Radio Emission by Plasma Oscillations in Nonuniform Plasmas*

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Equations of motion for small-amplitude plasma oscillations interacting with the electromagnetic field in slowly varying density or temperature gradients are set up. We then make a calculation of the radio noise excited by a wave packet of plasma oscillations traversing such gradients using the WKB approximation. A similar calculation is also made for a density discontinuity.

1. INTRODUCTION

In this paper we are concerned with the excitation of electromagnetic radiation inside a plasma by plasma oscillations. The source of radiation will be that which derives from the coupling between the longitudinal electric vector and the transverse electromagnetic field as a result of density or temperature gradients. We shall suppose that there is no externally applied magnetostatic field so that the electromagnetic fields present are those alternating fields associated with the electron oscillations. In the first section we set up an equation for the electric field in the plasma supposing that the temperature and density are slowly varying functions of position, and neglect all quantities second order in their gradients. We then calculate the radio noise excited by plasma oscillations for two extreme cases, namely, gradients with a length scale $L\gg\lambda$ for which we make use of the WKB approximation, and the case of density discontinuities $L=0$, where λ is the wavelength of the plasma oscillations. A discussion of the propagation of purely longitudinal waves using the WKB solutions for slowly varying gradients has been given by Watson.[1]

Field[2] previously considered this problem. He gave equations for the case of a slowly varying density ($L\gg\lambda$) and also explicitly calculated the radio emission produced by plasma oscillations striking a plasma-vacuum boundary ($L=0$). In his derivation of the electric field equations for large L, however, he omits the anisotropy of the pressure in a tenuous plasma and the restraining field $\mathbf{E}'$ discussed below. Gould,[3] starting with the same moment equations as Field, has made calculations of the radiation excited by plasma oscillations in a region of random density fluctuations which he characterized by a mean square fluctuation and a correlation length. He developed a theory in which the irregularities were treated as a perturbation from the uniform case and his approach can be considered as complementary to ours.

2. BASIC EQUATIONS

Consider a plasma which is so tenuous that collisions of the electrons may be neglected and in which the ion temperature $T_0(x_i)$ and density $N(x_i)$ are slowly varying functions of position. The plasma will, of course, not be in hydrostatic equilibrium as a result of the nonvanishing pressure gradient $2\nabla(NkT_0)$ of the electrons and ions. However, any disturbances in the electron component will propagate much more rapidly than the bulk streaming of the plasma which will proceed with about the ion thermal velocity. Thus one may approximately consider the electron plasma motions to take place in the presence of a static nonuniform ion distribution, and our basic equations become the Boltzmann equation for the electron distribution function $f(u_i,x_i,t)$,

$$\partial f/\partial t+\mathbf{u}\cdot\nabla f-e/m\left(\boldsymbol{\mathcal{E}}+\frac{1}{c}\mathbf{u}\times\mathbf{H}\right)\cdot\partial f/\partial\mathbf{u}=0, \quad (1)$$

together with Maxwell's equations. We shall use the following standard[4] kinetic theory definitions of the electron fluid velocity v_i, density Σ, pressure Π_{ij}, temperature T, and heat tensor Q_{ijk},

$$\begin{aligned}\Sigma&=\int fd\mathbf{u},\\ v_i&=\frac{1}{\Sigma}\int u_ifd\mathbf{u},\\ T&=\frac{m}{3k\Sigma}\int C^2fd\mathbf{u},\\ \Pi_{ij}&=m\int C_iC_jfd\mathbf{u},\\ Q_{ijk}&=\frac{m}{2}\int C_iC_jC_kfd\mathbf{u}.\end{aligned} \quad (2)$$

The random thermal velocity C_i is simply u_i-v_i. The first three moments of Eq. (1) become, neglecting terms in the square of the electron fluid velocity,

$$\frac{\partial\Sigma}{\partial t}+\frac{\partial}{\partial x_i}(v_i\Sigma)=0, \quad (3)$$

$$\frac{\partial v_i}{\partial t}+\frac{1}{m\Sigma}\frac{\partial}{\partial x_j}\Pi_{ij}=-\frac{e}{m}\left[\mathcal{E}_i+\frac{1}{c}\epsilon_{ijk}v_jH_k\right], \quad (4)$$

