representations permit (A1c) to be written as

$$I=\frac{(8\pi)^3b(1+a)}{2\pi^2}\int_0^1 dx\, x(1-x)$$

$$\times\int_0^1 dy\int\frac{d\mathbf{K}_1}{K_1^2+\epsilon^2}\chi_1^{-\frac{1}{2}}\Big\{360y^3(1-y)Z_1^{-6}$$

$$+72y^2(1-y)\chi_1^{-1}Z_1^{-5}+\frac{54}{4}y(1-y)\chi_1^{-2}Z_1^{-4}$$

$$-Q^2[1440y^4(1-y)Z_1^{-7}+240y^3(1-y)\chi_1^{-1}Z_1^{-6}\quad\text{(A1d)}$$

$$+36y^2(1-y)\chi_1^{-2}Z_1^{-5}]\Big\},$$

$$u_1=(1-y)(1+A_3^2)+y(1-x)(b^2+A_2^2)+xy[(1+a)^2+A_1^2],$$

$$\mathbf{R}=\mathbf{A}_2+x(\mathbf{A}_1-\mathbf{A}_2),\qquad \mathbf{T}=(1-y)\mathbf{A}_3+y\mathbf{R},$$

$$Q^2=|\mathbf{R}+\mathbf{K}_1|^2,\qquad Z_1=K_1^2+2\mathbf{T}\cdot\mathbf{K}_1+u_1.$$

The following relations are used to perform the $\mathbf{K}_1$-integration:

$$\frac{2\mathbf{K}_1\cdot\mathbf{R}}{Z_1^n}=-\frac{R}{(n-1)y}\frac{\partial}{\partial R}Z_1^{-(n-1)},$$

$$\int\frac{d\mathbf{K}_1}{[K_1^2+\epsilon^2]Z_1^n}=\frac{\pi^2(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial u_1^{n-2}}u_1^{-1}v_1^{-\frac{1}{2}},$$

$$\int\frac{d\mathbf{K}_1K_1^2}{[K_1^2+\epsilon^2]Z_1^n}=\frac{\pi^2(-1)^n}{(n-1)!}\frac{\partial^{n-2}}{\partial u_1^{n-2}}v_1^{-\frac{1}{2}},$$

$$v_1=u_1-T^2,\qquad R=|\mathbf{R}|.$$

With the new notation (A1d) reduces to

$$I=\frac{(8\pi)^3b(1+a)}{2}\int_0^1 dx\,x(1-x)\chi_1^{-\frac{1}{2}}\int_0^1 dy\,y(1-y)$$

$$\times\Bigg\{\bigg[3y^2\frac{\partial^4}{\partial u_1^4}-3y\chi_1^{-1}\frac{\partial^3}{\partial u_1^3}+\frac{9}{4}\chi_1^{-2}\frac{\partial^2}{\partial u_1^2}$$

$$+R^2\bigg(2y^3\frac{\partial^5}{\partial u_1^5}-2y^2\chi_1^{-1}\frac{\partial^4}{\partial u_1^4}+\frac{3y}{2}\chi_1^{-2}\frac{\partial^3}{\partial u_1^3}\bigg)$$

$$+R\frac{\partial}{\partial R}\bigg(2y^3\frac{\partial^4}{\partial u_1^4}-2y^2\chi_1^{-1}\frac{\partial^3}{\partial u_1^3}+\frac{3y}{2}\chi_1^{-2}\frac{\partial^2}{\partial u_1^2}\bigg)\bigg]u_1^{-1}v_1^{-\frac{1}{2}}$$

$$+\bigg[2y^3\frac{\partial^5}{\partial u_1^5}-2y^2\chi_1^{-1}\frac{\partial^4}{\partial u_1^4}+\frac{3y}{2}\chi_1^{-2}\bigg]v_1^{-\frac{1}{2}}\Bigg\}.\quad\text{(A1e)}$$

With the omission of the factor outside the integral sign (A1e) can be written in the form given for I_1' under Case II in the main body of this paper.

