equivalent to exclude the events which, upon reflection, would hit the unreflected AC counter. If the above conditions are met, each event will have a different normalizing factor, but these factors will be independent of P. The true likelihood of all of the events will then be expressed by (2′)

$$L=(\text{constant})\prod_{i=1}^{n}(1+PP_c(\theta,E)_i\sin\phi). \quad (2')$$

About 30% of the scatterings were discarded because they did not satisfy this reflection criterion.

APPENDIX II. $\pi^- - p$ POLARIZATION FORMULA

The derivation of the polarization formula for pion-proton scattering, in which only s and p orbital angular momentum states contribute, is given in reference 6. The resulting formula is given below.

If a normal right-hand set of axes, x, y, z, are defined, with the incident pion momentum along the $+z$ axis, and the recoil proton momentum in the (x,z) plane with a component in the $+x$ direction, the polarization will be given by Eq. (1″),

$$P=\frac{p(+y\text{ direction})-p(-y\text{ direction})}{p(+y\text{ direction})+p(-y\text{ direction})}, \quad (1'')$$

where the p's are probabilities of the spin being along the direction specified.

For the case of π^- mesons scattered from protons, there are two isotopic spin states, $T=\frac{3}{2}$ and $T=\frac{1}{2}$, and three total angular momentum states, $j=\frac{3}{2}$, $\frac{1}{2}$ for p waves, and $j=\frac{1}{2}$ for s waves. There are thus six phase shifts. The probability of spin-flip scattering will, in general, be different than for non-spin flip scattering, thus giving rise to a polarization,

$$P=i\frac{\sin\theta(X^*Z-XZ^*)+\sin\theta\cos\theta(Y^*Z-YZ^*)}{|X+Y\cos\theta|^2+|Z\sin\theta|^2}, \quad (2'')$$

where $X=a_3+2a_1$, $Y=(2a_{33}+a_{31})+2(2a_{13}+a_{11})$, and $Z=(a_{31}-a_{33})+2(a_{11}-a_{13})$. The a_{ij} are the p-wave scattering amplitudes, the a_i the s-wave amplitudes. In terms of the phase shifts, α_{ij},

$$a_{ij}=(1/2i)[\exp(2i\alpha_{ij})-1].$$

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Thirring Model with Variable Interaction

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The Thirring model is solved with a variable coupling constant, $\lambda=\lambda_0 f(x,t)$. It is found that the infrared divergence is eliminated if f tends to zero along the past light cone. The S matrix is no longer diagonal in the physical particle representation and is generally not well-defined in the sense of Haag's theorem. The ordered, renormalized Heisenberg operators for ψ, $\psi\,\psi^*$ are computed and production processes are analyzed by examining matrix elements.

1. INTRODUCTION

THE two-dimensional relativistic model introduced by Thirring[1] has served as a valuable tool for the exploration of the structure of quantum field theory. There are, however, two aspects of the original model that limit its usefulness. Because of the small number of dimensions, an infrared divergence is present in the wave function renormalization constant, and therefore some renormalized products of a finite number of field operators do not exist. Furthermore, the S matrix is diagonal in the physical particle representation so that creation of matter does not occur.

In this paper we discuss a modified, but still soluble, version of the Thirring model with variable coupling, $\lambda=\lambda_0 f(x,t)$. In general, the S matrix is not diagonal and for a large class of functions $f(x,t)$ no infrared divergence appears. Consequences of adiabatic variations or discontinuous changes in λ may also be examined with the extended model.

Section 2 contains a discussion of the equations of motion, their operator solutions and the construction of state vectors. Although energy and momentum are not conserved, the particle-number operators remain diagonal. It is shown that the S matrix is identical to the U matrix for certain forms of f, suggesting that the former is ill-defined in the sense of Haag's theorem.

In Sec. 3, the Heisenberg operators for ψ and $\psi\psi^*$ are ordered and renormalized by using configuration space techniques which are related to Glaser's methods for the original model. Matrix elements of these operators are employed to discuss the elementary production processes in Sec. 4, and the conclusions are summarized

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[1] W. E. Thirring, Ann. Phys. 9, 91 (1958).

in the last section. The integral equations encountered in the ordering of ψ and $\psi\psi^*$ are solved in the appendix.