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$$\frac{\partial}{\partial t}\Pi_{ij}+2\frac{\partial}{\partial x_k}Q_{ijk}+\frac{\partial}{\partial x_k}[v_i\Pi_{jk}+v_j\Pi_{ik}+v_k\Pi_{ij}]$$

$$+e\Sigma(\mathcal{E}_iv_j+\mathcal{E}_jv_i)+\frac{e}{mc}[\epsilon_{jkh}H_h\Pi_{ik}+\epsilon_{ikh}H_h\Pi_{jk}]=0, \quad (5)$$

where ϵ_{ijk} is the unit antisymetric tensor. Now we can close the set of coupled moment equations at the second moment only if we make some assumption about the third moment Q_{ijk} occurring in (5). In the following calculation we neglect this heat flow term in the plasma, as is frequently done.[1–3] We shall also in the following discussion use suffix notation for vectors only when they occur in equations with tensors.

Now consider an initial state in which the electron fluid is in equilibrium with the background ions of nonuniform density $N(\mathbf{x})$ and temperature $T_0(\mathbf{x})$. The momentum Eq. (4) then gives an expression for the electrostatic field $\mathbf{E}'$ which balances the gradient of the electron pressure in this equilibrium state,[5]

$$\nabla P=-e(N+n')\mathbf{E}'. \quad (6)$$

The small density deviation n' of the electrons from that of the background protons N is from Maxwell's equation $4\pi e(\Sigma-N)=-\nabla\mathcal{E}$,

$$n'=\frac{1}{4\pi e^2}\nabla\left(\frac{\nabla P}{N+n'}\right). \quad (7)$$

This is of order L^{-2} where L is the scale of the density or temperature nonuniformities and will be neglected. Thus insofar as we neglect such terms, the equilibrium state will be characterized by a distribution function f_0 with its corresponding density N, pressure $P\delta_{ij}$, fluid velocity zero, temperature T_0, and a restraining electrostatic field $\mathbf{E}'=-\nabla P/eN$ with zero magnetic field. Now suppose we perturb this system by writing the distribution function as f_0+f'. The perturbed moments will then be represented by

$$\Sigma=N+n=\int(f_0+f')d\mathbf{u}, \quad (8)$$

$$\Pi_{ij}=P\delta_{ij}+p_{ij}=m\int C_iC_j(f_0+f')d\mathbf{u}, \quad (9)$$

$$v_i=\int u_if'd\mathbf{u}, \quad (10)$$

$$T=T_0+T'=\frac{m}{3k(N+n)}\int C^2(f_0+f')d\mathbf{u}. \quad (11)$$

The change n in the electron density will produce an additional electric field E_i such that

$$(n+n')=-\frac{1}{4\pi e}\frac{\partial}{\partial x_i}(E_i+E_i'). \quad (12)$$

Using the expressions (8)–(11) in the set of Eqs. (3)–(5) and dropping terms in the perturbations squared, we obtain the linearized set of equations:

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x_i}(Nv_i)=0, \quad (13)$$

$$N\frac{\partial v_i}{\partial t}-\frac{n}{mN}\frac{\partial P}{\partial x_i}+\frac{1}{m}\frac{\partial}{\partial x_j}p_{ij}+\frac{Ne}{m}E_i=0, \quad (14)$$

$$\frac{\partial p_{ij}}{\partial t}+P\frac{\partial v_i}{\partial x_j}+P\frac{\partial v_j}{\partial x_i}+\frac{\partial}{\partial x_k}(v_kP\delta_{ij})=0, \quad (15)$$

together with Maxwell's equations,

$$\nabla\mathbf{E}=-4\pi en, \quad (16)$$

$$c\nabla\times\mathbf{H}=\partial\mathbf{E}/\partial t-4\pi Ne\mathbf{v}, \quad (17)$$

$$c\nabla\times\mathbf{E}=-\partial\mathbf{H}/\partial t. \quad (18)$$

The energy flux transported in the plasma by the electromagnetic field and electron fluid motions corresponding to the above equations is readily shown to be

$$S_i=\frac{c}{4\pi}\epsilon_{ijk}E_jH_k+\tfrac{1}{2}v_ip_{jj}+v_jp_{ij}. \quad (19)$$