PHYSICAL REVIEW VOLUME 117, NUMBER 2 JANUARY 15, 1960

Range Straggling of Charged Particles in Be, C, Al, Cu, Pb, and Air*

R. M. STERNHEIMER
Brookhaven National Laboratory, Upton, New York
(Received August 5, 1959)

The straggling of the range of charged particles due to fluctuations of the ionization loss has been evaluated for six substances (Be, C, Al, Cu, Pb, and air). The calculations extend up to $T/\mu c^2 \sim 100$, where T is the kinetic energy and μ is the mass of the incident particle. At high energies ($T/\mu c^2 \gtrsim 5$), the integral giving the range straggling becomes somewhat dependent on the ratio μ/m, where m is the electron mass. Two separate calculations have therefore been carried out, which apply to protons and μ mesons, respectively. The results for protons can also be used for π and K mesons in the energy range of interest ($T/\mu c^2 \lesssim 5$).

I. INTRODUCTION

RECENTLY tables of the range-energy relations for protons[1] have been obtained for several substances, which are based on accurate values[2] of the mean excitation potential I. These tables were calculated up to a maximum proton energy $T_p=100$ Bev, in order to enable one to obtain ranges of μ mesons up to an energy $T_\mu \sim 10$ Bev. The calculations were carried out for Be, C, Al, Cu, Pb, and air. In connection with these tables, it seemed of interest also to obtain the range straggling due to the fluctuations of the ionization loss process as given by the theory of Bohr.[3] In the present paper, we give the results of these calculations. It may be noted that the range straggling in nuclear emulsion has been previously investigated by Barkas,

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] R. M. Sternheimer, Phys. Rev. **115**, 137 (1959). This paper will be referred to as I.

[2] Bichsel, Mozley, and Aron, Phys. Rev. **105**, 1788 (1957); V. C. Burkig and K. R. MacKenzie, Phys. Rev. **106**, 848 (1957); D. O. Caldwell, Phys. Rev. **100**, 291 (1955).

[3] N. Bohr, Phil. Mag. **30**, 581 (1915).

Smith, and Birnbaum.[4] Calculations of the range straggling in copper, up to a proton energy $T_p=1$ Bev, have also been carried out by Millburn and Schecter.[5] Symon[6] has obtained results for the range straggling in iron (up to an energy $T/\mu c^2=10$), which are reported in Rossi's book.[7]

II. CALCULATIONS

The expression for the mean square deviation of the range, $\langle(\Delta R)^2\rangle$, which was used in the present calculations, is a slight modification of the expression derived by Lindhard and Scharff[8] for relativistic particles. Thus $\langle(\Delta R)^2\rangle$ is obtained from

$$\langle(\Delta R)^2\rangle=4\pi z^2e^4n\int_0^{T_1}\frac{(1-\frac{1}{2}\beta^2)}{(1-\beta^2)[1+(2m/\mu)\gamma]}\times\left(-\frac{dE}{dx}\right)^{-3}KdT, \quad (1)$$

where z= charge of incident particle, n= number of electrons per cm^3 in the stopping material, $\beta=v/c$, where v is the velocity of the particle, m= electron mass, μ= mass of incident particle, $\gamma\equiv(1-\beta^2)^{-\frac{1}{2}}$, dE/dx is the ionization loss (in ergs/cm), T= kinetic energy variable, and T_1 is the kinetic energy for which $\langle(\Delta R)^2\rangle$ is evaluated. In Eq. (1), the factor K takes into account the effects of binding on the atomic electrons at low energies of the incident particle. The expression for K has been derived by Bethe,[9] and will be discussed below. Equation (1) differs from the expression of Lindhard and Scharff[8] by the presence of the factor $[1+(2m/\mu)\gamma]$ in the denominator. This factor arises from the expression for the maximum energy transfer from the incident particle to an atomic electron[10]:

$$W_{\max}=\frac{E^2-\mu^2c^4}{\mu c^2[(\mu/2m)+(m/2\mu)+(E/\mu c^2)]}\cong\frac{2mv^2}{(1-\beta^2)[1+(2m/\mu)\gamma]}, \quad (2)$$

