Quasi-Particles and Gauge Invariance in the Theory of Superconductivity*

Yoichiro Nambu

The Enrico Fermi Institute for Nuclear Studies and the Department of Physics, The University of Chicago, Chicago, Illinois

(Received July 23, 1959)

Ideas and techniques known in quantum electrodynamics have been applied to the Bardeen-Cooper-Schrieffer theory of super-conductivity. In an approximation which corresponds to a generalization of the Hartree-Fock fields, one can write down an integral equation defining the self-energy of an electron in an electron gas with phonon and Coulomb interaction. The form of the equation implies the existence of a particular solution which does not follow from perturbation theory, and which leads to the energy gap equation and the quasi-particle picture analogous to Bogoliubov's.

The gauge invariance, to the first order in the external electro-magnetic field, can be maintained in the quasi-particle picture by taking into account a certain class of corrections to the charge-current operator due to the phonon and Coulomb interaction. In fact, generalized forms of the Ward identity are obtained between certain vertex parts and the self-energy. The Meissner effect calculation is thus rendered strictly gauge invariant, but essentially keeping the BCS result unaltered for transverse fields.

It is shown also that the integral equation for vertex parts allows homogeneous solutions which describe collective excitations of quasi-particle pairs, and the nature and effects of such collective states are discussed.

1. INTRODUCTION

A NUMBER of papers have appeared on various aspects of the Bardeen-Cooper-Schrieffer[1] theory of superconductivity. On the whole, the BCS theory, which leads to the existence of an energy gap, presents us with a remarkably good understanding of the general features of superconducivity. A mathematical formulation based on the BCS theory has been developed in a very elegant way by Bogoliubov,[2] who introduced coherent mixtures of particles and holes to describe a superconductor. Such "quasi-particles" are not eigenstates of charge and particle number, and reveal a very bold departure, inherent in the BCS theory, from the conventional approach to many-fermion problems. This, however, creates at the same time certain theoretical difficulties which are matters of principle. Thus the derivation of the Meissner effect in the original BCS theory is not gauge-invariant, as is obvious from the viewpoint of the quasi-particle picture, and poses a serious problem as to the correctness of the results obtained in such a theory.

This question of gauge invariance has been taken up by many people.[3] In the Meissner effect one deals with a linear relation between the Fourier components of the external vector potential A and the induced current J, which is given by the expression

$$J_i(q) = \sum_{j=1}^{3} K_{ij}(q) A_j(q),$$

with

$$K_{ij}(q) = -\frac{e^2}{m}\langle 0|\rho|0\rangle\delta_{ij} + \sum_n \left(\frac{\langle 0|j_i(q)|n\rangle\langle n|j_j(-q)|0\rangle}{E_n} + \frac{\langle 0|j_j(-q)|n\rangle\langle n|j_i(q)|0\rangle}{E_n} \right). \quad (1.1)$$

ρ and j are the charge-current density, and $|0\rangle$ refers to the superconductive ground state. In the BCS model, the second term vanishes in the limit $q \to 0$, leaving the first term alone to give a nongauge invariant result. It has been pointed out, however, that there is a significant difference between the transversal and longitudinal current operators in their matrix elements. Namely, there exist collective excited states of quasi-particle pairs, as was first derived by Bogoliubov,[2] which can be excited only by the longitudinal current.

As a result, the second term does not vanish for a longitudinal current, but cancels the first term (the longitudinal sum rule) to produce no physical effect; whereas for a transversal field, the original result will remain essentially correct.

If such collective states are essential to the gauge-invariant character of the theory, then one might argue that the former is a necessary consequence of the latter. But this point has not been clear so far.

Another way to understand the BCS theory and its problems is to recognize it as a generalized Hartree-Fock approximation.[4] We will develop this point a little further here since it is the starting point of what follows later as the main part of the paper.

[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).

[2] N. N. Bogoliubov, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 58, 73 (1958) [translation: Soviet Phys. **34**, 41, 51 (1958)]; Bogoliubov, Tolmachev, and Shirkov, *A New Method in the Theory of Superconductivity* (Academy of Sciences of U.S.S.R., Moscow, 1958). See also J. G. Valatin, Nuovo cimento **7**, 843 (1958).

[3] M. J. Buckingam, Nuovo cimento **5**, 1763 (1957). J. Bardeen, Nuovo cimento **5**, 1765 (1957). M. R. Schafroth, Phys. Rev. **111**, 72 (1958). P. W. Anderson, Phys. Rev. **110**, 827 (1958); **112**, 1900 (1958). G. Rickayzen, Phys. Rev. **111**, 817 (1958); Phys. Rev. Letters **2**, 91 (1959). D. Pines and R. Schrieffer, Nuovo cimento **10**, 496 (1958); Phys. Rev. Letters **2**, 407 (1958). G. Wentzel, Phys. Rev. **111**, 1488 (1958); Phys. Rev. Letters **2**, 33 (1959). J. M. Blatt and T. Matsubara, Progr. Theoret. Phys. (Kyoto) **20**, 781 (1958). Blatt, Matsubara, and May, Progr. Theoret. Phys. (Kyoto) **21**, 745 (1959). K. Yosida, *ibid*. **731**.

[4] Recently N. N. Bogoliubov, Uspekhi Fiz. Nauk **67**, 549 (1959) [translation: Soviet Phys.—Uspekhi **67**, 236 (1959)], has also reformulated his theory as a Hartree-Fock approximation, and discussed the gauge invariance collective excitations from this viewpoint. The author is indebted to Prof. Bogoliubov for sending him a preprint.

Take the Hamiltonian in the second quantization form for electrons interacting through a potential V:

$$H = \int \sum_{i=1}^{2} \psi_i^+(x) K_i \psi_i(x) d^3x + \tfrac{1}{2} \int \int \sum_{i,k} \psi_i^+(x)$$

$$\times \psi_k^+(y) V(x,y) \psi_k(y) \psi_i(x) d^3x d^3y$$

$$\equiv H_0 + H_{int}. \tag{1.2}$$

K is the kinetic energy plus any external field. $i=1, 2$ refers to the two spin states (e.g., spin up and down along the z axis).

The Hartree-Fock method is equivalent to linearizing the interaction H_{int} by replacing bilinear products like $\psi_i^+(x)\psi_k(y)$ with their expectation values with respect to an approximate wave function which, in turn, is determined by the linearized Hamiltonian. We may consider also expectation values $\langle \psi_i(x)\psi_k(y) \rangle$ and $\langle \psi_i^+(x)\psi_k^+(y) \rangle$ although they would certainly be zero if the trial wave function were to represent an eigenstate of the number of particles, as is the case for the true wave function.

We write thus a linearized Hamiltonian

$$H_0' = \int \sum_i \psi_i^+ K_i \psi_i d^3x + \int \int \sum_{i,k} [\psi_i^+(x)\chi_{ik}(xy)\psi_k(y)$$

$$+ \psi_i^+(x)\phi_{ik}(xy)\psi_k^+(y)$$

$$+ \psi_k(x)\phi_{ki}^+(xy)\psi_i(y)] d^3x d^3y$$

$$\equiv H_0 + H_s, \tag{1.3}$$

where

$$\chi_{ik}(xy) = \delta_{ik}\delta^3(x-y) \int V(xz) \sum_j \langle \psi_j^+(z)\psi_j(z) \rangle d^3z$$

$$- V(xy)\langle \psi_k^+(y)\psi_i(x) \rangle, \tag{1.4}$$

$$\phi_{ik}(xy) = \tfrac{1}{2} V(xy)\langle \psi_k(y)\psi_i(x) \rangle,$$

$$\phi_{ik}^+(xy) = \tfrac{1}{2} V(xy)\langle \psi_k^+(y)\psi_i^+(x) \rangle.$$

We diagonalize H_0' and take, for example, the ground-state eigenfunction which will be a Slater-Fock product of individual particle eigenfunctions. The defining equations (1.4) then represent just generalized forms of Hartree-Fock equations to be solved for the self-consistent fields χ and ϕ.

The justification of such a procedure may be given by writing the original Hamiltonian as

$$H = (H_0 + H_s) + (H_{int} - H_s) \equiv H_0' + H_{int},$$

and demanding that H_{int}' shall have no matrix elements which would cause single-particle transitions; i.e., no matrix elements which would effectively modify the starting H_0': to put it more precisely, we demand our approximate eigenstates to be such that

$$\langle n | H_{int}' | 0 \rangle = \langle n | H | 0 \rangle = 0, \tag{1.5}$$

if in $|n\rangle$ more than one particle change their states from those in $|0\rangle$. This condition is contained in Eq. (1.4).[5] Since in many-body problems, as in relativistic field theory, we often take a picture in which particles and holes can be created and annihilated, the condition (1.5) should also be interpreted to include the case where $|n\rangle$ and $|0\rangle$ differ only by such pairs. The significance of the BCS theory lies in the recognition that with an essentially attractive interaction V, a nonvanishing ϕ is indeed a possible solution, and the corresponding ground state has a lower energy than the normal state. It is also separated from the excited states by an energy gap $\sim 2\phi$.

The condition (1.5) was first invoked by Bogoliubov[2] in order to determine the transformation from the ordinary electron to the quasi-particle representation. He derived this requirement from the observation that H_{int}' contains matrix elements which spontaneously create virtual pairs of particles with opposite momenta, and cause the breakdown of the perturbation theory as the energy denominators can become arbitrarily small. Equation (1.5), as applied to such pair creation processes, determines only the nondiagonal part (in quasi-particle energy) of H_s in the representation in which $H_0 + H_s$ is diagonal. The diagonal part of H_s is still arbitrary. We can fix it by requiring that

$$\langle 1' | H_{int}' | 1 \rangle = 0, \tag{1.6}$$

namely, the vanishing of the diagonal part of H_{int} for the states where one more particle (or hole) having a Hamiltonian H_0' is added to the ground state. In this way we can interpret H_0' as describing single particles (or excitations) moving in the "vacuum," and the diagonal part of H_s represents the self-energy (or the Hartree potential) for such particles arising from its interaction with the vacuum.

The distinction between Eqs. (1.5) and (1.6) is not so clear when applied to normal states. On the one hand, particles and holes (negative energy particles) are not separated by an energy gap; on the other hand, there is little difference when one particle is added just above the ground state.

In the above formulation of the generalized Hartree fields, χ and ϕ will in general depend on the external field as well as the interaction between particles. There is a complication due to the fact that they are gauge dependent. This is because a phase transformation $\psi_i(x) \rightarrow e^{i\lambda(x)}\psi_i(x)$ applied on Eq. (1.3) will change χ and ϕ according to

$$\chi(xy) \rightarrow e^{-i\lambda(x)+i\lambda(y)}\chi(xy),$$

$$\phi(xy) \rightarrow e^{i\lambda(x)+i\lambda(y)}\phi(xy), \tag{1.6}$$

$$\phi^+(xy) \rightarrow e^{-i\lambda(x)-i\lambda(y)}\phi^+(xy).$$

[5] Equation (1.5) refers only to the transitions from occupied states to unoccupied states. Transitions between occupied states or unoccupied states are given by Eq. (1.6). These two together then are equivalent to Eq. (1.4). For the analysis of the Hartree approximation in terms of diagrams, see J. Goldstone, Proc.