2. EQUATION OF MOTION

The field operators for the Thirring model satisfy $i\gamma^\mu\partial_\mu\psi+2\lambda(\bar\psi\psi)\psi=0$. In the representation with $\gamma^{(1)}=i\sigma_1$, $\gamma^{(2)}=\beta=\sigma_2$, the equations of motion are

$$\partial\psi_1/\partial u=i\lambda\psi_2^*\psi_2\psi_1,\quad \partial\psi_2/\partial v=i\lambda\psi_1^*\psi_1\psi_2, \tag{1}$$

where $u=t+x$, $v=t-x$, and the operator relation $[\psi_\tau(x)]^2=0$ has been assumed. For $\lambda=\lambda_0 f(x,t)$, the general solutions to these equations are

$$\psi_1(x,t)=\phi_1(v)\exp\left[i\lambda_0\int_{-\infty}^u du'\, f(u',v)\phi_2^*(u')\phi_2(u')\right],$$
$$\psi_2(x,t)=\phi_2(u)\exp\left[i\lambda_0\int_{-\infty}^v dv'\, f(u,v')\phi_1^*(v')\phi_1(v')\right], \tag{2}$$

and ϕ_1,ϕ_2 are completely arbitrary. The stress-energy tensor, $T_\nu{}^\mu=\bar\psi\gamma^\mu(\partial_\nu\psi)-(\partial_\nu\bar\psi)\gamma^\mu\psi-g_\nu{}^\mu\mathfrak{L}$, no longer satisfies $\partial_\mu T_\nu{}^\mu=0$, reflecting the role of $f(x,t)$ as an external field which supplies energy and momentum to the system. However, the equations of motion still yield conservation laws, $\partial_\mu(\bar\psi\gamma^\mu\psi)=0$, $\partial_\mu(\bar\psi\gamma^\mu\gamma^{(5)}\psi)=0$, $\gamma^{(5)}=\gamma^{(1)}\gamma^{(2)}$, and these lead to two constants of motion,

$$N_{1,2}=\int_{-\infty}^\infty dx\,\psi_{1,2}^*(x,t)\psi_{1,2}(x,t)=\int_{-\infty}^\infty dy\,\phi_{1,2}^*(y)\phi_{1,2}(y). \tag{3}$$

The theory is quantized by treating $\phi_\tau(=\psi_\tau{}^{in})$ as free spinor operators, as in Glaser's treatment[2] for $f(x,t)=1$. Then ϕ has the properties

$$\phi_{1,2}=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^\infty dp\,C_{1,2}(p)\exp ip(u,v),$$
$$\{C_\tau(p),C_{\tau'}(p')\}=0,\ \{C_\tau(p),C_{\tau'}^*(p')\}=\delta_{\tau\tau'}\delta(p-p'), \tag{4}$$

and the C_τ may be decomposed into particle $[a(p)]$ and antiparticle $[b(p)]$ operators as follows

$$C_{1,2}(p)=[\theta(\pm p)a(p)+\theta(\mp p)b^*(-p)]. \tag{5}$$

With the definition $\rho_\tau=\phi_\tau^*\phi_\tau=\psi_\tau^*\psi_\tau$, Eq. (4) yields $[\rho_\tau(a),\rho_{\tau'}(b)]=[:\rho_\tau(a):,:\rho_{\tau'}(b):]=0$.[3] Since $[\rho_1,\phi_2]=[:\rho_1:,\phi_2]=0$, etc., Eqs. (2), (3) may be interpreted as operator equations as they stand, or with the replacement $\rho_\tau\to:\rho_\tau:$. We choose to make the latter identification so that Eq. (3) becomes

$$N_{1,2}=\int_{-\infty}^\infty dp\,\theta(\pm p)[a^*(p)a(p)-b^*(p)b(p)]. \tag{6}$$

[2] V. Glaser, Nuovo cimento **9**, 990 (1958).
[3] F. L. Scarf, Nuclear Phys. **11**, 475 (1959).