where E is the total energy of the particle (including the rest energy μc^2). Instead of (2), Lindhard and Scharff[8] used $W_{\max}=2mv^2/(1-\beta^2)$. This procedure is justified except at very high energies when $\gamma\gtrsim(\mu/2m)$. The integrand in the expression for $\langle(\Delta R)^2\rangle$ from the Bohr theory[3] involves the factor

$$\frac{1}{mv^2}\int_0^{W_{\max}}\left(1-\beta^2\frac{W}{W_{\max}}\right)dW=\frac{W_{\max}}{mv^2}(1-\tfrac{1}{2}\beta^2)=\frac{2(1-\frac{1}{2}\beta^2)}{(1-\beta^2)[1+(2m/\mu)\gamma]}, \quad (3)$$

where W is the variable energy transfer. Thus $[1+(2m/\mu)\gamma]$ enters as an additional factor in the denominator of the integrand of (1).

Equation (1) can be rewritten as follows:

$$\langle(\Delta R)^2\rangle_{(\mathrm{g\,cm^{-2}})^2}=1.022Az^2\times\int_0^{T_1}\frac{(1-\frac{1}{2}\beta^2)}{(1-\beta^2)[1+(2m/\mu)\gamma]}\times\left[-\left(\frac{1}{\rho}\right)\left(\frac{dE}{dx}\right)\right]^{-3}KdT, \quad (4)$$

where $A\equiv 2\pi ne^4/(mc^2\rho)$ is in units Mev/g cm^{-2}, (see Table I of I); ρ is the density of the material; 1.022 $=2mc^2/$(Mev). In Eq. (4), $[-(1/\rho)(dE/dx)]$ should be in units Mev/g cm^{-2}, and T_p in Mev. We have $A=0.1536(Z/A_0)$, where Z and A_0 are the atomic number and atomic weight of the substance. As indicated, Eq. (4) gives $\langle(\Delta R)^2\rangle$ in units (g cm^{-2})^2.

It may be noted that for very high energies, both the factors $[1+(2m/\mu)\gamma]^{-1}$ and $[-(1/\rho)(dE/dx)]^{-3}$ become significantly dependent on the mass μ of the incident particle. We will define the percentage range straggling ϵ as follows:

$$\epsilon\equiv100(\sigma/R), \quad (5)$$

where $\sigma\equiv[\langle(\Delta R)^2\rangle]^{\frac{1}{2}}$. Except for the μ dependence described above, ϵ for particles of a given velocity β is proportional to $\mu^{-\frac{1}{2}}$. Thus[11]

$$\epsilon_i(\beta)=\epsilon_p(\beta)(\mu_p/\mu_i)^{\frac{1}{2}}, \quad (6)$$

where $\epsilon_i\equiv\epsilon$ for particle i, $\epsilon_p\equiv$ corresponding ϵ for protons, and μ_p, μ_i are the proton mass and the mass of particle i, respectively.

For high energies, two separate calculations were performed for each substance: the integral of Eq. (4) was evaluated both for protons and μ mesons. The results for protons can also be used for K and π mesons [by means of Eq. (6)] in the energy range of interest which extends to $\gamma\sim3$ for K mesons and $\gamma\sim6$ for pions. These values of γ correspond to a range of ~4 geometrical mean free paths for nuclear interactions in copper (attenuation to 1.8%).

The factor K in Eqs. (1) and (4) will now be dis-

[4] Barkas, Smith, and Birnbaum, Phys. Rev. **98**, 605 (1955).

[5] G. P. Millburn and L. Schecter, University of California Radiation Laboratory Report UCRL-2234 (revised), 1953 (unpublished).

[6] K. R. Symon, Harvard University thesis, 1948 (unpublished).

[7] B. Rossi, *High-Energy Particles* (Prentice-Hall, Inc., New York, 1952), p. 37.

[8] J. Lindhard and M. Scharff, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 15 (1953).

