well occur at attainable fields.

V. PROPAGATION PARALLEL TO THE FIELD

$\boldsymbol{\sigma}$ for this case may be obtained from (3.9) by taking the limit as X goes to zero. In addition, it is convenient to let the transverse currents and fields be circularly polarized in the x-y plane, following the treatment of Kjeldaas.[15] The correspondence between the two coordinate systems is now $1 \leftrightarrow z$, $2 \leftrightarrow y$, and $3 \leftrightarrow -x$. We obtain

$$\sigma_\pm'=\sigma_\pm/\sigma_0=(\sigma_{33}\pm i\sigma_{23})/\sigma_0\equiv G^\pm,$$

where

$$G^\pm=\frac{3}{4}\int_0^\pi \frac{\sin^3\theta d\theta}{1+i(\pm\omega_c+qv_F\cos\theta-\omega)\tau}. \tag{5.1}$$

$$\sigma_{11}'=\frac{\frac{3}{2}\int_0^\pi \frac{\cos^2\theta \sin\theta d\theta}{1+i(ql\cos\theta-\omega\tau)}}{1-(ql/\omega\tau)\frac{1}{2}\int_0^\pi \frac{\cos\theta \sin\theta d\theta}{1+i(ql\cos\theta-\omega\tau)}}. \tag{5.2}$$

The quantity $G^\pm$ is precisely that obtained by Kjeldaas. We note that $|G^\pm|<1$ and that $G^\pm \to 0$ as the magnetic fields becomes infinite.

For the transverse wave we obtain for the attenuation,

$$S_\pm=\mathrm{Re}\left[\frac{(1+i\beta)^2}{G^\pm+i\beta}-1\right]. \tag{5.3}$$

For the case treated by Kjeldaas, for which β can be neglected, we obtain his result,

$$S_\pm=\mathrm{Re}[(1/G^\pm)-1] \quad \text{for} \quad \beta\ll 1, \quad \beta \ll |G^\pm|. \tag{5.4}$$

It should be noted that the condition $\beta \ll |G^\pm|$ will always break down at sufficiently high magnetic fields since $G^\pm$ can be made arbitrarily small by increasing the field. Thus the equation (5.4) should be supplemented with a high-field limit. We take the high-field limit for $G^\pm$, which is obtained from (5.1) by letting $\omega_c\tau$ become large.

$$G^\pm \approx 1/(1\pm i\omega_c\tau).$$

This may be substituted into (5.3) to obtain

$$S_\pm=\frac{(\beta\omega_c\tau)^2}{(1\mp\beta\omega_c\tau)^2+\beta^2}; \quad \beta\ll 1, \quad \text{high fields.} \tag{5.5}$$

We see that after the initial drop in attenuation discussed by Kjeldaas, the attenuation again begins to rise, reaches a peak value for S_+ of $\omega_c^2\tau^2$ when the field is sufficiently high that $\omega_c\tau$ is equal to about $1/\beta$ and then drops to a limiting value of unity. S_-, on the other hand, asymptotically increases towards unity.

The condition $\beta \ll |G^\pm|$ will also break down at sufficiently high frequencies; thus when $\omega\tau$ becomes large, $G^\pm$ becomes small, and S approaches the constant value unity and becomes independent of field.

These departures from the behavior discussed by Kjeldaas arise from the breakdown of the screening of the ionic current; as $\mathbf{H}$ becomes very large, the electronic current begins to lag the ionic current, causing a rising attenuation. Ultimately these currents drop to zero so the attenuation drops to the residual value which may be associated with the viscous drag of the electron gas on the scattering centers. The high-frequency behavior results from the breakdown of screening at all fields. When $\beta \gg 1$, we obtain

$$S_{\pm} = \mathrm{Re}(1 - G^{\pm}); \quad \beta \gg 1. \tag{5.6}$$

In this case S increases smoothly to unity as the field goes to infinity. This amounts to only a very slight increase in attenuation if ql is large, but as ql goes to zero, the zero-field attenuation (more precisely, the attenuation at fields such that $\pm\omega_c = \omega$) goes to zero.