In the representation with $a|0\rangle=b|0\rangle=0$, $|\alpha,\beta\rangle=(a^*)^\alpha(b^*)^\beta|0\rangle$, the operators $N_{1,2}$ are clearly diagonal and correspond to the net number of particles travelling to the right and left, respectively. The state vectors of the system are simply the eigenstates of N_τ.

The relations $\psi_\tau(x,t)=U^*(t)\psi_\tau{}^{in}(x,t)U(t)$, $U^*U=1$, define a unitary matrix which yields Eq. (1) if

$$U(t)=\exp\left[-i\lambda_0\int_{-\infty}^t dt'\int_{-\infty}^\infty dx' f(x',t')\times:\rho_1(t'-x')::\rho_2(t'+x'):\right], \tag{7}$$

and the S matrix, $S=U(t=+\infty)$, is not diagonal in the representation discussed above, although $[S,N_\tau]=0$. The reason for this becomes apparent if one sets $N=N(a)-N(b)$. Then, in a state with $|1,0\rangle=a^*|0\rangle$, $N(a)|1,0\rangle=|1,0\rangle$, $N(b)|1,0\rangle=0$. However, the same eigenvalue of N occurs for a state $|1+m,m\rangle$ with $N(a)|1+m,\ m\rangle=(1+m)|1+m,\ m\rangle$, $N(b)|1+m,\ m\rangle=m|1+m,m\rangle$. Thus, $|1,0\rangle^{out}=S|1,0\rangle$ is a state containing $|1,0\rangle$ plus an infinite number of real particle-antiparticle pairs (when $f=1$, such production is forbidden by energy-momentum conservation[4]). In principle, it is possible to diagonalize S and N_τ simultaneously, but it is more interesting to consider $|\alpha,\beta\rangle$ as a physical particle state and to allow production of real pairs.

Finally, we note that for many discontinuous changes in λ [for example, $f=\theta(T-t)$] the S matrix is identical to the U matrix of a problem with $f=1$. Since Haag's theorem[5] applies to the latter, it may be inferred that the S matrix is not a well-defined quantity when λ varies with space and time.

3. ORDERING AND RENORMALIZATION

In a representation with $\phi|0\rangle=0$, the above expressions for ψ_τ and U [Eqs. (2), (7)], or those for products such as $\psi_\tau(a)\psi_\tau^*(b)$, are well-defined, but in the physical representation, $a|0\rangle=b|0\rangle=0$, they are not. To obtain well-defined operators, the exponentials must be rewritten as ordered products so that, for example

$$\psi_1(a)\psi_1^*(b)=\phi_1(v_a)\phi_1^*(v_b)e_1(u_a,v_a)e_1^{-1}(u_b,v_b), \tag{8a}$$

$$e_1(u,v)=\exp\left[i\lambda_0\int_{-\infty}^u du' f(u',v):\rho_2(u'):\right], \tag{8b}$$

becomes

$$\psi_1(a)\psi_1^*(b)=\phi_1(v_a)\phi_1^*(v_b)\langle e_1(a)e_1^{-1}(b)\rangle_0\times:\left[\exp\int dz\int dy\ g_1(z,y;\,a,b)\phi_2^*(z)\phi_2(y)\right]:. \tag{9}$$

[4] W. E. Thirring, Nuovo cimento **9**, 1007 (1958).
[5] R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, 12 (1955).