[9] M. S. Livingston and H. A. Bethe, Revs. Modern Phys. **9**, 261 (1937).

[10] H. J. Bhabha, Proc. Roy. Soc. (London) **A164**, 257 (1937).

[11] See, for example, H. A. Bethe and J. Ashkin, in *Experimental Nuclear Physics*, edited by E. Segrè (John Wiley and Sons, New York, 1952), Vol. 1, p. 246.

cussed. From the work of Bethe,[9] K is given by

$$K=\frac{Z_{\rm eff}}{Z}+\sum_n \frac{k_n I_n Z_n}{mv^2}\ln\left(\frac{2mv^2}{I_n}\right), \tag{7}$$

where Z_n is the number of electrons in the nth shell of the atoms of the stopping material, I_n is the corresponding effective excitation potential for the nth shell, k_n is a constant which was taken as 4/3 for all shells,[9] the sum over n extends over the shells for which $I_n<2mv^2$, and $Z_{\rm eff}$ is the effective number of electrons which participate in the stopping process at the incident velocity v considered, i.e., $Z_{\rm eff}$ is the sum of the Z_n for the shells for which $I_n<2mv^2$.

The values of I_n which enter into Eq. (7) were obtained in the same manner as in the previous calculations of the density effect for the ionization loss.[12] The effective excitation potentials I_n for the various shells satisfy the condition:

$$\textstyle\sum_n f_n \ln I_n=\ln I, \tag{8}$$

where f_n is the oscillator strength of the nth shell ($f_n=Z_n/Z$), the sum goes over all shells, and I is the mean excitation potential which was used in calculating dE/dx in I (see Table I of I).

Typical values of K are as follows: For Be, $K=1.37$ for a proton energy $T_p=1$ Mev, $K=1.24$ at 2 Mev, 1.12 at 5 Mev, 1.07 at 10 Mev, and 1.02 at 50 Mev. As was pointed out by Bethe,[9,11] K decreases rapidly towards 1 with increasing kinetic energy. For Al, $K=1.33$ at $T_p=2$ Mev, 1.27 at 5 Mev, 1.19 at 10 Mev, 1.06 at 50 Mev. and 1.04 at 100 Mev. Similarly, for Pb, $K=1.31$ at $T_p=5$ Mev, 1.22 at 10 Mev, 1.19 at 50 Mev, 1.16 at 100 Mev, and 1.11 at 200 Mev.

TABLE I. Values of the percentage range straggling ϵ_p for protons in Be, C, Al, Cu, Pb, and air.