It should perhaps also be noted at this point that the zero-field limit for transverse waves can be obtained directly from the above expressions. We may set $\omega_c = 0$ in the expression for $G^{\pm}$ and if we drop terms of order $\omega\tau/ql = v_s/v_F$, we obtain

$$G = \frac{3}{2(ql)^2}\left(\frac{1+q^2l^2}{q^2l^2}\arctan ql - 1\right), \tag{5.7}$$

and (5.4) and (5.6) lead to the corresponding expressions (23) and (26) of Pippard.[6] These also agree with our expressions (4.17) and (4.18) for the zero-field limit. In addition, we have the high-frequency limit of unity for S when $\omega\tau \ll 1$, in agreement with Pippard.

Finally we consider the case of a longitudinal wave propagating parallel to the magnetic field. In this case the field does not enter, and we reproduce the results of Pippard.[6] From equation (2.46) we find that

$$S_{11} = \mathrm{Re}[(1/\sigma_{33}') - 1].$$

We then obtain, in agreement with Pippard, the attenuation

$$S_{11} = \tfrac{1}{3}\frac{q^2l^2 \arctan ql}{ql - \arctan ql} - 1 \tag{5.8}$$

for all values of β.

VI. DISCUSSION

An effort will be made in this section to present a coherent physical picture in terms of which the many diverse cases which have been treated may be understood. For the sake of clarity, this picture has been somewhat oversimplified; in particular, complications associated with the relative phase of currents, fields, and lattice velocities are largely overlooked, except where they become of primary importance. Furthermore, the intricacies of the screening problem will be discussed in a grosser fashion than is really warranted. Thus the picture presented should be regarded as an interpretation of the results of a careful calculation rather than an explanation of the phenomena in question. It would have been quite difficult to develop such a picture in the absence of the detailed calculations which appear in the preceding sections.

The physical effects associated with propagation parallel to the magnetic field have been covered in some detail by Kjeldaas[15] and in Sec. V, and will not be discussed further here. We restrict attention to waves propagating perpendicular to the magnetic field.

Before considering the oscillatory phenomena in detail, we should perhaps indicate again the origin of each. The geometric resonances in the attenuation are associated with the Bessel functions in the conductivity tensor. These have to do with the strength of the interaction between the particular orbits and the electric field rather than with resonant absorption of energy. In quantum-mechanical language geometric resonances correspond to variation in the matrix elements rather than in the resonance denominators also appearing in the conductivity tensor. It is the *cyclotron* resonance which corresponds to the variation of resonance denominators; i.e., to the resonant absorption of energy from the sound wave. A third oscillatory phenomenon, the de Haas-van Alphen oscillation, is associated with variations in the density of electron states at the Fermi surface. It is precisely the same phenomenon as the more familiar oscillations in magnetic susceptibility and has nothing to do directly with the interaction between the electrons and the lattice wave. This corresponds to the fact that it is the one oscillatory phenomenon which does not scale with frequency. Although Reneker has observed de Haas-van Alphen oscillations in ultrasonic attenuation,[12] we do not discuss the phenomenon here because it lies beyond the reach of semiclassical transport theory.

The geometric resonances can be understood in quite simple physical terms at least for the transverse waves polarized perpendicular to the field at low frequencies. The attenuation is given schematically by

$$\alpha = (Nm/\rho v_s\tau)\ \mathrm{Re}[(\sigma_0/\sigma_{\mathrm{eff}}) - 1], \tag{6.1}$$

where σ_{eff} stands for the appropriate combination of components of the conductivity tensor. The important point is, of course, that the conductivity appears in the *denominator*. This is because with nearly complete screening, we have a constant *current* system rather than a constant *voltage* system, the electron current being forced to equal $N_0e\mathbf{u}$. Let us now consider two values of magnetic field corresponding to (a) orbit diameters equal to a half wavelength (or an odd number of half wavelengths) and (b) orbit diameters equal to an integral number of whole wavelengths. These are illustrated in Fig. 7, where the vertical arrows correspond to the self-consistent electric field associated with the lattice wave. Since the wave frequency is much less than the cyclotron frequency, $\omega \approx (v_s/v_F)\omega_c$, we may regard the wave as fixed.