It is especially serious for ϕ (and ϕ^+) since, even if $\phi(xy) = \delta^3(x-y)$ times a constant in some gauge, it is not so in other gauges. Therefore, unless we can show explicitly that physical quantities do not depend on the gauge, any calculation based on a particular ϕ is open to question. It would not be enough to say that a longitudinal electromagnetic potential produces no effect because it can be transformed away before making the Hartree approximation. A natural way to reconcile the existence of ϕ, which we want to keep, with gauge invariance would be to find the dependence of ϕ on the external field explicitly. If the gauge invariance can be maintained, the dependence must be such that for a longitudinal potential $A = -\mathrm{grad}\lambda$, it reduces to Eq. (1.6). This should not be done in an arbitrary manner, but by studying the actual influence of H_{int} on the primary electromagnetic interaction when ϕ is first determined without the external field.

After these preliminaries, we are going to study the points raised here by means of the techniques developed in quantum electrodynamics. We will first develop the Feynman-Dyson formulation adapted to our problem, and write down an integral eauation for the self-energy part which corresponds to the Hartree approximation. It is observed that it can possess a nonperturbational solution, and the existence of an energy gap is immediately recognized.

Next we will introduce external fields. Guided by the well-known theorems about gauge invariance, we are led to consider the so-called vertex parts, which include the "radiative corrections" to the primary charge-current operator. When an integral equation for the general vertex part is written down, certain exact solutions are obtained in terms of the assumed self-energy part, leading to analogs of the Ward identity.[6] They are intimately related to inherent invariance properties of the theory. Among other things, the gauge invariance is thus strictly established insofar as effects linear in the external field are concerned, including the Meissner effect.

Later we look into the collective excitations. A very interesting result emerges when we observe that one of the exact solutions to the vertex part equations becomes a homogeneous solution if the external energy-momentum is zero, and expresses a bound state of a pair with zero energy-momentum. Then by perturbation, other bound states with nonzero energy-momentum are obtained, and their dispersion law determined. Thus the existence of the bound state is a logical consequence of the existence of the special self-energy ϕ and the gauge invariance, which are seemingly contradictory to each other.

When the Coulomb interaction is taken into account, the bound pair states are drastically modified, turning

into the plasma modes due to the same mechanism as in the normal case. This situation will also be studied.

2. FEYNMAN-DYSON FORMULATION

We start from the Lagrangian for the electron-phonon system, which is supposed to be uniform and isotropic.[7]

$$\mathcal{L} = \sum_p \sum_i \left[i\psi_i^+(p)\dot{\psi}_i(p) - \psi_i^+(p)\epsilon_p\psi_i(p) \right]$$
$$+ \sum_k \tfrac{1}{2}\left[\dot{\varphi}(k)\dot{\varphi}(-k) - c^2\varphi(k)\varphi(-k) \right]$$
$$- g\frac{1}{\sqrt{\mathcal{U}}} \sum_{p,k} \psi_i^+(p+k)\psi_i(p)h(k)\varphi(k). \quad (2.1)$$

p is the phonon field, with the momentum k (energy $\omega_k = ck$) running up to a cutoff value $k_m(\omega_m)$; c is the phonon velocity. ϵ_p is the electron kinetic energy relative to the Fermi energy; $gh(k)$ represents the strength of coupling.[8] ($\mathcal{U}$ is the volume of the system.)

The Coulomb interaction between the electrons is not included for the moment in order to avoid complication. Later we will make remarks whenever necessary about the modifications when the Coulomb interaction is taken into account.

It will turn out to be convenient to introduce a two-component notation[9] for the electrons

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2^+(x) \end{pmatrix} \quad \text{or} \quad \Psi(p) = \begin{pmatrix} \psi_1(p) \\ \psi_2^+(-p) \end{pmatrix}, \quad (2.2)$$

and the corresponding 2×2 Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.3)$$

The Lagrangian then becomes:

$$\mathcal{L} = \sum_p \Psi^+(p)\left(i\frac{\partial}{\partial t} - \epsilon_p\tau_3 \right)\Psi(p)$$
$$+ \sum_k \tfrac{1}{2}\left[\dot{\varphi}(k)\dot{\varphi}(-k) - c^2\varphi(k)\varphi(-k) \right]$$
$$- g\frac{1}{\sqrt{\nabla}} \sum_{p,k} \Psi^+(p+k)\tau_3\Psi(p)h(k)\varphi(k) + \sum_p \epsilon_p$$
$$= \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}} + \mathrm{const.}$$

The last infinite c-number term comes from the rearrangement of the kinetic energy term. This is certainly uncomfortable, but will not be important except for the calculation of the total energy.

The fields obey the standard commutation relations.

Roy. Soc. (London) A239, 267 (1957). Compare also T. Kinoshita and Y. Nambu, Phys. Rev. 94, 598 (1953).

[6] J. C. Ward, Phys. Rev. 78, 182 (1950).

[7] We use the units $\hbar = 1$.

[8] For convenience, we have included in $h(k)$ the frequency factor: $h(k) = h_1(k)k_0$.

[9] P. W. Anderson [Phys. Rev. 112, 1900 (1958)], has also introduced this two-component wave function.

Especially for Ψ, we have

$$\{\Psi_i(x),\Psi_j{}^+(y)\}\equiv\Psi_i(x)\Psi_j{}^+(y)+\Psi_j{}^+(y)\Psi_i(x)$$
$$=\delta_{ij}\delta^3(x-y),\qquad(2.5)$$

$$\{\Psi_i(p),\Psi_j{}^+(p')\}=\delta_{ij}\delta_{pp'}.$$

We may now formally treat H_{int} as perturbation, using the formulation of Feynman and Dyson.[10] The unperturbed ground state (vacuum) is then the state where all individual electron states $\epsilon_p<0(>0)$ are occupied (unoccupied) in the representation where $\psi_i{}^+(p)\psi_i(p)$ is the occupation number.

Having defined the vacuum, the time-ordered Green's functions for free electrons and phonons

$$\langle T(\Psi_i(xt),\Psi_j{}^+(x't'))\rangle=[G_0(x-x',\,t-t')]_{ij},$$
$$\langle T(\varphi(xt),\varphi(x't'))\rangle=\Delta_0(x-x',\,t-t')\qquad(2.6)$$

are easily determined. We get for their Fourier representation (in the limit $\mathcal{V}\to\infty$)[10a]

$$G_0(xt)=(1/(2\pi)^4)\int G_0(pp_0)e^{i\mathbf{p}\cdot\mathbf{x}-ip_0t}d^3pdp_0,$$

$$\Delta_0(xt)=\frac{1}{(2\pi)^4}\int_{|k|<k_m}\Delta_0(kk_0)e^{i\mathbf{k}\cdot\mathbf{x}-ik_0t}d^3kdk_0,$$

$$G_0(pp_0)=i\left[P\frac{1}{p_0-\epsilon_p\tau_3}-i\pi\,\mathrm{sgn}(\tau_3\epsilon_p)\delta(p_0-\tau_3\epsilon_p)\right]\quad(2.7)$$

$$=i(p_0+\epsilon_p\tau_3)/(p_0{}^2-\epsilon_p{}^2+i\epsilon),$$

$$\Delta_0(kk_0)=i\left[P\frac{1}{k_0{}^2-c^2k^2}-i\pi\delta(k_0{}^2-c^2k^2)\right]$$

$$=i/(k_0{}^2-c^2k^2+i\epsilon).$$

With the aid of these Green's functions, we are able to calculate the S matrix and other quantities according to a well-defined set of rules in perturbation theory.

We will analyze in particular the self-energies of the electron and the phonon. In the many-particle system, these energies express (apart from the self-interaction of the electron) the average interaction of a single particle or phonon placed in the medium. Because the phonon spectrum is limited, there will be no ultraviolet divergences, unlike the case of quantum electrodynamics.

These self-energies may be obtained in a perturbation expansion with respect to H_{int}. We are, however, interested in the Hartree method which proposes to take account of them in an approximate but nonperturbational way. It is true that the self-energies are in general complex due to the instability of single par-

[10] F. J. Dyson, Phys. Rev. 75, 486, 1736 (1949); R. P. Feynman, Phys. Rev. 76, 769 (1949); J. Schwinger, Phys. Rev. 74, 1439 (1948). Although we followed here the perturbation theory of Dyson, there is no doubt that the relations obtained in this paper can be derived by a nonperturbational formulation such as J. Schwinger's: Proc. Natl. Acad. Sci. U. S. 37, 452, 455 (1951).
[10a] P stands for the principal value; $i\epsilon$ in the denominator is a small positive imaginary quantity.
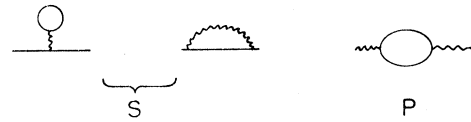

FIG. 1. Second order self-energy diagrams. Solid and curly lines represent electron and phonons, respectively, themselves being under the influence of the self-energies Σ and Π. All diagrams are to be interpreted in the sense of Feynman, lumping together all topologically equivalent processes.

ticles. But to the extent that the single-particle picture makes physical sense, we will ignore the small imaginary part of the self-energies in the following considerations.

Let us thus introduce the approximate self-energy Lagrangian $\mathcal{L}_s$, and write

$$\mathcal{L}=(\mathcal{L}_0+\mathcal{L}_s)+(\mathcal{L}_{int}-\mathcal{L}_s)$$
$$\equiv\mathcal{L}_0'+\mathcal{L}_{int}',$$
$$\mathcal{L}_0=\sum_p\Psi_p{}^+L_0\Psi_p+\sum_k\tfrac{1}{2}\varphi_kM_0\varphi_{-k},\qquad(2.8)$$
$$\mathcal{L}_s=-\sum_p\Psi_p{}^+\Sigma\Psi_p-\sum_k\tfrac{1}{2}\varphi_k\Pi\varphi_{-k},$$
$$L_0-\Sigma\equiv L,\qquad M_0-\Pi\equiv M.$$

The free electrons with "spin" functions u and phonons obey the dispersion law

$$L_0(\mathbf{p},\,p_0=\epsilon_p)u_p=0,\qquad M_0(\mathbf{k},\,k_0=\omega_k)=0,\qquad(2.9)$$

whereas they obey in the medium

$$L(\mathbf{p},\,p_0=E_p)u_p=0,\qquad M(\mathbf{k},\,k_0=\Omega_k)=0.\qquad(2.9')$$

Σ will be a function of momentum p and "spin." Π will consist of two parts: $\Pi(k_0k)=\Pi_1(k)k_0{}^2+\Pi_2(k)$ in conformity with the second order character (in time) of the phonon wave equation.[11]

The propagators corresponding to these modified electrons and phonons are

$$G(pp_0)=i/(L(pp_0)+i\,\mathrm{sgn}(p_0)\epsilon),$$
$$\Delta(kk_0)=i/(M(kk_0)+i\epsilon).\qquad(2.10)$$

We now determine Σ and Π self-consistently to the second order in the coupling g. Namely the second order self-energies coming from the phonon-electron interaction have to be cancelled by the first order effect of $\mathcal{L}_{int}$.