We next require an equation for the electric field in the nonuniform plasma. First differentiate (14) twice with respect to time and use (13) and (15) to eliminate $\partial n/\partial t$ and $\partial p_{ij}/\partial t$, respectively. Then differentiating (17) with respect to time, and using (18) to eliminate $\partial\mathbf{H}/\partial t$, we obtain

$$N\frac{\partial v_i}{\partial t}=\frac{\partial^2E_i}{\partial t^2}+Q_i, \quad (20)$$

where

$$Q_i=c^2\epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{kmn}\frac{\partial E_n}{\partial x_m}. \quad (21)$$

We now use (20) to eliminate $\partial\mathbf{v}/\partial t$ obtaining finally an equation for the electric field, namely

$$\frac{\partial^2}{\partial t^2}\left\{\frac{\partial^2E_i}{\partial t^2}+\omega_e^2(\mathbf{x})E_i-\frac{kT}{m}\left[\frac{\partial^2E_i}{\partial x_j^2}+2\frac{\partial^2E_j}{\partial x_i\partial x_j}+\frac{1}{T_0}\frac{dT_0}{dx_j}\left(\frac{\partial E_i}{\partial x_j}+2\frac{\partial E_j}{\partial x_i}\right)-\frac{1}{N}\frac{dN}{dx_j}\frac{\partial E_i}{\partial x_j}-\frac{2}{N}\frac{dN}{dx_i}\frac{\partial E_j}{\partial x_j}\right]\right\}$$

$$=\frac{kT_0}{m}\left[\frac{\partial^2Q_i}{\partial x_j^2}+\frac{1}{T_0}\frac{dT_0}{dx_j}\frac{\partial Q_j}{\partial x_i}-\frac{1}{N}\frac{dN}{dx_j}\frac{\partial Q_i}{\partial x_j}\right]$$

$$-\frac{\partial^2Q_i}{\partial t^2}+\frac{1}{T_0}\frac{dT_0}{dx_j}\left(\frac{\partial Q_i}{\partial x_j}+\frac{\partial Q_j}{\partial x_i}\right). \quad (22)$$

[5] The assumption of a scalar pressure for the quasi-equilibrium state in the presence of arbitrary density and pressure gradients is correct to order L^{-2}.

3. RADIATION EQUATIONS

In order to exhibit the coupling which occurs between the electromagnetic field and plasma oscillations, we shall first split the electric field $\mathbf{E}$ into its longitudinal and transverse parts $\mathbf{E}_L$ and $\mathbf{E}_T$ defined by

$$\mathbf{E}=\mathbf{E}_L+\mathbf{E}_T,\quad \nabla\cdot\mathbf{E}_T=0,\quad \nabla\times\mathbf{E}_L=0. \tag{23}$$

Using these, and denoting the root mean square thermal velocity in the electron plasma by

$$V=(3kT_0/m)^{\frac{1}{2}}$$

and the plasma frequency

$$\omega_e=\left(\frac{4\pi Ne^2}{m}\right)^{\frac{1}{2}},$$

Eq. (22) for the electric field becomes

$$\left[\left(\frac{\partial^2}{\partial t^2}+\omega_e^2-c^2\frac{\partial^2}{\partial x_k^2}\right)\frac{\partial^2}{\partial t^2}+\frac{V^2}{3}\frac{\partial^2}{\partial x_k^2}\left(c^2\frac{\partial^2}{\partial x_j^2}-\frac{\partial^2}{\partial t^2}\right)\right]E_{Ti}$$
$$+\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial t^2}+\omega_e^2-V^2\frac{\partial^2}{\partial x_k^2}\right]E_{Li}=\frac{\partial^2}{\partial t^2}(S_{Ni}+S_{Ti}), \tag{24}$$