In case (a), the component of field in the direction of electron motion is always positive, and therefore the electron speed increases with each passage. On the other hand, the speed of an electron following an orbit displaced by a half wavelength decreases. In both cases, there is a significant increase in the current in phase with the field. This corresponds to a large current response, a large conductivity, and hence a low

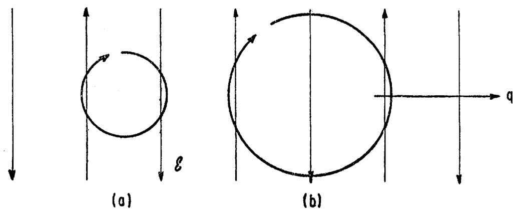

FIG. 7. Schematic representation of a transverse wave moving perpendicular to a magnetic field pointing into the plane of the paper. The polarization of the wave is also perpendicualr to the field. The case (a) represents an electron orbit corresponding to high conductivity (low attenuation); the case (b), an electron orbit corresponding to low conductivity (high attenuation).

attenuation. In case (b), the electron is alternately accelerated and decelerated and has no net increase in speed per cycle. This corresponds to a small current response, a low conductivity and a high attenuation. A geometric analysis like the present one was first proposed by Pippard,[8] who inferred from it, however, an erroneous resonance condition.

Now all electrons at the Fermi surface contribute to the attenuation, but electrons moving nearly perpenducular to the magnetic field and hence having large orbital diameter are heavily weighted. Over half the electrons, in fact, have orbit diameters within 13% of the maximum. Thus the oscillations in the attenuation through minima and maxima correspond to variations of the field which take the *maximum* orbit diameter alternately through the conditions (a) and (b) above.

When a transverse wave is polarized parallel to the field, on the other hand, the electrons with maximum orbit diameter move perpendicular to the electric fields and cannot contribute to the conductivity. Thus a much wider range of orbit diameters become important, and the oscillations tend to wash out.

An argument such as that given above applied to longitudinal waves gives no speeding or slowing of any electrons around an orbit when only the longitudinal component of the field is considered, since they all move in a conservative field. In this approximation the conductivity is zero, and it becomes essential to include the transverse electric fields self consistently. A study of the orbits then shows a nonvanishing current, but one which is out of phase with the electric fields and gives zero attenuation. It therefore becomes essential to include the effects of scattering as well, which is indicated by the appearance of the q^2l^2 factor in the attenuation of longitudinal waves. The physical picture is complicated considerably by the inclusion of scattering and will not be attempted here.

The curve for $\beta \ll 1$ in Fig. 3 and the curves in Fig. 4 show these geometric resonances for the usual situation of nearly complete screening. The location of the first few maxima are seen not to occur precisely at the values of $qR=n\pi$ indicated by the simple picture, but the change in qR between maxima is seen to approach π quite rapidly (see also Table I).

The fact that these curves are limited to the case in which the cyclotron frequency is much greater than the electron scattering frequency must be kept in mind when these curves are compared with experimental curves. This condition will always break down at sufficiently low fields; that is, at sufficiently large qR. At qR larger than this break-down value, the curves should depart from the calculated curves and go to a constant equal to the zero-field attenuation. This is illustrated by the earliest experimental curves given by Morse, Bohm, and Gavenda[9] for which the breakdown occurs at qR of the order of 2. In these curves only traces of the oscillatory behavior remain, and agreement with the theoretical curves is obtained only at the highest fields. More recent data on longitudinal waves in copper by Morse and Gavenda[10] show very good qualitative agreement with the curve in Fig. 3. Seven maxima are discernable, and the drop in attenuation with increasing qR is apparent at the high fields. It should be mentioned, however, that the period of the oscillations they observe yields smaller orbit radii than are consistent either with a free-electron sphere or the Fermi surface proposed by Pippard.[31] The source of this difficulty has not been determined. The effect of the breakdown of the condition $\omega_c\tau \gg 1$ is also nicely illustrated by the calculations of Kjeldaas and Holstein,[18] who have calculated the attenuation for the same free-electron model but for intermediate scattering times.

There is another feature of the earlier work by Morse, Bohm, and Gavenda[9] which is anomalous with respect to what has been found here. This is a prominent peak in the attenuation for the case S_{33}. It is difficult to reconcile this peak with our calculation. Further experimental work is needed to see if this peak is associated with a geometric resonance or whether it arises in some other way.