These second order self-energies are represented by the nonlocal operators[12] (Fig. 1)

$$S((t+t')/2)=\iiint\Psi^+(xt)S(x-x',\,t-t')$$
$$\times\Psi(x't')d^3xd^3x'd(t-t'),\qquad(2.11)$$

$$\mathcal{P}((t+t')/2)=\tfrac{1}{2}\iiint\varphi(xt)P(x-x',\,t-t')$$
$$\times\varphi(x't')d^3xd^3x'd(t-t'),$$

[11] In the same spirit Σ should actually be in the form $\Sigma_1(p)p_0+\Sigma_2(p)$. Here we neglect the renormalization term Σ_1 since the two conditions (2.13) can be met without it.
[12] We use the word nonlocal here for nonlocality in time.

where S and P have the Fourier representation

$$S(pp_0) = -ig^2\tau_3\delta^3(p)\delta(p_0)h^2(0)\Delta(0)$$

$$\times \int \text{Tr}[\tau_3 G(p'p_0)]d^3pdp_0$$

$$-ig^2 \int \tau_3 G(p-k, \, p_0-k_0)\tau_3 h^2(kk_0)$$

$$\times \Delta(kk_0)d^3kdk_0, \quad (2.12)$$

$$P(kk_0) = ig^2 h(kk_0)^2 \int \text{Tr}[\tau_3 G(pp_0)$$

$$\times G(p+k, \, p_0+k_0)]d^3pdp_0.$$

In Eq. (2.11) we have chosen more or less arbitrarily $(t+t')/2$ as the fixed time to which we refer the nonlocal operators $\mathcal{S}$ and $\mathcal{P}$. The self-consistency requirements (1.5) and (1.6) mean in the present case that Σ, Π must be identical with S, P (a): for the diagonal elements [on the energy shell, Eq. (2.9)], and (b): for the non-diagonal matrix elements for creating a pair out of the vacuum.

The pair creation of electrons is possible because Ψ, being a two-component wave function, can have in general two eigenfunctions u_{ps} ($s=1, 2$) with different energies E_{ps} for a fixed momentum, p, only one of which is occupied in the ground state.

Thus taking particular plane waves $u_{ps}^*e^{-ip\cdot x+ip_0 t}$, $u_{p's'}e^{ip'\cdot x'-ip_0't'}$ for Ψ^+ and Ψ in (2.11), we easily find that the diagonal matrix element of Σ corresponds to $u_{ps}^*S(p,E_{ps})u_{ps}$, while the nondiagonal part corresponds to $u_{ps}^*S(p,0)u_{ps'}$, $s\neq s'(p_0' = -p_0)$.

A similar situation holds also for the photon self-energy Π. Since Π consists of two parts, the diagonal and off-diagonal conditions will fix these.

With this understanding, the self-consistency relations may be written

$$\Sigma(\mathbf{p}E_p)_\text{D} = S(\mathbf{p},E_p)_\text{D}, \quad \Sigma(\mathbf{p}0)_\text{ND} = S(\mathbf{p}0)_\text{ND},$$
$$\Pi(\mathbf{k}\Omega_k) = P(\mathbf{k}\Omega_k), \quad \Pi(\mathbf{k}0) = P(\mathbf{k}0), \quad (2.13)$$

where D, ND signify the diagonal and nondiagonal parts in the "spin" space. As stated before, we have agreed to omit possible imaginary parts in S and P. (The nondiagonal components, however, will turn out to be real.)

Before discussing the general solutions, let us consider the meaning of Eq. (2.13) in terms of perturbation theory. Suppose we expand G occurring in Eq. (2.12), with respect to Σ:

$$G = G_0 - iG_0\Sigma G_0 - G_0\Sigma G_0\Sigma G_0 + \cdots,$$

and expand Σ itself with respect to g^2, then we easily realize that Eq. (2.13) defines an infinite sum of a particular class of diagrams, which are illustrated in Fig. 2. The first term in S of Eq. (2.12) corresponds to the ordinary Hartree potential which is just a constant,
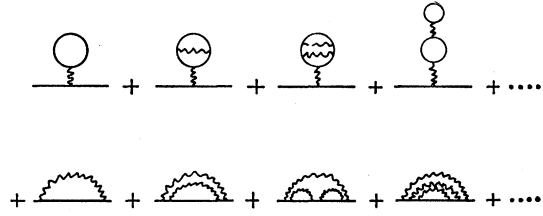

Fig. 2. Expansion of the self-consistent self-energy $\Sigma \sim S$ in terms of bare electron diagrams.

whereas the second term gives an exchange effect. In the latter, the approximation is characterized by the fact that no phonon lines cross each other.

It must be said that the Hartree approximation does not really sum the series of Fig. 2 completely since we equate in Eq. (2.13) only special matrix elements of both sides. For in the perturbation series the Σ obtained to any order is a function of p_0, whereas in Eq. (2.13) it is replaced by a p_0-independent quantity. Hence there will be a correction left out in each order (analogous to the radiative correction after mass renormalization in quantum electrodynamics).

In this perturbation expansion, S in Eq. (2.13) is always proportional to τ_3 on the energy shell since $H_0 \propto \tau_3$. Accordingly Σ will be $\propto \tau_3$ and commute with H_0, so that no off-diagonal part exists.[11]

It is important, however, to note the possibility of a nonperturbational solution by assuming that Σ contains also a term proportional to τ_1 or τ_2. Thus, take

$$\Sigma(p) = \chi(p)\tau_3 + \phi(p)\tau_1,$$
$$H_0' = (\epsilon+\chi)\tau_3 + \phi\tau_1 \quad (2.14)$$
$$\equiv \bar{\epsilon}\tau_3 + \phi\tau_1.$$

This form bears a resemblance to the Dirac equation. Its eigenvalues are

$$E = \pm E_p \equiv \pm(\bar{\epsilon}_p^2 + \phi_p^2)^{\frac{1}{2}}. \quad (2.15)$$

Since H_0' describes by definition excited states, we have to adopt the hole picture and conclude that the ground state (vacuum) is the state where all negative energy "quasi-particles" ($E<0$) are occupied and no positive energy particles exist. If ϕ remains finite on the Fermi surface, the positive and negative states are separated by a gap $\sim 2|\phi|$. The corresponding Green's function G now has the representation

$$G(pp_0) = i\frac{p_0 + \bar{\epsilon}_p\tau_3 + \phi_p\tau_1}{p_0^2 - E_p^2 + i\epsilon}. \quad (2.16)$$

In order to extract the diagonal and nondiagonal parts in spin space, we will use the trick

$$O_\text{D} = \tfrac{1}{2}\text{Tr}(\Lambda O),$$
$$O_\text{ND} = -(i/2)\text{Tr}(\Lambda O\tau_2), \quad (2.17)$$
$$\Lambda = [E_p + H_0'(p)]/2E_p.$$

Applying this to Eq. (2.13a) with Eqs. (2.12), (2.14), and (2.15), we finally obtain the following equations for χ and ϕ

$$\frac{\epsilon_p \chi_p + \phi_p{}^2}{E_p} = \frac{g^2 \pi}{(2\pi)^4} P \int \left[\frac{E_p}{\Omega_k} + \frac{E_{p-k} + \Omega_k}{E_p E_{p-k} \Omega_k} \right]$$

$$\times (\bar{\epsilon}_p \bar{\epsilon}_{p-k} - \phi_p \phi_{p-k}) \frac{h(k)^2}{E_p{}^2 - (E_{p-k} - \Omega_k)^2} d^3k,$$

$$\epsilon_p \phi_p = \frac{g^2 \pi}{(2\pi)^4} \int (\bar{\epsilon}_{p-k} \phi_p + \bar{\epsilon}_p \phi_{p-k})$$

$$\times \frac{h(k)^2 d^3k}{E_{p-k} \Omega_k (E_{p-k} + \Omega_k)}. \quad (2.18)$$

The second equation, coming from the nondiagonal condition, has a trivial solution $\phi = 0$. If a finite solution ϕ exists, it cannot follow from perturbation treatment since there is no inhomogeneous term to start with.

Equation (2.18) is equivalent to, but slightly different from, the corresponding conditions of Bogoliubov because of a slightly different definition of the nondiagonal part of the self-energy operator, which is actually due to an inherent ambiguity in approximating nonlocal operators by local ones. (This is the same kind of ambiguity as one encounters in the derivation of a potential from field theory. The difference between the local operator Σ and the nonlocal one S shows up in a situation like that in Fig. 3, and the compensation between Σ and S is not complete.) We may avoid this unpleasant situation, by extending the Hartree self-consistency conditions to all virtual matrix elements, but this would mean that ϕ (and χ) must be treated as nonlocal. We will discuss this situation in a separate section since such a generalization brings simplification in dealing with the problem of gauge invariance and collective excitations.

For the moment we consider the second equation of (2.18) and rewrite it

$$\phi_p = A_p \frac{g^2 \pi}{(2\pi)^4} \int \frac{\phi_{p-k}}{E_{p-k}} \frac{h(k) d^3k}{\Omega_k (E_{p-k} + \Omega_k)},$$

$$A_p = \bar{\epsilon}_p \Big/ \left(\epsilon_p - \frac{g^2 \pi}{(2\pi)^4} \int \frac{\bar{\epsilon}_p h(k)^2 d^3k}{E_{p-k} \Omega_k (E_{p-k} + \Omega_k)} \right). \quad (2.19)$$

This is essentially the energy gap equation of BCS if $g^2 A_p h(k)^2 / \Omega_k (E_{p-k} + \Omega_k)$ is identified with the effective interaction potential V, and if $\bar{\epsilon}_p \sim \epsilon_p (\chi_p \sim 0)$. It has a solution

$$\phi \sim \Omega_m \exp(-1/VN),$$

if $VN \ll 1$, N being the density of states: $N = dn/d\epsilon_p$ on the Fermi surface.

The phonon self-energy Π may be studied similarly from Eq. (2.13), which should determine the renormalization of the phonon field. It does not play an

essential role in superconductivity, though it gives rise to an important correction when the Coulomb effect is taken into account. (See the following section.)

From the nature of Eq. (2.12), it is clear that $\tau_1 \phi$ can actually be pointed in any direction in the 1–2 plane of the τ space: $\tau_1 \phi_1 + \tau_2 \phi_2$. It was thus sufficient to take $\phi_1 \neq 0$, $\phi_2 = 0$. Any other solution is obtained by a transformation

$$\Psi \to \exp(i\alpha\tau_3/2)\Psi,$$

$$(\phi, 0) \to (\phi \cos\alpha, \phi \sin\alpha). \quad (2.20)$$

In view of the definition of Ψ, Eq. (2.20) is a gauge transformation with a constant phase. Thus the arbitrariness in the direction of ϕ is the 1–2 plane is a reflection of the gauge invariance.

For later use, we also mention here the particle-antiparticle conjugation C of the quasi-particle field Ψ. This is defined by

$$C: \quad \Psi \to \Psi^C = C\Psi^+ = \tau_2 \Psi^+,$$

or

$$\begin{pmatrix} \psi_1{}^C \\ \psi_2{}^{+C} \end{pmatrix} = \begin{pmatrix} -i\psi_2 \\ i\psi_1{}^+ \end{pmatrix}, \quad (2.21)$$

and changes quasi-particles of energy-momentum (p_0, p) into holes of energy-momentum $(-p_0, -p)$, or interchanges up-spin and down-spin electrons. Under C, the τ operators transform as

$$C: \quad \tau_i \to C^{-1}\tau_i C = -\tau_i{}^T, \quad i = 1, 2, 3 \quad (2.22)$$

where T means transposition.