T_p (Mev)	$T/\mu c^2$	Be	C	Al	Cu	Pb	Air
2	0.00213	1.704	1.867	1.968	2.293	2.659	1.981
4	0.00426	1.550	1.631	1.814	2.030	2.331	1.679
6	0.00640	1.469	1.526	1.720	1.875	2.187	1.564
8	0.00853	1.419	1.466	1.649	1.779	2.079	1.498
10	0.01066	1.382	1.424	1.597	1.749	1.994	1.452
15	0.01599	1.322	1.357	1.507	1.667	1.865	1.376
20	0.02132	1.286	1.315	1.450	1.609	1.828	1.335
25	0.02665	1.259	1.285	1.408	1.562	1.783	1.304
30	0.03198	1.238	1.263	1.377	1.526	1.742	1.280
40	0.0426	1.206	1.230	1.330	1.468	1.674	1.244
50	0.0533	1.183	1.203	1.297	1.425	1.619	1.218
60	0.0640	1.165	1.183	1.271	1.390	1.574	1.197
70	0.0746	1.149	1.166	1.249	1.363	1.536	1.180
80	0.0853	1.135	1.152	1.231	1.339	1.505	1.165
100	0.1066	1.112	1.128	1.201	1.300	1.464	1.141
120	0.1279	1.094	1.109	1.178	1.271	1.431	1.121
140	0.1493	1.078	1.093	1.157	1.245	1.401	1.104
160	0.1706	1.065	1.078	1.140	1.225	1.376	1.087
200	0.2132	1.041	1.054	1.112	1.190	1.333	1.060
250	0.2665	1.017	1.029	1.084	1.155	1.289	1.036
300	0.3198	0.997	1.009	1.060	1.127	1.254	1.016
350	0.373	0.980	0.991	1.040	1.104	1.225	0.999
400	0.426	0.966	0.976	1.023	1.085	1.200	0.984
500	0.533	0.942	0.952	0.996	1.052	1.160	0.959
600	0.640	0.924	0.932	0.974	1.028	1.127	0.939
800	0.853	0.897	0.904	0.943	0.991	1.081	0.909
1000	1.066	0.879	0.886	0.921	0.966	1.049	0.889
1500	1.599	0.856	0.862	0.891	0.928	1.002	0.860
2000	2.132	0.850	0.855	0.882	0.915	0.981	0.850
2500	2.665	0.853	0.857	0.882	0.913	0.973	0.849
3000	3.198	0.860	0.864	0.888	0.917	0.972	0.853
4000	4.26	0.881	0.884	0.907	0.933	0.982	0.869
5000	5.33	0.907	0.910	0.931	0.957	1.000	0.889
7000	7.46	0.964	0.965	0.985	1.010	1.047	0.935
10 000	10.66	1.048	1.049	1.067	1.092	1.122	1.005
15 000	15.99	1.178	1.179	1.195	1.220	1.244	1.115
20 000	21.32	1.295	1.295	1.311	1.336	1.357	1.214
25 000	26.65	1.401	1.401	1.416	1.442	1.460	1.304
30 000	31.98	1.499	1.498	1.512	1.540	1.555	1.384
40 000	42.63	1.674	1.674	1.686	1.716	1.727	1.508
60 000	63.95	1.969	1.968	1.979	2.013	2.018	1.772
80 000	85.27	2.214	2.212	2.223	2.260	2.262	1.983
100 000	106.6	2.426	2.425	2.435	2.475	2.475	2.168

[12] R. M. Sternheimer, Phys, Rev. **88**, 851 (1952); **103**, 511 (1956).

III. RESULTS

The results of the calculations are given in Tables I and II. For the calculations for protons (Table I), the

TABLE II. Values of the percentage range straggling ϵ_μ for μ mesons in Be, C, Al, Cu, Pb, and air.

T_μ (Mev)	Be	C	Al	Cu	Pb	Air
0.225	5.081	5.566	5.866	6.835	7.927	5.904
0.450	4.621	4.862	5.408	6.052	6.948	5.005
0.676	4.379	4.550	5.127	5.588	6.520	4.662
0.901	4.229	4.371	4.917	5.302	6.198	4.467
1.126	4.120	4.244	4.761	5.213	5.944	4.328
1.689	3.941	4.044	4.491	4.968	5.558	4.101
2.252	3.834	3.921	4.322	4.798	5.448	3.981
2.815	3.753	3.831	4.197	4.657	5.315	3.888
3.378	3.690	3.764	4.104	4.548	5.193	3.815
4.50	3.596	3.666	3.966	4.375	4.991	3.709
5.63	3.527	3.587	3.867	4.247	4.826	3.630
6.76	3.472	3.527	3.788	4.144	4.693	3.569
7.88	3.425	3.477	3.724	4.063	4.580	3.519
9.01	3.384	3.435	3.670	3.992	4.487	3.474
11.26	3.316	3.364	3.580	3.876	4.365	3.401
13.51	3.260	3.306	3.511	3.788	4.265	3.341
15.76	3.214	3.258	3.450	3.713	4.178	3.290
18.02	3.174	3.214	3.399	3.652	4.102	3.240
22.52	3.104	3.142	3.315	3.548	3.972	3.160
28.15	3.031	3.069	3.231	3.442	3.844	3.090
33.78	2.973	3.006	3.161	3.360	3.737	3.029
39.4	2.922	2.955	3.101	3.291	3.653	2.977
45.0	2.878	2.910	3.050	3.234	3.577	2.932
56.3	2.809	2.837	2.970	3.136	3.456	2.858
67.6	2.754	2.780	2.904	3.064	3.360	2.798
90.1	2.673	2.695	2.809	2.953	3.221	2.709
112.6	2.621	2.641	2.744	2.879	3.124	2.648
168.9	2.536	2.553	2.641	2.749	2.969	2.549
225.2	2.511	2.526	2.607	2.703	2.900	2.512
281.5	2.515	2.526	2.602	2.692	2.871	2.504
337.8	2.530	2.542	2.614	2.699	2.864	2.511
450	2.586	2.596	2.661	2.740	2.884	2.548
563	2.655	2.663	2.725	2.800	2.930	2.600
788	2.803	2.808	2.867	2.938	3.048	2.719
1126	3.022	3.025	3.079	3.150	3.239	2.899
1689	3.350	3.352	3.400	3.471	3.542	3.171
2252	3.633	3.635	3.679	3.753	3.812	3.408
2815	3.882	3.881	3.923	4.000	4.051	3.614
3378	4.101	4.100	4.140	4.218	4.264	3.795
4504	4.475	4.475	4.510	4.592	4.628	4.103
6756	5.044	5.041	5.072	5.163	5.183	4.565
9008	5.461	5.457	5.487	5.584	5.597	4.904
11 260	5.786	5.783	5.813	5.913	5.921	5.174