An analysis equivalent to that already made for the Thirring model with $f=1$ [6] shows that $\mathcal{G}_1$ is given by

$$[\phi_2(z),e_1(a)e_1^{-1}(b)]=G_1;\phi_2(z)e_1e_1^{-1};$$
$$=\int dy\,\mathcal{G}_1(z,y;a,b)$$
$$\times[\phi_2^{(+)}(y)e_1e_1^{-1}+e_1e_1^{-1}\phi_2^{(-)}(y)].\quad(10)$$

When Eq. (8) is differentiated with respect to λ_0, and the expectation value is taken, one finds [using Eq. (9)]

$$\frac{\partial\langle e_1e_1^{-1}\rangle_0}{\partial\lambda_0}=\frac{i\langle e_1e_1^{-1}\rangle_0}{(2\pi)^2}\times\left[\int_{-\infty}^{u_a}du'\,f(u',v_a)-\int_{-\infty}^{u_b}du'f(u',v_b)\right]\times\int dz\int\frac{dy\,\mathcal{G}_1(z,y;a,b)}{(u'-z-i\epsilon)(u'-y-i\epsilon)}.\quad(11)$$

The ordering of $\psi_1(a)$ is carried out by setting $u_b=-\infty$ in Eqs. (8)–(11) [for this calculation, the second integral over u' must be omitted from Eq. (11) *before* the integration is performed]. Equation (11) shows that the renormalization "constant," $Z_1^{\frac{1}{2}}=\langle e_1(a)\rangle_0$, formally depends on x, t, if $f\neq1$.

The kernel $\mathcal{G}_1$ is computed by using

$$e_1\phi_2(z)e_1^{-1}=\exp[-i\lambda_0f(z,v)\theta(u-z)]\phi_2(z),\quad(12)$$

which leads to

$$[\phi_2(z),e_1(a)e_1^{-1}(b)]=\{[1-\exp i\lambda_0f(z,v_a)]\theta(u_a-z)+\exp i\lambda_0f(z,v_a)[1-\exp-i\lambda_0f(z,v_b)]\theta(u_b-z)\}\times\phi_2(z)e_1(a)e_1^{-1}(b).\quad(13)$$

[Equation (12) is derived by noting that both sides obey the same first order differential equation in λ_0 and have the same initial value.] The transition from Eq. (13) to Eq. (10) is effected by applying positive and negative frequency projection operators to (13) and by rearranging terms. The algebraic details are identical to those already encountered in the calculation with $f=1$,[6] and we merely quote the results: the integral operator G_1 equals $G_1^{(-)}-G_1^{(+)}$ where $G_1^{(\pm)}$ are defined by the solutions $g=[1+G_1^{(\pm)}]k$ to the following integral equations

$$g(z)\mp\int_{-\infty}^{u_a}dy\frac{[1-\exp\mp i\lambda_0f(y,v_a)]g(y)}{2\pi i(y-z\mp i\epsilon)}\mp\int_{-\infty}^{u_b}dy\frac{[1-\exp\pm i\lambda_0f(y,v_b)][\exp\mp i\lambda_0f(y,v_a)]g(y)}{2\pi i(y-z\mp i\epsilon)}=k(z).\quad(14)$$

[6] F. L. Scarf, Phys. Rev. **115**, 463 (1959).

These equations are solved in the appendix and the kernels of $G_1^{(\pm)}$ are

$$\mathcal{G}_1^{(+)}(z,y)=\frac{\{[Q(z_+)/Q(y_+)]-[Q(z_+)/Q(y_-)]\}}{2\pi i(y-z-i\epsilon)}=[\mathcal{G}_1^{(-)}(z,y)]^*,\quad(15)$$

where

$$Q(z_+)=\exp\frac{\lambda_0}{2\pi}\left(\int_{-\infty}^{u_a}\frac{dx\,f(x,v_a)}{(x-z-i\epsilon)}-\int_{-\infty}^{u_b}\frac{dx\,f(x,v_b)}{(x-z-i\epsilon)}\right)=[Q(z_-)]^*.\quad(16)$$

When $u_a>u_b$, $\mathcal{G}_1=0$ if $(z,y)>u_a$, etc. At this point, the calculation of ψ_1 and $\psi_1\psi_1^*$ as explicit ordered and renormalized functionals of the incoming field operators, ϕ_r, has been reduced to a sequence of quadratures. The corresponding expressions for ψ_2, $\psi_2\psi_2^*$ follow by space reflection; $\psi_1\psi_2$ and $\psi_1\psi_2^*$ are already ordered if ψ_1, ψ_2 are.