As a consequence, we have also

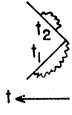$$C: \quad L(p) \to L^C(-p) = -L(-p)^T. \quad (2.23)$$

Finally we make a remark about the Coulomb interaction. When this is taken into account, the phonon interaction factor $g^2 h(k)^2 \Delta(k, k_0)$ in Eq. (2.12a) has to be replaced by

$$[g^2 h(k)^2 \Delta(kk_0) + ie^2/k^2] /$$
$$\{1 - i\Pi(kk_0)[\Delta(kk_0) + ie^2/g^2 h(k)^2 k^2]\}.$$

As is well known, the denominator represents the screening of the Coulomb interaction. Discussion about this point will be made later in connection with the plasma oscillations.

3. NONLOCAL (ENERGY-DEPENDENT) SELF-CONSISTENCY CONDITIONS

In the last section we remarked that the self-consistency conditions Eq. (2.13) may be extended to all virtual matrix elements, namely, not only on the energy shell (diagonal) and for the virtual pair creation out of

the vacuum, but also for the self-energy effects which appear in intermediate states of any process.

This simply means that ϕ and π are now nonlocal; i.e., depend both on energy and momentum arbitrarily, and are to be completely equated with S and P, respectively,

$$\Sigma(pp_0)=S(pp_0), \quad \Pi(kk_0)=P(kk_0). \quad (3.1)$$

Actually, these self-energies can no more be incorporated in H_0' as the zeroth order Lagrangian since they contain infinite orders of time derivatives.[13] Nevertheless, Eq. (3.1) has a precise meaning in the bare particle perturbation theory. It defines the (proper) self-energy parts (in the sense of Dyson) as an infinite sum of the special class of diagrams illustrated in Fig. 2.

The earlier condition of Eq. (2.13) represented, as was noted there, only an approximation to this sum. In other words, Eqs. (2.13) and (3.1) are not exactly identical even on the energy shell.

The Hartree-Fock approximation based on Eq. (3.1) could be interpreted as a nonperturbation approximation to determine the "dressed" single particles (together with the "dressed vacuum") or the Green's function $\langle 0|T(\Psi(xt),\Psi^+(x't'))|0\rangle$ for the true interacting system. Such single particles will satisfy

$$L(p,p_0)u\cong 0, \quad M(k,k_0)\cong 0. \quad (3.2)$$

We use the approximate equality since a really stable single particle may not exist.

Let us assume that these determine the approximate renormalized dispersion law

$$p_0^2=E_r(p)^2, \quad k_0^2=\Omega_r(k)^2. \quad (3.3)$$

If we write for Σ

$$\Sigma(pp_0)=p_0\zeta(pp_0)+\chi(pp_0)\tau_3+\phi(pp_0)\tau_1, \quad (3.4)$$

where ζ, χ, ϕ are even functions of p_0, then

$$E_r^2(p)=[\bar{\epsilon}(pp_0)^2+\phi(pp_0)^2]/[1-\zeta(pp_0)]^2|_{p_0^2=E_r(p)^2}$$
$$\equiv E(pp_0)^2/Z(pp_0)^2|_{p_0^2=E_r(p)^2}. \quad (3.5)$$

The Green's functions G and Δ will be given by

$$G(pp_0)=i/L(pp_0)$$

$$=i\int_0^\infty \frac{dx}{p_0^2-x+i\epsilon}$$

$$\times \text{Im}\frac{p_0Z(px)+\bar{\epsilon}(px)\tau_3+\phi(px)\tau_1}{x^2Z(px)^2-E(px)^2}, \quad (3.6)$$

$$\Delta(kk_0)=i/M(kk_0)$$

$$=i\int_0^\infty \frac{dx}{k_0^2-x+i\epsilon}\,\text{Im}\frac{1}{M(kx)}.$$

This representation assumes that $G(p_0)[\Delta(k_0)]$ is analytic except for a branch cut on the real axis. The imaginary part in the integrand is expected to have a delta function or a sharp peak at $x=E_r^2(p)$ $[\Omega_r(k^2)]$. These properties are necessary in order that the vacuum is stable and the quasi-particles and phonons have a valid physical meaning as excitations.[14] In the following, we will generally consider this quasi-particle peak only, and write

$$G(pp_0)=i\frac{p_0Z(pp_0)+\bar{\epsilon}(pp_0)\tau_3+\phi(pp_0)\tau_1}{p_0Z(pp_0)^2-E(pp_0)^2+i\epsilon}, \quad \text{etc.}$$

The Hartree equations now take the form

$$\Sigma(pp_0)=-i\frac{g^2}{(2\pi)^4}\int \tau_3G(p-k,\,p_0-k_0)\tau_3h(kk_0)^2d^3kdk_0,$$

$$\Pi(kk_0)=i\frac{g^2}{(2\pi)^4}\int \text{Tr}\,[\tau_3G(k-p,\,k_0-p_0)$$
$$\times\tau_3G(pp_0)]d^3pdp_0. \quad (3.7)$$

This equation for Σ is much simpler than the previous one (2.18) since we may just equate the coefficients of 1, τ_3, τ_1 on both sides. In particular, we get the energy gap equation

$$\phi(pp_0)=-\frac{ig^2}{(2\pi)^4}\int \frac{\phi(p'p_0')}{p_0^2Z(p'p_0')^2-E(p'p_0')^2+i\epsilon}$$

$$\times h(\mathbf{p}-\mathbf{p}',\,p_0-p_0')^2\Delta(\mathbf{p}-\mathbf{p}',\,p_0-p_0')d^3p'dp_0', \quad (3.8)$$

which is to be compared with Eq. (2.19).

Although the existence of a solution to Eq. (3.6) may be difficult to establish, the solution, if it exists, should not be much different from the older solution to Eq. (2.19). At any rate, our assumption about the analyiticity of G and Π is consistent with Eq. (3.6) or (3.7) which implies that Σ and Π are also analytic except for a cut on the real axis.

In later calculations we shall encounter various integrals which we may classify into three types regarding their sensitivity to the energy gap. First, a normal self-energy part, for example, represents the effect of the bulk of the surrounding electrons on a particular electron, and is insensitive to the change of the small fraction $\sim\phi/E_F$ of the electrons near the Fermi surface in a superconductor. Such a quantity is

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[13] It would seem then that we lose the advantage of the generalization since we cannot find the Bogoliubov transformation. However, we could still start from the older solution (2.13) as the zeroth approximation to Eq. (3.1), and then calculate the correction; namely, the "radiative" correction to the Bogoliubov vacuum and the Bogoliubov quasi-particle. These corrections would take account of the single-particle transitions which remain after the Bogoliubov condition (2.13) is imposed.

[14] This is a representation of the Lehmann type [H. Lehmann, Nuovo cimento 11, 342 (1954)] which can be derived by defining the Green's functions in terms of Heisenberg operators. See also V. M. Galizkii and A. B. Migdal, J. Exptl. Theoret. Phys. U.S.S.R. 34, 139 (1958) [translation: Soviet Phys. JETP 7, 96 (1958)].
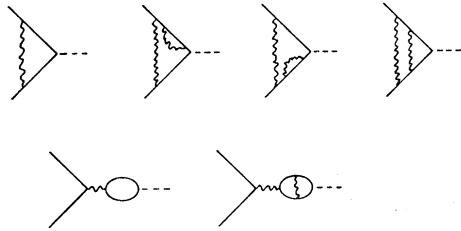
FIG. 4. Construction of the vertex part Γ in bare particle picture. The second line represents the polarization diagrams.

given by an integral like

$$g^2 \int \frac{\epsilon_k}{E_k} f(\mathbf{p}-\mathbf{k}) d^3k, \qquad (3.9)$$

where the region $\epsilon_k \lesssim E_k = (\epsilon_k^2 + \phi^2)^{\frac{1}{2}}$ makes little contribution if $f(p-k)$ is a smooth function.

Second, the energy gap itself is determined from an equation of the form

$$g^2 \int \frac{d^3k}{E_k} f(\mathbf{p}-\mathbf{k}) \sim g^2 \int_{E_k \lesssim \omega_m} \frac{d^3k}{E_k} f(\mathbf{p}-\mathbf{k}) \sim 1, \quad (3.10)$$

which means that even if g^2 is small, such an expression is always of the order 1.

Finally we meet with integrals like

$$g^2 \int \frac{\epsilon_k \phi}{E_k^3} f(\mathbf{p}-\mathbf{k}) d^3k, \quad g^2 \int \frac{\phi^2}{E_k^3} f(\mathbf{p}-\mathbf{k}) d^3k, \quad \text{etc.} \quad (3.11)$$

They have an extra cutoff factor $\sim 1/E$, $1/E^2$, etc., in the integrand which restricts the contribution to an energy interval $\sim 2\phi$ near the Fermi surface. The integrals are thus of the order

$$g^2 N \phi / \omega_m, \quad g^2 N, \quad \text{etc.}$$

In the following, we will not be primarily concerned with the ordinary self-energy effects. We will assume that proper renormalization has been carried out, or else simply disregard it unless essential. When we carry out perturbation type calculations, we will arrange things so that quantities of the second type are taken into account rigorously, and treat quantities of the third type as small, and hence negligible ($g^2 N \ll 1$).

4. INTEGRAL EQUATIONS FOR VERTEX PARTS[15]

In the presence of an electromagnetic potential, the original Lagrangian $\mathcal{L}$ has to be modified according to the rule

$$i\frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t} + eA_0, \quad \mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\mathbf{A}$$

for the electron. Going to the two-component repre-

sentation, this corresponds to the prescription

$$i\frac{\partial}{\partial t} \rightarrow i\frac{\partial}{\partial t} + e\tau_3 A_0, \quad \mathbf{p} \rightarrow \mathbf{p} - \frac{e}{c}\tau_3 \mathbf{A} \qquad (4.1)$$

acting on Ψ. It can also be inferred from the gauge transformation $\Psi \rightarrow \exp(i\alpha\tau_3)\Psi$ as was observed previously. So the ordinary charge-current operator turns out to be in our form given by

$$\rho(x) = \frac{e}{2}([\Psi^+(x), \tau_3\Psi(x)] + \{\Psi^+(x), \Psi(x)\}),$$

$$\mathbf{j}(x) = \frac{-ie}{4m}([\Psi^+(x), (\nabla - ie\tau_3\mathbf{A})\Psi(x)]$$

$$+ [(-\nabla - ie\tau_3 A)\Psi^+(x), \Psi(x)]$$

$$+ \{\Psi^+(x), (\nabla\tau_3 - ieA)\Psi(x)\}$$

$$- \{(\tau_3\nabla - ieA)\Psi^+(x), \Psi(x)\}). \qquad (4.2)$$

The second terms on the right-hand side, being infinite C numbers, arise from the rearrangement of ψ and ψ^+, and will actually be compensated for by the first terms.

This expression, however, has to be modified when we go to the quasi-particle picture.

For we have seen that the self-energy ϕ of a quasi-particle is a gauge-dependent quantity. If we want to have the quasi-particle picture and gauge invariance at the same time, then it is clear that the electromagnetic current of a quasi-particle must contain, in addition to the normal terms given by Eq. (4.2), terms which would cause a physically unobservable transformation of ϕ if the electromagnetic potential is replaced by the gradient of a scalar. In other words, the complete charge current of a quasi-particle has to satisfy the continuity equation, which Eq. (4.2) does not, since

$$\partial\rho/\partial t + \nabla \cdot \mathbf{j} = 2\Psi^+\phi\tau_2\Psi.$$

In order to find such a conserving expression for charge current, it is instructive to go back to the bare electron picture, in which the self-energy is represented by a particular class of diagrams discussed in the previous sections.