The curves associated with very large β are also of interest, although, as has been indicated earlier, they correspond to experimental conditions which have not yet been achieved. In the qualitative discussion above, it has been assumed that the ion currents are almost completely screened. This is always true for longitudinal waves except at the highest magnetic fields. (Note that the only appreciable differences between the $\beta \gg 1$ and the $\beta \ll 1$ curves in Fig. 3 occurs at very small qR.) This "quasi-neutrality" (almost complete screening) can break down for transverse waves, on the other hand, in three distinct ways: (1) At sufficiently high frequencies β becomes large and the electric fields drop to a low value simply because the wavelength becomes short, and ion currents moving in opposite directions become quite close together; (2) $\omega\tau$ can become large, and the relevant conductivity tensors become small. These two points were first made by Pippard[6]; (3) At sufficiently high fields the conductivity

[31] A. B. Pippard, Phil. Trans. Roy. Soc. **A250**, 325 (1957).

tensors become small. The third condition is related to the only condition under which quasi-neutrality is violated in the case of longitudinal waves. Under any of these three circumstances the transverse electron current tends to drop to a low value, and the attenuation becomes small. The high-field limit will be discussed later; we consider now the conditions (1) and (2), appropriate to transverse waves.

Both β and $\omega\tau$ become large as the frequency is increased; their relative magnitude depends primarily upon the scattering time. Consider first the case in which the scattering time is sufficiently short that β becomes large while $\omega\tau$ remains small. The electric field then drops to a low value, and the only observed attenuation may be associated with a viscous drag on the scattering centers by the stationary electron gas. This tends to give an attenuation independent of field, but with suitable orbit-wavelength matching it is still possible for the scattering alone to bring the electrons along with the lattice. This partially restores quasi-neutrality even though the electric field is negligible and hence reduces the attenuation associated with the viscous drag and gives rise to the rather weak oscillations seen in Fig. 5.

If, on the other hand, the scattering time is sufficiently long that $\omega\tau\gg1$, while again $\beta\gg1$, the scattering becomes ineffective in restoring quasi-neutrality, and the attenuation is simply a constant. The final case which must be considered is that appropriate to a good metal in which $\omega\tau$ becomes large before β does. This is of particular interest with regard to cyclotron resonance and will, therefore, be discussed in that connection.

In this case of cyclotron resonances, as we have indicated, variations of the interaction between the electrons and the lattice become unimportant, and we focus attention upon the energy denominator. These resonances will be observable only when $\omega\tau$ is large, so we consider only that case. For longitudinal waves, a prominent resonance absorption is found as indicated in Fig. 4. The peak absorption occurs at resonance as expected physically; the constant-current arguments which have been used in discussing the geometric resonances are not relevant, since the large currents generated are out of phase with the ionic current in resonance absorption. For transverse waves no cyclotron resonance is observed if β is large, since the relevant components of the conductivity tensor have become so small that any effect is swamped by the constant attenuation associated with viscous drag on the scattering centers. If, however, β remains sufficiently small, the electric fields remain large enough that an observable resonant absorption is superimposed on the viscous term.

We turn finally to the high field behavior of the attenuation in transverse fields. At fields such that $qR\ll1$, the electric field associated with the sound wave is effectively constant over an electron orbit. Thus for σ_{33} where the electric and magnetic fields are parallel, the conductivity takes on a value appropriate to a uniform-electric field in the absence of a magnetic field, Eq. (4.19). For the remaining components σ_{11}, σ_{12}, σ_{21}, and σ_{22}, the situation is that of crossed, uniform electric and magnetic fields. The electrons drift in a direction perpendicular to both at a rate proportional to $1/H$. Thus σ_{21} and σ_{12} are proportional to $1/H$, whereas σ_{11} and σ_{22} fall off more rapidly; i.e., as $1/H^2$.

The electronic currents are small at high fields because the components of the conductivity tensor are, and hence S_{22} and S_{33} have a small limiting value, Eq. (4.29), according to Eq. (2.26). For longitudinal waves, the longitudinal electric field produced by the ion cores is enormous, and the tendency towards complete screening is very strong. The component of $\mathbf{j}_e$ in phase with $\boldsymbol{\mathcal{E}}$ is correspondingly small. As the components of the conductivity tensor decrease with field at high fields, the magnitude of $\mathbf{j}_e$ decreases from N_0eu, but the component of $\mathbf{j}_e$ in phase with $\boldsymbol{\mathcal{E}}$ increases enormously. Finally the magnitude of $\mathbf{j}_e$ decreases to the point where the attenuation is again small.