In general $\langle ee^{-1}\rangle_0^R(=\langle ee^{-1}\rangle_0Z^{-\frac{1}{2}}Z^{*-\frac{1}{2}})$ does not exist, but if $f\to0$ as $u,v\to-\infty$, there is no infrared divergence and the renormalized matrix element is finite. In order to proceed, the functional form of $f(u,v)$ must be specified. The choice $f(u,v)=\theta(u+L)\theta(v+L)$ leads to particularly simple expressions for $\langle ee^{-1}\rangle_0$ and $Z^{\frac{1}{2}}$ [see reference 6]; this value yields

$$S_{12}'(a|b)=i\langle\{\psi_1(a),\psi_1^*(b)\}\rangle_0^R=i\delta(v_a-v_b)\left(\frac{(L+u_a)(L+u_b)}{(u_a-u_b)^2}\right)^m,\quad(17)$$

where $m=\lambda_0\lambda_0'/8\pi^2$, $\lambda_0'=\lambda_0+2\pi n$ and $|\lambda_0'/2\pi|<1$. All other causal functions have the same λ_0-dependent singular factor. Inspection of the renormalized operator $\{\psi_1,\psi_1^*\}^R$ shows that it can be written as $c+O$ (c is a singular c-number, O is a bounded operator) near $u_a=u_b$ but that it has the form $c\cdot O$ for $v_a\simeq v_b$. Thus, the decomposition $\{\psi,\psi^*\}^R=c+O$ is not valid on the entire light cone. It also follows from Eq. (17) that an expansion in u/L or λ_0 is not meaningful as $L\to\infty$. These results are insensitive to the precise form of $f(u,v)$ as long as $f\to0$, u, $v\to-\infty$.

4. MATRIX ELEMENTS

In order to understand how real pairs of physical particles are created, we examine the temporal development of the bare particle operators. Consider $\langle0|\psi_1^R(x,t)|n\rangle$, the amplitude for finding one bare particle in a state of n physical particles. If $|1\rangle=c_1(p)|0\rangle=\theta(-p)b^*(-p)|0\rangle$, then $\langle0|\psi_1^R|1\rangle=\langle0|\phi_1(v)|1\rangle$, and a bare particle in state $\tau=1$ "contains" a physical particle going to the left at all times.

For $n=3$, the only nonvanishing matrix element has $|3\rangle=c_1(p)c_2^*(q)c_2(k)|0\rangle$; this gives

$$\langle 0|\psi_1^R|3\rangle=\frac{\theta(k)\theta(-q)}{2\pi}\langle 0|\phi_1(v)|1\rangle \times\int_{-\infty}^{u}dz\int_{-\infty}^{u}dy\,\mathcal{G}_1(y,z)\exp i(qy-kz), \quad (18)$$

showing that ψ_1^R also develops into pairs of physical particles going to the right. The amplitudes for $n=5$, 7, etc., simply represent iterations of the fundamental radiation process and no pairs going to the left appear; the diagram which changes a right-pair into a left-pair is not prohibited by energy-momentum conservation in this model, but it is still ruled out by the differential conservation laws, $\partial\rho_1/\partial u=0$, $\partial\rho_2/\partial v=0$. Thus, at a finite time a bare particle going to the left is a physical particle going to the left plus radiated pairs of bare or physical ($\psi_2^*\psi_2=\phi_2^*\phi_2$) particles going to the right.