It is well known[16] in quantum electrodynamics that, in any process involving electromagnetic interaction, perturbation diagrams can be grouped into gauge-invariant subsets, such that the invariance is maintained by each subset taken as a whole. Such a subset can be constructed by letting each photon line in a diagram interact with a charge of all possible places along a chain of charge-carrying particle lines. The gauge-invariant interaction of a quasi-particle with an electromagnetic potential should then be obtained by attaching a photon line at all possible places in the diagrams of Fig. 2. The result is illustrated in Fig. 4,

[15] Hereafter we will often use the four-dimensional notation $x = (\mathbf{x}, t)$, $p = (\mathbf{p}, p_0)$, $d^4p = d^3p\, dp_0$.

[16] Z. Koba, Progr. Theoret. Phys. (Kyoto) 6, 322 (1951).

which consists of the "vertex" part Γ and the self-energy part Σ.

In this way we are led to consider the modification of the vertex due to the phonon interaction in the same approximation as the self-energy effect is included in the quasi-particle. It is not difficult to see that it corresponds to a "ladder approximation" for the vertex part, and we get an integral equation[17]

$$\Gamma_i(p',p) = \gamma_i(p',p) - g^2 \int \tau_3 G(p'-k)\Gamma_i(p'-k,\, p-k)$$

$$\times G(p-k)\tau_3 h(k)^2 \Delta(k) d^4k, \quad (4.3)$$

where γ_i, $i=0, 1, 2, 3$ stand for the free particle charge current $[\tau_3, (1/2m)(\mathbf{p}+\mathbf{p'})]$ which follows from Eq. (4.2). Similar equations may be set up for any type of vertex interactions.

Equation (4.3) is the basis of the rest of this paper. It expresses a clear-cut approximation procedure in which the "free" charge-current operator γ_i of a quasi-particle is modified by a special class of "radiative corrections" due to H_{int}'.

As the next important step, we observe that there exist exact solutions to Eq. (4.3) for the following four types of vertex interactions

(a) $\gamma^{(a)}(p',p) = L_0(p') - L_0(p)$
$$= (p_0'-p_0) - \tau_3(\epsilon_{p'}-\epsilon_p),$$
    $\Gamma^{(a)}(p',p) = L(p') - L(p)$
$$= \gamma_a(p') - [\Sigma(p')-\Sigma(p)],$$

(b) $\gamma^{(b)}(p',p) = L_0(p')\tau_3 - \tau_3 L_0(p)$
$$= (p_0'-p_0)\tau_3 - (\epsilon_{p'}-\epsilon_p), \quad (4.4)$$
    $\Gamma^{(b)}(p',p) = L(p')\tau_3 - \tau_3 L(p),$

(c) $\gamma^{(c)}(p',p) = L_0(p')\tau_1 + \tau_1 L_0(p'),$
    $\Gamma^{(c)}(p',p) = L(p')\tau_1 + \tau_1 L(p),$

(d) $\gamma^{(d)}(p',p) = L_0(p')\tau_2 + \tau_2 L(p),$
    $\Gamma^{(d)}(p',p) = L(p')\tau_2 + \tau_2 L(p).$

The verification is straightforward by noting that $G(p) = i/L(p)$, and making use of Eq. (3.7).

The fact that there are simple solutions is not accidental. These solutions express continuity equations and other relations following from the four types of operations, which do not depend on the presence or

[17] This equation may also be derived simply by considering the self-energy equation (3.7) in the presence of an external field, and expanding Σ in A. Σ should be now a function of initial and final momenta, and we define

$$\Sigma^{(A)}(p',p) \equiv \Sigma(p)\delta^4(p'-p) + \sum_{i=0}^{3} (\Gamma_i(p',p) - \gamma_i(p',p))$$
$$\times A^i(p'-p) + O(A^2).$$

In the limit $p'-p=0$, $\Gamma_i-\gamma_i = \partial\Sigma/\partial A^i$, which is the content of the Ward identity.[6] Investigation of the higher order terms in A is beyond the scope of this paper.

absence of the interaction:

(a) $\Psi(x) \to e^{i\alpha(x)}\Psi(x),$ $\Psi^+(x) \to \Psi^+(x)e^{-i\alpha(x)},$

(b) $\Psi \to e^{i\tau_3\alpha}\Psi,$ $\Psi^+ \to \Psi^+ e^{-i\tau_3\alpha},$

(c) $\Psi \to e^{\tau_1\alpha}\Psi,$ $\Psi^+ \to \Psi^+ e^{\tau_1\alpha},$ (4.5)

(d) $\Psi \to e^{\tau_2\alpha}\Psi,$ $\Psi^+ \to \Psi^+ e^{\tau_2\alpha},$

where $\alpha(x)$ is an arbitrary real function.

(a) and (b) correspond, respectively, to the spin rotation around the z axis, and the gauge transformation. The entire Lagrangian is invariant under them, so that we obtain continuity equations for the z component of spin and charge, respectively:

$$\frac{\partial}{\partial t}\Psi^+\Psi + \nabla \cdot \Psi^+ \frac{\mathbf{p}}{m}\tau_3\Psi = 0,$$
$$\frac{\partial}{\partial t}\Psi^+\tau_3\Psi + \nabla \cdot \Psi^+ \frac{\mathbf{p}}{m}\Psi = 0, \quad (4.6)$$

where Ψ is the true Heisenberg operator.

These equations are identical with

$$\Psi^+\gamma^{(a)}\Psi = 0, \quad \Psi^+\gamma^{(b)}\Psi = 0. \quad (4.7)$$

Taken between two "dressed" quasi-particle states, the left-hand side of Eq. (4.7) will become

$$e^{-i(p'-p)\cdot x}\langle |p'| \Psi^+(x)\gamma^{(n)}\Psi(x) |p\rangle$$
$$= u_{p'}^*\Gamma^{(n)}(p',p)u_p$$
$$= 0, \quad (n=a, b) \quad (4.7')$$

where u_p, $u_{p'}$ are single-particle wave functions satisfying $L(p)u_p = u_{p'}^* L(p') = 0$.

In this way we have shown the existence of spin and charge currents $\Gamma_i^{(a)}(p',p)$ and $\Gamma_i^{(b)}(p',p)$ for a quasi-particle, for which the continuity equations

$$(p_0'-p_0)\cdot\Gamma_0^{(n)} - \sum_{i=1}^{3}(p'-p)_i\cdot\Gamma_i^{(n)} = \Gamma^{(n)}(p',p) = 0$$

will hold.

The last two transformations of Eq. (4.4) are not unitary, but mix ψ_1 and ψ_2^+ in such a way as to keep $\Psi^+\tau_3\Psi$ invariant. From infinitesimal transformations of these kinds we get

$$i\Psi^+\tau_1\left(\frac{\vec{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t}\right)\Psi + \nabla \cdot \Psi^+\tau_2\left(\frac{\vec{\mathbf{p}}}{m} + \frac{\overleftarrow{\mathbf{p}}}{m}\right)\Psi = 0,$$

$$-i\Psi^+\tau_2\left(\frac{\vec{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t}\right)\Psi + \nabla \cdot \Psi^+\tau_1\left(\frac{\vec{\mathbf{p}}}{m} + \frac{\overleftarrow{\mathbf{p}}}{m}\right)\Psi = 0,$$

$$(4.8)$$

which bear the same relations to $\gamma^{(c),(d)}$ and $\Gamma^{(c),(d)}$ as Eq. (4.6) did to $\gamma^{(a),(b)}$ and $\Gamma^{(a),(b)}$. Note that the above equations are unaffected by the presence of the phonon interaction.

The fact that we can find a conserved charge-current

FIG. 5. The diagram for the kernel $K^{(2)}$.

FIG. 6. Graphical derivation of Eq. (5.5). The thick lines represent quasi-particles.

for a quasi-particle is rather surprising. A quasi-particle cannot be an eigenstate of charge since it is a linear combination of an electron and a hole, tending to an electron well above the Fermi surface, and to a hole well below. We must conclude than that an accelerated wave packet of quasi-particles, whose energy is confined to a finite region of space, continuously picks up charge from, or deposits it with, the surrounding medium which extends to infinity. This situation will be studied in Sec. 7, where we will derive the charge current operators Γ_i explicitly.

5. GAUGE INVARIANCE IN THE MEISSNER EFFECT

We will next discuss how the gauge invariance is maintained in the problem of the Meissner effect when the external magnetic field is static. We calculate the Fourier component of the current $J(q)$ induced in the superconducting ground state by an external vector potential $A(q)$:

$$J_i(q) = \sum_{j=1}^{3} K_{ij}(q) A_j(q), \qquad (5.1)$$

where q is kept finite.

For free electrons, K is represented by

$$K_{ij} = K_{ij}^{(1)} + K_{ij}^{(2)}, \quad K_{ij}^{(1)} = -\delta_{ij} ne^2/m, \quad (5.2)$$

where n is the number of electrons inside the Fermi sphere. $K^{(1)}$ comes from the expectation value of the current operator Eq. (4.2), whereas $K^{(2)}$ corresponds to the diagram in Fig. 5. [Compare also Eq. (1.1).] It is well known that in this case K_{ij} is of the form

$$K_{ij}(q) = (\delta_{ij} q^2 - q_i q_j) K(q^2), \qquad (5.3)$$

so that for a longitudinal vector potential $A_i(q) \sim q_i \lambda(q)$, we have

$$J_i(q) = K_{ij} q_j \lambda(q) = 0, \qquad (5.4)$$

establishing the unphysical nature of such a potential.

In the case of a superconducting state, the free electron lines in Fig. 5 will be replaced by quasi-particle lines. But then we have $K^{(2)}(q) \to 0$ as $q \to 0$ since the intermediate pair formation is suppressed due to the finite energy gap, whereas $K^{(1)}$ is essentially unaltered. Thus Eq. (5.2) takes the form of the London equation, except that even a longitudinal field creates a current.

According to our previous argument, this lack of gauge invariance should be remedied by taking account of the vertex corrections. Starting again from the free electron picture, and inserting the phonon interaction
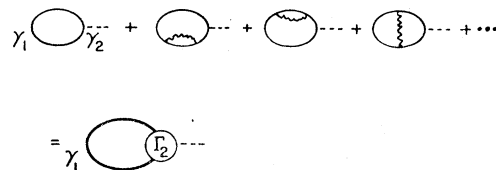
effects, as indicated in Fig. 6, we arrive at the conclusion that either one of the vertices γ in Fig. 5 has to be replaced by the full Γ^{10}. In addition, there is the polarization correction represented by a string of bubbles. Let us, however, first neglect this correction. $K_{ij}^{(2)}$ is then

$$K_{ij}^{(2)}(q) = \frac{-ie^2}{(2\pi)^4} \int \mathrm{Tr}[\gamma_i(p-q/2,\ p+q/2) G(p+q/2)$$
$$\times \Gamma_j(p+q/2,\ p-q/2) G(p-q/2)] d^4p. \quad (5.5)$$

Although we do not know $\Gamma_j(p,p)$ explicitly, we can establish Eq. (5.4) easily. For

$$-\sum_{j=1}^{3} \Gamma_j(p+q,\ q) q_j$$

is exactly the solution $\Gamma^{(b)}(p+q,\ p)$ of Eq. (4.4) where q_0 is equal to zero. Substituting this solution in Eq. (4.5) we find

$$K_{ij}^{(2)}(q) q_j$$

$$= \frac{-1}{(2\pi)^4} \int \mathrm{Tr}\{\gamma_i(p-q/2,\ p+q/2)$$
$$\times [\tau_3 G(p+q/2) - G(p-q/2)\tau_3]\} d^4p$$

$$= \frac{-1}{(2\pi)^4} \int \mathrm{Tr}\{[\gamma_i(p-q,\ p) - \gamma_i(p,\ p+q)]$$
$$\times \tau_3 G(p)\} d^4p$$

$$= \frac{1}{(2\pi)^4} \frac{q_i}{m} \int \mathrm{Tr}[\tau_3 G(p)] d^4p, \qquad (5.6)$$

where the properties of γ_i and G under particle conjugation and a translation in p space were utilized in going from the first to the second line.