The maximum in S_{11} at high fields can be better understood in terms of the magnetoplasma oscillations of the electron gas. The conditions under which (4.22) holds are such that diffusion is negligible, that σ_{22}' is also negligible, and that relaxation effects may be treated as perturbations. A Drude-Lorentz type of treatment is therefore indicated. We consider the electrons in a magnetic field, but without the sound wave. Their equation of motion is

$$m d\mathbf{v}/dt=-e(\boldsymbol{\mathcal{E}}+(\mathbf{v}/c)\times\mathbf{H}), \tag{6.2}$$

where $\boldsymbol{\mathcal{E}}$ now arises solely from the electronic current $-N_0e\mathbf{v}$ via Eq. (2.1). We now suppose there to be a density fluctuation of wave number $\mathbf{q}$ perpendicular to $\mathbf{H}$ and solve Eq. (6.2) to find its natural frequency. The result for the frequency of this magnetoplasma oscillation, ω_μ, is

$$\omega_\mu^2=\omega_p^2+\omega_c^2/[1+(\omega_p/cq)^2]. \tag{6.3}$$

The longitudinal electric field generated by the sound wave excites this magnetoplasma oscillation. The electronic current excited is proportional to $[\omega_\mu^2-\omega^2]^{-1}$ but out of phase with the electric field. When relaxation effects are included, an in phase component and hence an attenuation occurs; one expects a result for S_{11} much like (4.22). This becomes obvious upon rewriting S_{11} as

$$S_{11}=\frac{\omega_p^4\omega_c^2/[1+(\omega_p/cq)^2]^2\omega^2}{\omega_\mu^4+\omega_p^4/[1+(\omega_p/cq)^2]^2\omega^2\tau^2}. \tag{6.4}$$

Thus the maximum in S_{11} at high fields is associated with the field dependence of the low-frequency tail of the magneto-plasma resonance, $\omega^2\lll\omega_\mu^2$.

In passing from the free electron gas to a real metal, modifications may be expected to arise from a deformation potential which need not be the same for all electrons and from the averaging over orbits on a nonspherical Fermi surface. As has been shown in the discussion of S_{33} the averaging over orbits has a profound effect on the strength of the geometric resonances. The kinetic arguments are similar, however, and a generalization of the present work should be possible along the lines of Blount's general theory of the zero field attenuation.[32]

VII. ACKNOWLEDGMENT

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APPENDIX I

The numerical evaluation of the functions, $g_n(X)$, $s_n(X)$, and $r_n(X)$ can be greatly simplified by making use of a few mathematical identities.

First, using the relation[33]

$$\int_0^{\pi/2} J_\nu^2(z\sin\theta)\sin\theta d\theta=\frac{1}{z}\sum_{m=0}^{\infty} J_{2\nu+2m+1}(2z),$$

we may write

$$g_n(X)=\frac{1}{X}\sum_{m=0}^{\infty} J_{2m+2n+1}(2X). \qquad \text{(A1)}$$

This expression may readily be evaluated using tables of Bessel functions.[34] Furthermore, using the familiar Bessel function identity, $2J_\nu'=J_{\nu-1}-J_{\nu+1}$, we find

$$g_n'(X)=(1/X)[J_{2n}(2X)-g_n(X)]. \qquad \text{(A2)}$$

Now

$$r_n(X)=\int_0^{\pi/2} J_n^2(X\sin\theta)\cos^2\theta\sin\theta d\theta.$$

We differentiate this with respect to X and perform a partial integration to obtain

$$\frac{\partial r_n(X)}{\partial X}=\frac{1}{X}g_n(X)-\frac{3}{X}r_n(X),$$

from which

$$X^3 r_n(X)=\int_0^X x^2 g_n(x)dx.$$

(A1) is used to expand $g_n(X)$ in the integrand; we integrate term by term by parts and obtain finally:

$$r_n(X)=\frac{g_n(X)}{2}-\frac{1}{2X^3}\int_0^X x^2 J_{2n}(2x)dx.$$

TABLE II. Values of $g_0(X)$, $\frac{1}{2}g_0'(X)$, $s_0(X)$, and $r_0(X)$ for $X\leqslant 12$.