The limit of $\langle 0|\psi_1^R|3\rangle$ as $u\rightarrow-\infty$ is zero since $\psi^{in}=\phi$. For the *out*-fields, $Q(z_\pm)=\exp[\pm i\lambda_0 f^{(\pm)}(z,v)]$ [Eq. (16) with $u_a=+\infty$, $v_a=v$, $u_b=-\infty$] where $f^{(\pm)}$ are the positive and negative frequency parts of f, and

$$\frac{\langle 0|\psi_1^R|3\rangle^{out}}{\langle 0|\psi_1^R|1\rangle^{in}}=\theta(k)\theta(-q)\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dy \times\frac{\exp i\{\lambda_0[f^{(+)}(z,v)-f^{(+)}(y,v)]+qy-kz\}}{(2\pi)^2 i(y-z+i\epsilon)} \quad (19)$$

[the matrix element for ψ_2 depends on $f^{(-)}$]. In the interaction representation, $|t\rangle=U(t)|in\rangle$ has virtual pairs for any f, but if $f=1$, they are all reabsorbed as $t\rightarrow+\infty$ so that $|out\rangle=|in\rangle$ to within a phase factor. If f is not constant, the reabsorption is incomplete; $|out\rangle$ contains real pairs and Eq. (19) is a measure of the pair population. In the high-energy limit, $k\rightarrow\infty$, $(k+q)<\infty$, the distribution tends to zero since $\langle 0|\psi_1^R|3\rangle\rightarrow 0$.

The behavior of the wave function $\chi_{12}^R(a,b)=\langle 2|\psi_1(a)\psi_2(b)|0\rangle$ illustrates the effect of matter creation on the two-body interaction. Equation (12) enables one to factor χ_{12}^R as

$$\chi_{12}^R(a,b)=\chi_{12}^{in}\exp[-i\lambda_0 f(u_b,v_a)\theta(u_a-u_b)] \times F_1(a,b)F_2(b,a), \quad (20)$$

where

$$F_\tau(x,y)=\langle 1|\phi_\tau(x)e_\tau(y)|0\rangle/\langle 1|\phi_\tau(x)|0\rangle.$$

For $\langle 2|=\theta(-p)\theta(q)\langle 0|b(-q)b(-p)$ the matrix elements can be evaluated with the ordered expressions for e_τ, giving

$$F_1(a,b)=\left(1-\frac{\theta(-p)}{2\pi i}\int_{-\infty}^{v_b}dz\int_{-\infty}^{v_b}dy\frac{\mathcal{G}_2(z,y;-\infty,b)\exp ip(y-v_a)}{(v_a-z+i\epsilon)}\right),$$
$$F_2(b,a)=\left(1+\frac{\theta(q)}{2\pi i}\int_{-\infty}^{u_a}dz\int_{-\infty}^{u_a}dy\frac{\mathcal{G}_1(z,y;a,-\infty)\exp iq(y-u_b)}{(u_b-z-i\epsilon)}\right). \quad (21)$$

The *in*-operators are defined by $u_a=v_b=-\infty$, and χ^{out} has $u_a=v_b=+\infty$. The phase shift [Eq. (20)] contains $\lambda_0 f(u,v)$ at $u=u_b$, $v=v_a$, which is the point of collision.

When f is constant, $F^{out}=1$, $|\chi_{12}^{out}/\chi_{12}^{in}|=1$, and the phase shift is the only effect of the scattering.[7] However, when f is variable the final state has

$$F_1^{out}(a,b)=\left\{1+\frac{\theta(-p)}{2\pi i}\int_{-\infty}^{\infty}dy\frac{\exp i[p(y-v_a)-\lambda_0 f^{(+)}(y,u_b)+\lambda_0 f^{(+)}(v_a,u_b)]}{(v_a-y-i\epsilon)}\right\},$$
$$F_2^{out}(b,a)=\left\{1-\frac{\theta(q)}{2\pi i}\int_{-\infty}^{\infty}dy\frac{\exp i[q(y-u_b)+\lambda_0 f^{(-)}(y,v_a)-\lambda_0 f^{(-)}(u_b,v_a)]}{(u_b-y-i\epsilon)}\right\}. \quad (22)$$

Thus, $|\chi_{12}^{out}/\chi_{12}^{in}|$ is not unity and real particles are radiated during the two-body scattering. For high-energy collisions, the matter production is again negligible and the above ratio tends to its value for constant f.