On the other hand, the part $K^{(1)}$ is, according to Eq. (4.2) given by

$$K_{ij}^{(1)} = -\delta_{ij} \frac{e^2}{2m} (\langle 0 | [\Psi^+(x), \tau_3 \Psi(x)] | 0 \rangle + \{\Psi^+(x), \Psi(x)\})$$

$$= K_{ij}^{(1a)} + K_{ij}^{(1b)}. \qquad (5.7)$$

The first term becomes further

$$-\delta_{ij}\frac{e^2}{2m}\langle 0|[\Psi^+(x),\tau_3\Psi(x)]|0\rangle$$

$$=-\delta_{ij}\frac{e^2}{m}\tfrac{1}{2}\lim_{\epsilon\to 0}\sum_{\pm}\langle 0|T(\Psi^+(x,t\pm\epsilon)\tau_3\Psi(x,t))|0\rangle$$

$$=-\delta_{ij}\frac{e}{m}\operatorname{Tr}[\tau_3 G(xt=0)]$$

$$=-\delta_{ij}\frac{e}{m}\frac{1}{(2\pi)^4}\int \operatorname{Tr}[\tau_3 G(p)]x^4 p. \quad (5.8)$$

Thus

$$[K_{ij}^{(1a)}(q)+K_{ij}^{(2)}(q)]q_j=0. \quad (5.9)$$

The second term $K^{(1b)}$ comes from the c-number term of the current operator (4.2), and is just the anticommutator of the electron field, which does not depend on the quasi-particle picture, nor on the presence of interaction. Therefore we may write for this contribution

$$K_{ij}^{(1b)}(q)A_j(q)=\frac{-ie}{2m}\frac{1}{(2\pi)^3}\int e^{-iqx}d^3x$$

$$\times\{\Psi^+(x),(\tau_3\nabla-ieA(x))_i\Psi(x)\} \quad (5.10)$$

to show its formal gauge invariance since $\tau_3\nabla-ieA(x)$ is certainly a gauge-invariant combination for free electron field.

As for the polarization correction, we can easily show in a similar way that it vanishes for the static case $(q_0=0)$ because

$$\int \operatorname{Tr}\Gamma_i(p-q/2,\,p+q/2)G(p+q/2)$$

$$\times\gamma_0(p+q/2,\,p-q/2)G(p-q/2)d^4p=0.$$

Thus the above proof is complete and independent of the Coulomb interaction which profoundly influences the polarization effect. Although the proof is thus rigorous, it is still somewhat disturbing since $K^{(1a)}$, $K^{(1b)}$ and $K^{(2)}$ are all infinite. Actually there is a certain ambiguity in the evaluation of $K^{(2)}$, Eq. (5.6), which is again similar to the one encountered in quantum electrodynamics.[18] An alternative way would be to expand quantities in q without making translations in p space. In this case we may write

$$-\Gamma^{(b)}(p+q/2,\,p-q/2)=\bar\epsilon(p+q/2)-\bar\epsilon(p-q/2)$$
$$-i\tau_2[\phi(p+q/2)+\phi(p-q/2)]$$
$$\approx p\cdot q/m-2i\tau_2\phi. \quad (5.11)$$

The first term then gives

$$\frac{e^2}{(2\pi)^3}\int \frac{\phi^2}{4E_p{}^5}\left(\frac{\mathbf{p}\cdot\mathbf{q}}{m}\right)^2 p_i\mathbf{p}\cdot\mathbf{q}d^3p \propto q^2 q_i, \quad (5.12)$$

18 H. Fukuda and T. Kinoshita, Progr. Theoret. Phys. (Kyoto) 5, 1024 (1950).

which is convergent and the same as the one obtained from Eq. (1.1) using the bare quasi-particle states. The second term also is finite and equal to

$$\frac{e^2}{(2\pi)^3}\int \frac{\phi^2}{E_p{}^3}\frac{p^2}{3m^2}q_id^3p+O(q^2)q_i\approx N\alpha^2 q_i=n(e^2/m)q_i. \quad (5.13)$$

The last line follows from Eqs. (6.11) and (6.11') below.

The calculation of $K^{(1)}$ from Eqs. (5.7) and (5.10), gives, on the other hand, the same value as Eq. (5.2), so that we get $(K_{ij}^{(1)}+K_{ij}^{(2)})q_j=0$ in the limit of small q. (The polarization correction is again zero.)

Since Eq. (5.13) is a contribution from the collective intermediate state (see Secs. 6 and 7), we may say that the collective state saves gauge invariance, as has been claimed by several people.[3,19]

It goes without saying that the effect of the vertex correction on K_{ij} will be felt also for real magnetic field. But as we shall see later, it is a small correction of order g^2N (except for the renormalization effects), and not as drastic as for the longitudinal case.

6. THE COLLECTIVE EXCITATIONS

In order to understand the mechanism by which gauge invariance was restored in the calculation of the Meissner effect, and also to solve the integral equations for general vertex interactions, it is necessary to examine the collective excitations of the quasi-particles. In fact, people[3] have shown already that the essential difference between the transversal and longitudinal vector potentials in inducing a current is due to the fact that the latter can excite collective motions of quasi-particle pairs.

We see that the existence of such collective excitations follows naturally from our vertex solutions Eq. (4.4). For taking $p=p'$, the second solution $\Gamma^{(b)}(p',p)$ becomes

$$\Gamma^{(b)}(p,p)=L(p)\tau_3-\tau_3 L(p)$$
$$=2i\tau_2\phi, \quad (6.1)$$

$$\gamma^{(b)}=0.$$

In other words $\tau_2\phi(p)\equiv\Phi_0(p)$ satisfies a homogeneous integral equation:

$$\Phi_0(p)=-\frac{g^2}{(2\pi)^4}\int \tau_3 G(p')\Phi(p')G(p')$$

$$\times\tau_3 h(p-p')^2\Delta(p-p')d^4p. \quad (6.2)$$

We interpret this as describing a pair of a particle and an antiparticle interacting with each other to form a bound state with zero energy and momentum $q=p'-p=0$.

19 On the other hand, the way in which the collective mode accomplishes this end seems to differ from one paper to another. We will not attempt to analyze this situation here.

In fact, by defining

$$F(p, -p) \equiv -G(p)\Phi_0(p)G(p), \qquad (6.3)$$

Eq. (6.2) becomes

$$L(p)F(p, -p)L(p) = -\frac{g^2}{(2\pi)^4} \int \tau_3 F(p', -p')$$

$$\times \tau_3 h(p-p')^2 \Delta(p-p') d^4 p',$$

or

$$\sum_{j,l=1}^{2} L(p)_{ij} L^C(-p)_{kl} F(p, -p)_{jl}$$

$$= \frac{-q^2}{(2\pi)^4} \int \sum_{j,l} (\tau_3)_{ij} (\tau_3)_{kl} F(p', -p')_{jl}$$

$$\times h(p-p')^2 \Delta(p-p') d^4 p. \qquad (6.4)$$

The particle-conjugate quantity L^C was defined in Eq. (2.23).

Equations (6.2) and (6.4) are the analog of the so-called Bethe-Salpeter equation[20] for the bound pair of quasi-particles with zero total energy-momentum. $F_{ij}(p, -p)$ is the four-dimensional wave function with the spin variables i, j and the relative energy-momentum $(p_0, \mathbf{p})$.

Since there, thus, exists a bound pair of zero momentum, there will also be pairs moving with finite momentum and kinetic energy. In other words, there will be a continuum of pair states with energies going up from zero. We have to determine their dispersion law.

For a finite total energy-momentum q, the homogeneous integral equation takes the form

$$\Phi_q(p) \equiv L(\tfrac{1}{2}q+p)F(\tfrac{1}{2}q+p, \tfrac{1}{2}q-p)L(p-\tfrac{1}{2}q)$$

$$= -g^2 \frac{1}{(2\pi)^4} \int \tau_3 F(\tfrac{1}{2}q+p', \tfrac{1}{2}q-p')$$

$$\times \tau_3 h(p-p')^2 \Delta(p-p') d^4 p. \qquad (6.5)$$

From here on we carry out perturbation calculation. Let us expand F and L in terms of the small change $L(p\pm q/2) - L(p)$, thus

$$F(\tfrac{1}{2}q+p, \tfrac{1}{2}q-p) = F^{(0)}(p) + F^{(1)}(p,q/2) + \cdots, \qquad (6.6)$$
$$L(p\pm q/2) = L(p) + \Delta L(p, \pm q/2).$$

Collecting terms of the first order, we get

$$L(p)F^{(1)}(p,q/2)L(p) + U^{(1)}(p,q/2)$$

$$= -g^2 \frac{1}{(2\pi)^4} \int \tau_3 F^{(1)}(p',q/2)$$

$$\times \tau_3 h(p-p')^2 \Delta(p-p') d^4 p', \qquad (6.7)$$

$$U^{(1)}(p,q/2) = \Delta L(p,q/2)F^{(0)}(p)L(p)$$

$$+ L(p)F^{(0)}(p)\Delta L(p, -q/2).$$

[20] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).

This is an inhomogeneous integral equation for $F^{(1)}$. In order that it has a solution, the inhomogeneous term $U(p)$ must be orthogonal to the solution $\Phi_0(p)$ of the homogeneous equation. This condition can be derived as follows:

We multiply Eq. (6.7) by $F^{(0)}(p) = -G(p)\Phi_0(p)G(p)$, and integrate thus:

$$\int \text{Tr } F^{(0)}(p)L(p)F^{(1)}(p,q/2)L(p)d^4 p$$

$$+ \int \text{Tr } F^{(0)}(p)U^{(1)}(p,q/2)d^4 p$$

$$= -g^2 \frac{1}{(2\pi)^4} \int\int \text{Tr } F^{(0)}(p)\tau_3 F^{(1)}(p',q/2)$$

$$\times \tau_3 h(p-p')^2 \Delta(p-p')d^4 p d^4 p'.$$

In view of Eq. (6.5) the last line is

$$= \int \text{Tr } L(p')F^{(0)}(p')L(p')F^{(1)}(p',q/2)d^4 p',$$

so that

$$(F^{(0)}, U^{(1)}) \equiv \int \text{Tr } F^{(0)}(p)U^{(1)}(p,q/2)d^4 p = 0. \qquad (6.8)$$

This is the desired condition.