values of $-(1/\rho)(dE/dx)$ used in Eq. (4) were the same as those listed in Table II of I. Table I also gives the values of $T/\mu c^2$, which are useful in converting the table to pions or K mesons, using Eq. (6).

For the μ-meson calculations (Table II), the values of $-(1/\rho)(dE/dx)$ are slightly smaller at high energies than the values of $-(1/\rho)(dE/dx)$ for protons given in I, as a result of the somewhat smaller maximum energy transfer $W_{\max}$ for μ mesons. In view of Eq. (15) of I, we have

$$-\left(\frac{1}{\rho}\right)\left(\frac{dE}{dx}\right)_\mu=-\left(\frac{1}{\rho}\right)\left(\frac{dE}{dx}\right)_p -\frac{A}{\beta^2}\ln\left[\frac{1+(2m/\mu_\mu)\gamma}{1+(2m/\mu_p)\gamma}\right], \quad (9)$$

where $(dE/dx)_\mu$ and $(dE/dx)_p$ represent the ionization loss for μ mesons and protons of the same velocity β; μ_μ is the μ-meson mass.

Concerning the evaluation of (4), it may be noted that the fact that the quantity $G\equiv[1+(2m/\mu)\gamma]^{-1}$ is smaller for μ mesons than for protons accounts for the predominant part of the deviation of $\epsilon_\mu(\beta)$ from Eq. (6) at very high energies. If we denote $(\epsilon_\mu/\epsilon_p)(\mu_\mu/\mu_p)^{\frac{1}{2}}$ by ξ, one finds for $T_\mu=11.26$ Bev (corresponding to $T_p=100$ Bev): $\xi=0.801$ for Be and 0.802 for Pb. As is expected, the deviation $1-\xi$ decreases rapidly with decreasing energy. Thus for $T_\mu=1.126$ Bev ($T_p=10$ Bev), one obtains $\xi=0.968$ both for Be and Pb. We note that the ratio ξ also includes two other small effects besides the influence of the factor G: (A) The fact that $-(1/\rho)(dE/dx)_\mu$ is slightly smaller than $-(1/\rho)\times(dE/dx)_p$ [Eq. (9)] tends to increase ξ towards 1. However, this effect is very small. For $T_\mu=11.26$ Bev, the change of dE/dx would by itself (without G) increase ϵ_μ by a factor 1.024 for Be, and 1.027 for Pb, as compared to the values of $\xi\equiv0.801$ and 0.802, respectively, which include G. (B) The values of ξ also include a factor $1/F_\mu$, where F_μ has been defined in Eq. (14) of I. The quantity F_μ represents the lengthening of the range for μ mesons due to the slight decrease of $-(1/\rho)(dE/dx)$, as given in Eq. (9) above. The effect of F_μ is a slight decrease of ϵ_μ. Numerically this effect is even smaller than that described above under (A). Thus for $T_\mu=11.26$ Bev, $F_\mu=1.012$ for Be and 1.013 for Pb (see Table IV of I).