5. CONCLUSIONS

When $\lambda=\lambda_0 f(x,t)$, the Thirring model has several new features of interest. For instance, the wave function renormalization constant depends on x,t, although Z^{-1} is still represented by a divergent integral. For a large class of couplings the divergence is merely of the ultraviolet variety, so that renormalized propagators, such as $S^{(+)\prime}$, are well-defined and contain no infinite constants. With a particular choice of f, it has been shown that the anticommutator does not have the form predicted by Heisenberg[8] [i.e., $\{\psi,\psi^*\}^R\neq c+O$ near the

[7] F. L Scarf, Phys. Rev. **111**, 1433 (1958).

[8] W. E. Heisenberg, Revs. Modern Phys. **29**, 269 (1957).

light cone]. Moreover, arguments previously advanced for the ill-defined $f=1$ case[6] now can be applied rigorously to show that perturbation theory and expansion over intermediate states give poor approximations to the exact causal functions.

When f is not constant, production of real physical particle pairs occurs. This creation can be attributed directly to the variation of λ as a source of energy and momentum. Although the S matrix is ill-defined, examination of the matrix elements of $\psi(x,t)$ shows how the production occurs and yields the energy spectrum.

We conclude that the Thirring model with variable interaction provides a soluble relativistic quantum field theory which contains production processes, and for which all renormalized quantities exist if $\lambda \to 0$ along the past light cone.

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APPENDIX

Let $p(z)=g(z)-k(z)$, $A(y)=\exp[-i\lambda_0 f(y,v_a)]$, $B(y)=\exp[i\lambda_0 f(y,v_b)]$. Then the first of the equations given by (14) is $(u_a>u_b)$

$$p(z)-\int_{-\infty}^{u_a}\frac{dy[1-A(y)][p(y)+k(y)]}{2\pi i(y-z-i\epsilon)} -\int_{-\infty}^{u_b}dy\frac{A(y)[1-B(y)][p(y)+k(y)]}{2\pi i(y-z-i\epsilon)}=0. \quad \text{(A.1)}$$

We assume that $p(z)$, as given by (A.1), is the limiting value of a function, $p(z_+)$, analytic in the entire complex plane except for branch cuts between $-\infty$ and u_a, $-\infty$ and u_b. We define a function $p(z_-)$ by letting $i(y-z-i\epsilon)$ go to its complex conjugate in (A.1). Then, using $[(y-z-i\epsilon)^{-1}-(y-z+i\epsilon)^{-1}]=2\pi i\delta(y-z)$, Eq. (A.1) becomes

$$\begin{aligned} p(z_+)-p(z_-)&=0, \quad z>u_a,\\ A(z)p(z_+)-p(z_-)&=[1-A(z)]k(z), \quad u_a>z>u_b, \quad \text{(A.2)}\\ A(z)B(z)p(z_+)-p(z_-)&=[1-A(z)B(z)]k(z), \quad u_b>z,\end{aligned}$$

and $p(z_-)$ may be interpreted as the continuation of $p(z_+)$ below the branch cuts.

We now wish to construct a function, $Q(z)$, which is analytic everywhere except for the above branch cuts and has

$$Q(z_-)=[1;A(z);A(z)B(z)]Q(z_+) \quad \text{for} \quad [z>u_a;\, u_a>z>u_b;\, u_b>z].$$

If $f(y,v)$ is sufficiently well-behaved, the function given in Eq. (16) has all of these properties. [I am indebted to Professor V. Glaser for suggesting the form of Q.] With this definition, the relation $p(z)=Q(z)s(z)$ allows one to rewrite Eq. (A.2) as $s(z_+)-s(z_-)=\{[1/Q(z_-)]-[1/Q(z_+)]\}k(z)$, and contour integration gives

$$s(z_+)=\int_{-\infty}^{u_a}\left[\frac{1}{Q(y_-)}-\frac{1}{Q(y_+)}\right]\frac{k(y)dy}{2\pi i(y-z-i\epsilon)}. \quad \text{(A.3)}$$

Equation (15) follows after the substitution $s(z_+)=(g-k)/Q(z_+)$ is made. A similar analysis leads to $\mathcal{G}_1^{(-)}$ when the lower set of signs in Eq. (14) is used.