For the evaluation of Eq. (6.8), we will neglect the p dependence of the self-energy terms. Thus

$$F^{(0)}(p) = \tau_2 \phi / (p_0^2 - E_p^2 + i\epsilon), \quad E_p^2 = \epsilon_p^2 + \phi^2, \qquad (6.9)$$
$$\Delta L(p,q/2) = q_0/2 - \tau_3(\mathbf{p}\cdot\mathbf{q}/2m + (q/2)^2/2m).$$

We then obtain

$$(F^{(0)}, U^{(1)}) = 2\pi i \int \frac{\phi^2}{E_p^3}\left[\left(\frac{q_0}{2}\right)^2 - \left(\frac{\mathbf{p}\cdot\mathbf{q}}{2m}\right)^2\right.$$

$$\left. -\frac{\epsilon_p}{m^2}\left(\frac{q}{2}\right)^2\right]d^3 p = 0,$$

or

$$\left(\frac{q_0}{2}\right)^2 - \left(\frac{q}{2}\right)^2\left[\frac{1}{3}\frac{\bar{p}^2}{m^2} - \frac{\bar{\epsilon_p}}{m}\right] = 0, \qquad (6.10)$$

where the average $\bar{f}$ is defined

$$\bar{f} = \int f(p)\frac{\phi^2}{E^3}d^3 p \Big/ \int \frac{\phi^2}{E^3}d^3 p. \qquad (6.10')$$

The weight function $\phi^2/E_p^3 = \phi^2/(\epsilon_p^2+\phi^2)^{\frac{3}{2}}$ peaks around the Fermi momentum, so that $p^2 \sim p_F^2$, $\epsilon_p \sim 0$. Thus

$$q_0^2 \approx q^2 \frac{1}{3}\frac{\bar{p}^2}{m^2} \equiv \alpha^2 q^2, \quad \alpha^2 \approx p_F^2/3m^2, \qquad (6.11)$$

which is the dispersion law for the collective excita-
tions.[2,3] We also note, incidentally, that

$$\frac{1}{(2\pi)^3} \int \frac{\phi^2}{E^3} d^3p \approx N = mp_F/\pi^2,$$ (6.11')

$$\alpha^2 N \approx p_F{}^3/3\pi^2 m = n.$$

We would like to emphasize here that these collective
excitations are based on Eq. (6.2), which takes account
of the phonon-Coulomb scattering of the quasi-particle
pairs, but does not take into account the annihilation-
creation process of the pair due to the same interaction.

It is well known that this annihilation-creation
process is very important in the case of the Coulomb
interaction, and plays the role of creating the plasma
mode of collective oscillations. We will consider it in a
later section.

As for the wave function $F^{(1)}$ itself, we have still to
solve the integral equation (6.7). But this can be done
by perturbation because on substituting $U^{(1)}$ in the
integrand, we find that all the terms are of the type
(3.11). In other words, to the zeroth order we may
neglect the integral entirely and so

$$F^{(1)}(p,q/2) = -G(p)U^{(1)}(p,q/2)G(p).$$ (6.12)

The original function

$$\Phi_q(p) = -L(p+q/2)F(p,q/2)L(p-q/2)$$

is even simpler. We get

$$\Phi_q(p) \approx \Phi_0(p)$$ (6.13)

to this order.

7. CALCULATION OF THE CHARGE-CURRENT VERTEX FUNCTIONS

In this section we determine explicitly the charge-
current vertex functions Γ_i, $(i=0, 1, 2, 3)$ from their
integral equations. Only the particular combination $\Gamma^{(b)}$
of these was given before.

Let us first go back to the integral equation for Γ_0
generated by τ_3:

$$\Gamma_0(p+q/2, p-q/2)$$

$$= \tau_3 - g^2 \int \tau_3 G(p'+q/2)\Gamma_0(p'+q/2, p'-q/2)$$

$$\times G(p'-q/2)\tau_3 h(p-p')^2\Delta(p-p')d^4p',$$

or

$$L(p+q/2)F_0(p+q/2, p-q/2)L(p-q/2)$$

$$= \tau_3 + g^2 \int \tau_3 F_0(p'+q/2, p'-q/2)$$

$$\times \tau_3 h(p-p')^2\Delta(p-p')d^4p'.$$ (7.1)

For small g^2, the standard approach to solve the equa-
tion would be the perturbation expansion in powers of g^2.

We know, however, that there are low-lying collective
excitations, discussed before, to which τ_3 can be
coupled, and these excitations do not follow from per-
turbation.[21]

Fortunately, if we assume $q=0$, $q_0 \neq 0$, then we have
an exact solution to Eq. (7.1) in terms of $\Gamma^{(b)}$ of Eq.
(4.4). Namely,

$$\Gamma_0(p+q/2, p-q/2) = \Gamma^{(b)}(p+q/2, p-q/2)/q_0$$

$$= \tau_3\{[Z(p+q/2)+Z(p-q/2)]/2$$

$$+ (p_0/q_0)[Z(p+q/2)-Z(p-q/2)]\}$$

$$- [\chi(p+q/2)-\chi(p-q/2)]/q_0$$

$$+ i\tau_2[\phi(p+q/2)+\phi(p-q/2)]/q_0,$$ (7.2)

which can readily be verified.

The second term is the result of the coupling of τ_3
to the collective mode. This can be understood in the
following way. Γ_0 contains matrix elements for creation
or annihilation of a pair out of the vacuum. These
processes can go through the collective intermediate
state with the dispersion law (6.11), so that Γ will
contain terms of the form

$$R_\pm/(q_0 \pm \alpha q).$$

The residues $R_\pm$ can be obtained by taking the limit

$$R_\pm = \lim_{q_0 \pm \alpha q \to 0} \Gamma_0(p+q/2, p-q/2)(q_0 \pm \alpha q).$$ (7.3)

Applying this procedure to the integral equation (7.1)
for Γ_0, we find that $R_\pm$ must be a solution of the
homogeneous equation; namely,

$$R_\pm = C_\pm\Phi_q(p),$$ (7.4)

under the condition $q_0 \pm \alpha q = 0$.

For the particular case $q=0$, $\Phi_q(p)$ reduces to $\tau_2\phi(p)$,
which in fact agrees with Eq. (7.2) if

$$C_\pm = -2i.$$ (7.5)

This observation enables us to write down Γ_0 for
$q \neq 0$. According to the results of Sec. 6, $\Phi_q(p) = \Phi_0(p)$
in the zeroth order in g^2N. Since corrections to the non-
collective part of Γ_0 also turn out to be calculable by
perturbation, we may now put

$$\Gamma_0(p+q/2, p-q/2) \approx \tau_3\bar{Z} + 2i\tau_2\bar{\phi}q_0/(q_0^2-\alpha^2q^2),$$

$$\bar{\phi} \equiv [\phi(p+q/2)+\phi(p-q/2)]/2,$$

$$\bar{Z} \equiv [Z(p+q/2)+Z(p-q/2)]/2$$

to the extent that terms of order g^2N and/or the p-de-
pendence of the renormalization constants are neglected.

In quite a similar way the current vertex Γ may be
constructed. This time we start from the longitudinal

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[21] If we proceeded by perturbation theory, we would find in
each order terms of order 1.

component for $q_0=0$, $q\neq 0$, which has the exact solution

$$\Gamma(p+q/2,\ p-q/2)\cdot q/q = -\Gamma^{(b)}(p+q/2,\ p+q/2)/q$$

$$= \frac{p\cdot q}{mq}\left\{1+\frac{\chi(p+q/2)-\chi(p-q/2)}{p\cdot q/m}\right\}$$

$$-\tau_3 p_0[\zeta(p+q/2)-\zeta(p-q/2)]/q$$

$$-2i\tau_2\frac{\phi(p+q/2)+\phi(p-q/2)}{2q}. \quad (7.7)$$

For $q_0\neq 0$, then, we get

$$\Gamma(p+q/2,\ p-q/2)\cdot q/q$$

$$\approx (p\cdot q/q)\bar{Y}+2i\tau_2\bar{\phi}\alpha^2 q/(q_0{}^2-\alpha^2 q^2), \quad (7.8)$$

$$\bar{Y}\equiv 1+[\chi(p+q/2)-\chi(p-q/2)]/(p\cdot q/m).$$

Combining (7.6) and (7.8), the continuity equation takes the form

$$q_0\Gamma_0-q\cdot\Gamma=q_0\tau_3\bar{Z}+(p\cdot q/m)\bar{Y}+2i\tau_2\bar{\phi}$$

$$\approx\Gamma^{(b)},$$

which is indeed zero on the energy shell.

The transversal part of Γ, on the other hand, is not coupled with the collective mode because the latter is a scalar wave.[22] We may, therefore, write instead of Eq. (7.8)

$$\Gamma(p+q/2,\ p-q/2)\approx(p/m)\bar{Y}$$

$$+2i\tau_2\bar{\phi}\alpha^2 q/(q_0{}^2-\alpha^2 q^2). \quad (7.10)$$

Equations (7.6) and (7.10) for Γ_i have a very interesting structure. The noncollective part is essentially the same as the charge current for a free quasi-particle except for the renormalization $\bar{Z}$ and $\bar{Y}$, whereas the collective part is spread out both in space and time. Neglecting the momentum dependence of $\bar{Z}$, $\bar{Y}$, and ϕ, we may thus write the charge-current density (ρ,j) as

$$\rho(x,t)\cong e\Psi^+\tau_3 Z\Psi(x,t)+\frac{1}{\alpha^2}\frac{\partial f(x,t)}{\partial t}\equiv\rho_0+\frac{1}{\alpha^2}\frac{\partial f}{\partial t}, \quad (7.11)$$

$$j(x,t)\cong e\Psi^+(p/m)Y\Psi(x,t)-\nabla f(x,t)\equiv j_0-\nabla f,$$

where f satisfies the wave equation

$$\left(\Delta-\frac{1}{\alpha^2}\frac{\partial^2}{\partial t^2}\right)f\approx -2e\Psi^+\tau_2\phi\Psi. \quad (7.12)$$

(ρ_0,j_0) is the charge-current residing in the "core" of a quasi-particle. The latter is surrounded by a cloud of the excitation field f. In a static situation, for example, f will fall off like $1/r$ from the core. When the particle is accelerated, a fraction of the charge is exchanged between the core and the cloud.

The total charge residing in a finite volume around a core is not constant because the current $-\nabla f$ reaches out to infinity.

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[22] There may be transverse collective excitations (Bogoliubov, reference 2), but they do not automatically follow from the self-energy equation nor affect the energy gap structure.