X	$g_0(X)$	$\frac{1}{2}g_0'(X)$	$s_0(X)$	$r_0(X)$
0	1	0	0	0.3333
0.5	0.9197	−0.1545	0.0315	0.3171
1	0.7129	−0.2445	0.1073	0.2734
1.5	0.4625	−0.2409	0.1822	0.2148
2	0.2562	−0.1634	0.2142	0.1567
2.5	0.1431	−0.0641	0.1891	0.1107
3	0.1177	+0.0055	0.1266	0.0815
3.5	0.1364	+0.0234	0.0642	0.0669
4	0.1531	+0.0026	0.0312	0.0608
4.5	0.1391	−0.0255	0.0330	0.0573
5	0.1067	−0.0353	0.0522	0.0529
5.5	0.0763	−0.0225	0.0653	0.0472
6	0.0645	−0.0014	0.0604	0.0416
6.5	0.0703	+0.0105	0.0421	0.0374
7	0.0800	+0.0065	0.0250	0.0350
7.5	0.0803	−0.0063	0.0203	0.0335
8	0.0688	−0.0152	0.0274	0.0320
8.5	0.0537	−0.0132	0.0366	0.0301
9	0.0452	−0.0033	0.0386	0.0279
9.5	0.04668	+0.00526	0.03128	0.0260
10	0.05291	+0.00571	0.02102	0.02464
10.5	0.05566	−0.00091	0.01568	0.02378
11	0.05102	−0.00780	0.01806	0.02302
11.5	0.04206	−0.00889	0.02419	0.02208
12	0.03536	−0.00382	0.02754	0.02097

The integral may be evaluated by substituting in Bessel's differential equation and integrating twice by parts. We obtain

$$r_n(X)=\frac{g_n(X)}{2}-\frac{1}{8X^2}J_{2n}(2X)+\frac{1}{8X}\frac{d}{dX}\times J_{2n}(2X)+\frac{(1-4n^2)}{8X^3}\int_0^X J_{2n}(2x)dx.$$

But from (A1) we may obtain[35] the relation,

$$g_n(X)=\frac{1}{X}\int_0^X J_{2n}(2x)dx, \qquad \text{(A3)}$$

so we have finally

$$r_n(X)=\frac{g_n(X)}{2}+\frac{1-4n^2}{8X^2}g_n(X)-\frac{1}{8X^2}J_{2n}(2X)+\frac{1}{8X}\frac{d}{dX}J_{2n}(2X). \qquad \text{(A4)}$$

In order to obtain $s_n(X)$, we differentiate the integral expression for $g_n(X)$ given in (4.2) twice with respect to X and use Bessel's differential equation.

$$g_n''(X)=2s_n(X)-(1/X)g_n'(X)-2[g_n(X)-r_n(X)]+(2n^2/X^2)g_n(X).$$

We then evaluate $g_n''(X)$ from (A2) and solve for $s_n(X)$

$$s_n(X)=3r_n(X)-(1-n^2/X^2)g_n(X). \qquad \text{(A5)}$$

[32] E. I. Blount, Phys. Rev. **114**, 418 (1959).

[33] See reference 27, p. 91 and errata.

[34] T. Kjeldaas has pointed out that, noting the form for $g_n(X)$ given in (A3), we may find $Xg_n(X)$ tabulated directly, at least for $n=0$.

[35] We also need a relation given in reference 27, p. 45, Eq. (3).

The numerical values for g_0, $\frac{1}{2}g_0'$, r_0, and s_0 which have been calculated by these procedures are collected in Table II.

APPENDIX II

The asymptotic formulas for $g_n(X)$, $s_n(X)$, and $r_n(X)$ may be determined from relations given in Appendix I. (A3) may be rewritten[36] in the form

$$g_n(X)=(1/2X)-(1/X)\int_X^\infty J_{2n}(2x)dx.$$

If X is somewhat greater than n, we may use the asymptotic form[37] for $J_{2n}(2x)$ so the integral is of the order of

$$\frac{1}{X}\int_X^\infty (\pi x)^{-\frac{1}{2}}\cos x dx=O(X^{-\frac{3}{2}}),$$

and

$$g_n(X)\cong(1/2X)+O(X^{-\frac{3}{2}}). \tag{A6}$$

It follows also that

$$g_n'(X)\cong O(X^{-\frac{3}{2}}). \tag{A7}$$

Further, from (A4), (A5) and (A6) we obtain

$$r_n(X)\cong s_n(X)\cong(1/4X)+O(X^{-\frac{3}{2}}), \tag{A8}$$

for X greater than n.

If n is somewhat larger than X the Bessel functions become small, and $g_n(X)$, $g_n'(X)$, $r_n(X)$, and $s_n(X)$ go to zero. It is certainly possible to derive these expressions in terms of the Weber and Lommel functions used by Kjeldaas and Holstein.[18] The foregoing method, however, has been found convenient.

[36] See reference 27, p. 92, Eq. (30).

[37] See reference 27, p. 85, Eq. (3).