As an example of the use of Table II, we obtain σ in Cu for μ mesons of energy $T_\mu=2.252$ Bev. We have $\epsilon_\mu=3.753$, and Table III of I gives for the proton range R_p at the equivalent energy $T_p=20$ Bev: $R_p=12\,604$ g cm^{-2}. The factor F_μ is 1.0041 (see Table IV of I). Thus the μ-meson range R_μ is given by:

$$R_\mu=12\,604\times0.1126\times1.0041=1425 \text{ g cm}^{-2}, \quad (10)$$

where 0.1126 is the ratio $\mu_\mu/\mu_p=105.6/938.2$. Finally from the definition of ϵ_μ [Eq. (5)], we find

$$\sigma_\mu=10^{-2}\epsilon_\mu R_\mu=0.03753\times1425=53.48 \text{ g cm}^{-2}. \quad (11)$$

It may be noted that R_μ and σ_μ correspond to 159. 4 cm and 5.98 cm of Cu, respectively.

The straggling parameter S, which has been used by Bethe and Ashkin,[11] is given by

$$S\equiv(\pi/2)^{\frac{1}{2}}\sigma=1.253\times53.48=67.01 \text{ g cm}^{-2}. \quad (12)$$

In terms of S, the distribution of ranges R is given by

$$P(R)dR=\frac{1}{2S}\exp\left[-\left(\frac{\pi}{4S^2}\right)(R-R_0)^2\right]dR, \quad (13)$$

where $P(R)dR$ is the probability that the actual (measured) range lies between R and $R+dR$; R_0 is the mean range as calculated in I. It may be noted that in Eq. (13), the small difference between the mean range and the most probable range has been neglected.[13] This difference produces a small deviation from the (symmetric) Gaussian distribution of ranges, as has been discussed by Lewis.[13]

The values of Tables I and II show that ϵ at first decreases with increasing energy, and has a minimum at $T/\mu c^2\sim2.5$, which is followed by an increase at higher energies. These results are in essential agreement with those obtained previously by Barkas *et al.*[4] and by Symon.[6,7] It may be noted that the minimum of ϵ lies in the general region of the minimum of the ionization loss dE/dx. However, the rise of ϵ beyond the minimum is somewhat more rapid than the relativistic rise of dE/dx. The reason is that, although both increases are due to the increase of $W_{\max}$ with T, $W_{\max}$ enters directly into the integral for σ^2 [Eq. (3)], whereas it appears only in the logarithmic term of the expression for dE/dx. As an example of the rise of ϵ, we note that for μ mesons in Cu, the minimum of ϵ is: $\epsilon_{\min}=2.69$, and occurs at $T_\mu=280$ Mev. At the highest energy of Table II, $T_\mu=11.26$ Bev, ϵ is 5.913. Thus the ratio $\epsilon(11.26\text{ Bev})/\epsilon_{\min}$ is 2.20.

It can also be seen from Tables I and II that ϵ is almost independent of Z, showing only a very small increase in going from Be to Pb (at constant energy T). Thus for 100-Mev protons, the ratio $\epsilon_p(Z)/\epsilon_p(\text{Cu})$ is 0.855 for Be, 0.868 for C, 0.878 for air, 0.924 for Al, and 1.126 for Pb. For μ mesons of energy 1.126 Bev, $\epsilon_\mu(Z)/\epsilon_\mu(\text{Cu})$ is 0.959 for Be, 0.960 for C, 0.977 for Al, and 1.028 for Pb. Similar results concerning the variation of ϵ with Z were obtained in reference 5.

IV. ACKNOWLEDGMENT

I wish to thank Dr. David Berley for helpful discussions.

[13] H. W. Lewis, Phys. Rev. **85**, 20 (1952).