8. THE PLASMA OSCILLATIONS

The inclusion of the annihilation-creation processes in the equations of the previous sections means that the vertex parts get multiplied by a string of closed loops, which represent the polarization (or shielding effect) of the surrounding medium. We will call the new quantities Λ, which now satisfy the following type of integral equations

$$\Lambda(p',p)=\gamma-i\int\tau_3 G(p'-k)\Lambda(p'-k,\ p-k)$$

$$\times G(p-k)\tau_3 D(k)d^4k$$

$$+iD(p'-p)\tau_3\int\mathrm{Tr}[\tau_3 G(p'-k) \quad (8.1)$$

$$\times\Lambda(p'-k,\ p-k)G(p-k)]d^4k,$$

$$D(q)\equiv -ig^2\bar{h}(q)^2\Delta(q)+e^2/q^2.$$

$D(q)$ includes the effect of the Coulomb interaction [see Eq. (2.24)]. Putting

$$\bar{X}(p'-p)\equiv i\int\mathrm{Tr}[\tau_3 G(p'-k)\Lambda(p'-k,\ p'-k)$$

$$\times G(p-k)]d^4k, \quad (8.2)$$

Eq. (8.1) takes the same form as Eq. (4.3) for Γ with the inhomogeneous term replaced by $\gamma+\tau_3 D\bar{X}$, so that Λ is a linear combination of the Γ corresponding to γ and Γ_0:

$$\Lambda=\Gamma+\Gamma_0 D\bar{X}. \quad (8.3)$$

Substitution in Eq. (8.2) then yields

$$\bar{X}(p'-p)=i\int\mathrm{Tr}[\tau_3 G(p'-k)$$

$$\times\Gamma(p'-k,\ p-k)G(p-k)]d^4k$$

$$+iD(p'-p)\bar{X}(p'-p)\int\mathrm{Tr}[\tau_3 G(p'-k)$$

$$\times\Gamma_0(p'-k,\ p-k)G(p-k)]d^4k,$$

or

$$\bar{X}(p'-p)=i\int\mathrm{Tr}[\tau_3 G(p'-k)$$

$$\times\Gamma(p'-k,\ p-k)G(p-k)]d^4k$$

$$\times\left\{1-iD(p'-p)\int\mathrm{Tr}[\tau_3 G(p'-k)\right.$$

$$\left.\times\Gamma_0(p'-k,\ p-k)G(p-k)]d^4k\right\}^{-1}$$

$$\equiv X(p'-p)/[1-D(p'-p)X_0(p'-p)]. \quad (8.4)$$

Especially for $\gamma = \tau_3$, we get

$$\bar{X}_0(p'-p) = X_0(p'-p)/[1-D(p'-p)X_0(p'-p)],$$
$$\Lambda_0(p',p) = \Gamma_0(p',p)/[1-D(p'-p)X_0(p',p)]. \quad (8.5)$$

To obtain the collective excitations, let us next write down the homogeneous integral equation:

$$\Theta_q(p) = -i \int \tau_3 G(p'+q/2)\Theta_q(p')$$

$$\times G(p'-q/2)\tau_3 D(p-p')d^4p'$$

$$+ i\tau_3 D(q) \int \text{Tr} \left[\tau_3 G(p'+q/2)\Theta_q(p') \right.$$
$$\left. \times G(p'-q/2) \right] d^4p', \quad (8.6)$$

which means

$$\Theta_q(p) = \Gamma_0(p+q/2, \ p-q/2)D(q)\chi(q),$$

$$\chi(q) \equiv i \int \text{Tr}[\tau_3 G(p'+q/2)$$
$$\times \Theta_q(p')G(p'-q/2)\tau_3]d^4p'. \quad (8.7)$$

Substituting Θ_q in the second equation from the first, we get

$$1 = D(q)X_0(q), \quad (8.8)$$

where $X_0(q)$ is defined in Eq. (8.4).

The solutions to Eq. (8.8) determine the new dispersion law $q_0 = f(q)$ for the collective excitations.

With the solution (7.6), the quantity X_0 in Eq. (8.8) can be calculated. After some simplifications using Eq. (6.11), we obtain

$$X_0 = \frac{1}{(2\pi)^3}\left[\frac{\alpha^2 q^2}{q_0^2 - \alpha^2 q^2} \int \frac{\phi^2 d^3 p}{E_p(E_p^2 - \alpha^2 q^2/4)} \right.$$

$$+ \frac{q^2}{4} \int \frac{\phi^2 d^3 p}{E_p(q_0^2/4 - E_p^2)} \left(\frac{p^2}{3m^2 E_p^2} \right.$$

$$\left. \left. - \frac{\alpha^2}{E_p^2 - \alpha^2 q^2/4} \right) \right] + O(q^4). \quad (8.9)$$

For $\alpha q \ll \phi$, and $q_0 \gg \phi$ or $\ll \phi$, the second integral may be dropped and

$$X_0 \cong \alpha^2 q^2 N/(q_0^2 - \alpha^2 q^2). \quad (8.10)$$

For small q^2, the dominant part of $D(q)$ in Eq. (8.8) is the Coulomb interaction e^2/q^2. Equation (8.8) then becomes

$$q_0^2 = e^2 \alpha^2 N = e^2 n \quad (q^2 \to 0), \quad (8.11)$$

where n is the number of electrons per unit volume. This agrees with the ordinary plasma frequency for free electron gas.

We see thus that the previous collective state with $q_0^2 = \alpha^2 q^2$ has shifted its energy to the plasma energy as a result of the Coulomb interaction.

On the other hand, if Coulomb interaction is neglected, Eq. (8.8) leads to[23]

$$q_0^2 = \alpha^2 q^2[1 - ig^2\Delta(q,q_0)h(q,q_0)^2N]. \quad (8.12)$$

The correction term, however, is of the order $g^2 N$, hence should be neglected to be consistent with our approximation.

We can also study the behavior of X_0 in the limit $q_0 \to 0$ for small but finite q^2:

$$X_0 \approx \frac{1}{(2\pi)^3} \int \frac{\phi^2}{E^3} d^3 p \approx N, \quad (8.13)$$

which comes entirely from the noncollective part of Γ_0, but again agrees with the free electron value.

Another observation we can make regarding $\bar{X}_0(q,q_0)$ is the following. $\bar{X}_0$ represents the charge density correlation in the ground state:

$$\bar{X}_0(q,q_0) = \int \langle 0 | T(\rho(xt),\rho(0)) | 0 \rangle e^{-i\mathbf{q}\cdot\mathbf{x}+iq_0 t}d^3x dt.$$

If $|0\rangle$ is an eigenstate of charge, $\bar{X}_0$ should vanish for $q \to 0$, $q_0 \neq 0$ since the right-hand side then consists of the nondiagonal matrix elements of the total charge operator Q:

$$\bar{X}_0(0,q_0) \propto \sum_n \left(\frac{1}{q_0 - E_n} - \frac{1}{q_0 + E_n} \right) |\langle n | Q | 0 \rangle|^2.$$

The converse is also true if $E_n > |q_0|$, $n \neq 0$ for some $q_0 \neq 0$. Our result for $\bar{X}_0$, as is clear from Eqs. (8.5) and (8.9), has indeed the correct property in spite of the fact that the "bare" vacuum, from which we started, is not an eigenstate of charge.

9. CONCLUDING REMARKS

We have discussed here formal mathematical structure of the BCS-Bogoliubov theory. The nature of the approximation is characterized essentially as the Hartree-Fock method, and can be given a simple interpretation in terms of perturbation expansion. In the presence of external fields, the corresponding approximation insures, if treated properly, that the gauge invariance is maintained. It is interesting that the quasi-particle picture and charge conservation (or gauge invariance) can be reconciled at all. This is possible because we are taking account of the "radiative corrections" to the bare quasi-particles which are not eigenstates of charge. These corrections manifest themselves primarily through the existence of collective excitations.

There are some questions which have been left out. We would like to know, for one thing, what will happen if we seek corrections to our Hartree-Fock approximation by including processes (or diagrams) which have not been considered here. Even within our ap-

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[23] Compare Anderson, reference 7.

proximation, there is an additional assumption of the weak coupling ($g^2N\ll1$), and the importance of the neglected terms (of order g^2N and higher) is not known.

Experimentally, there has been some evidence[24] regarding the presence of spin paramagnetism in superconductors. This effect has to do with the spin density induced by a magnetic field and can be derived by means of an appropriate vertex solution. However, this does not seem to give a finite spin paramagnetism at 0°K.[25]

[24] Knight, Androes, and Hammond, Phys. Rev. 104, 852 (1956); F. Reif, Phys. Rev. 106, 208 (1957); G. M. Androes and W. D. Knight, Phys. Rev. Letters 2, 386 (1959).
[25] K. Yosida, Phys. Rev. 110, 769 (1958).

The collective excitations do not play an important role here as they are not excited by spin density. [$\Gamma^{(a)}$, Eq. (4.4), does not have the characteristic pole.]

It is desirable that both experiment and theory about spin paramagnetism be developed further since this may be a crucial test of the fundamental ideas underlying the BCS theory.

ACKNOWLEDGMENT

Optical Constants of Metals

MACIEJ SUFFCZYNSKI*
Department of Mathematics, Imperial College, London, England
(Received August 17, 1959)

A calculation of the interband contribution to the frequency dependent dielectric constant of metals is attempted based on a specific model. The frequency region near the threshold for the interband transitions is considered. Emphasis in the model is laid on the bending of the energy bands near the Brillouin zone boundary. Attention is focused on cases when the Fermi surface approaches the zone boundary or has a finite area of contact with it. The momentum matrix element is taken as constant, which is fitted so as to achieve agreement with the experimentally found dip in the dispersion curve of the extinction coefficient. The values of the square of the matrix element for the noble metals, copper, silver, and gold, which fit the experimental data of Schulz, are found to be in the ratio 0.43:0.69:0.69.

1. INTRODUCTION

THE theoretical calculation of the optical constants of metals might be interesting as there is an accumulating amount of experimental data which now becomes sufficiently consistent to allow comparison with the computed curves. Moreover the theory might indicate where further measurements of the optical constants are needed to give more information on the band structure of metals. The present knowledge of the optical properties of metals is reviewed by Givens,[1] Schulz,[2] Ginsburg and Motulevich,[3] and Roberts.[3a]

Roughly speaking in the wavelength region below 1000 μ down to 10 μ (microns) the notions of the skin effect theory are more appropriate to describe the optical properties of metals.[1,4] It is mainly between 10 μ and 0.01 μ where the classical concepts of the two optical constants, the refractive index n and the extinction coefficient k, apply best. In the region between 10 μ and 1 μ the simple theory of Drude is

[1] M. P. Givens, *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1958), Vol. 6, p. 313.
[2] L. G. Schulz, Suppl. Phil. Mag. 6, 102 (1957).
[3] V. L. Ginsburg and G. P. Motulevich, Uspekhi Fiz. Nauk 55, 469 (1955).
[3a] S. Roberts, Phys. Rev. 100, 1667 (1955); 114, 104 (1959).
[4] R. B. Dingle, Physica 19, 311, 348, 729, 1187 (1953).

able to predict these constants, at least the extinction coefficient, quantitatively. The refractive index predicted is always too small.

In many metals pronounced deviations from the Drude theory are observed mainly in the region of wavelengths shorter than 1 μ. These are due to the effect of the interband electronic transitions between the occupied bands and the higher empty bands. This volume effect is neglected in the classical theory of Drude. It is the purpose here to propose a simple model which would allow the calculation of the contribution of the interband transitions to the optical constants of metals.

2. CLASSICAL EXPRESSIONS

The classical electromagnetic theory relates the experimentally measured refractive index n and the extinction coefficient k to the frequency dependent dielectric constant $\epsilon(\nu)$ and the electrical conductivity $\sigma(\nu)$. Instead of σ alternatively the imaginary part of the dielectric constant may be used: $\epsilon' = 2\sigma/\nu$

$$n = \{\tfrac{1}{2}[(\epsilon^2 + \epsilon'^2)^{\frac{1}{2}} + \epsilon]\}^{\frac{1}{2}},$$
$$k = \{\tfrac{1}{2}[(\epsilon^2 + \epsilon'^2)^{\frac{1}{2}} - \epsilon]\}^{\frac{1}{2}}. \tag{2.1}$$

The dielectric constant and the conductivity both consist of two parts: Drude and interband, as we shall