where

$$\frac{\partial^2 S_{Ni}}{\partial t^2}=-\frac{2V^2}{3N}\frac{\partial^2}{\partial t^2}\frac{\partial N}{\partial x_i}\frac{\partial E_{Lj}}{\partial x_j}-\frac{V^2}{3N}\frac{\partial^2}{\partial t^2}\frac{\partial N}{\partial x_j}\frac{\partial E_i}{\partial x_j}$$
$$+\frac{V^2c^2}{3N}\frac{\partial N}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial^2}{\partial x_k^2}E_{Ti}, \tag{25}$$

and

$$\frac{\partial^2 S_{Ti}}{\partial t^2}=\frac{2V^2}{3T_0}\frac{\partial^2}{\partial t^2}\frac{\partial T_0}{\partial x_j}\frac{\partial E_j}{\partial x_i}-\frac{2V^2c^2}{3T_0}\frac{\partial T_0}{\partial x_j}\frac{\partial}{\partial x_i}\frac{\partial^2}{\partial x_k^2}E_{Tj}$$
$$+\frac{V^2}{3T_0}\frac{\partial^2}{\partial t^2}\frac{\partial T_0}{\partial x_j}\frac{\partial E_i}{\partial x_j}-\frac{V^2c^2}{3T_0}\frac{\partial T_0}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial^2}{\partial x_k^2}E_{Ti}. \tag{26}$$

Consider first the case in which the density and temperature gradients are zero, i.e., $\partial^2S_{Ni}/\partial t^2=\partial^2S_{Ti}/\partial t^2=0$. Then (24) gives for the equations describing the propagation of a pure longitudinal plasma wave, or a pure transverse electromagnetic wave, respectively:

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2-V^2\nabla^2\right)\mathbf{E}_L=0, \tag{27}$$

and

$$\left[\left(\frac{\partial^2}{\partial t^2}+\omega_e^2-c^2\nabla^2\right)\frac{\partial^2}{\partial t^2}+\frac{V^2}{3}\nabla^2\left(c^2\nabla^2-\frac{\partial^2}{\partial t^2}\right)\right]\mathbf{E}_T=0. \tag{28}$$

Equation (27) yields the familiar dispersion relation

$$\omega^2=\omega_e^2+k^2V^2, \tag{29}$$

for the plasma oscillations of a uniform plasma. Further, the second term in the square bracket of (28) is only important for a relativistically hot electron gas, and will be neglected here since we did not start with a relativistic Boltzmann equation anyway. Thus insofar as $c^2\gg V^2$, Eq. (28) yields the usual dispersion relation for electromagnetic waves in a uniform plasma,

$$\omega^2=\omega_e^2+k^2c^2. \tag{30}$$

Now we shall make a number of simplifications. First consider separately the two cases of a temperature gradient only and a density gradient only, setting the gradients along the x direction $\mathbf{i}$ in each case. Then taking the curl of (24) for these two cases, we obtain a pair of inhomogeneous wave equations for $\nabla\times(\partial^2\mathbf{E}_T/\partial t^2)$, namely:

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2(x)-c^2\nabla^2\right)\nabla\times\frac{\partial^2\mathbf{E}_T}{\partial t^2}$$
$$=\nabla\times\frac{\partial^2\mathbf{S}_N}{\partial t^2}-(\nabla\omega_e^2)\times\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{31}$$

for the case of constant temperature, and

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2-c^2\nabla^2\right)\nabla\times\frac{\partial^2\mathbf{E}_T}{\partial t^2}$$
$$=\nabla\times\frac{\partial^2\mathbf{S}_T}{\partial t^2}+(\nabla V^2)\times\nabla^2\frac{\partial^2\mathbf{E}_L}{\partial t^2}, \tag{32}$$

for constant density. In the source terms on the right of the radiation Eqs. (31) and (32) we can neglect the transverse field. This is because terms involving the transverse field are of order λ_e/λ_T times those for the longitudinal field, where λ_e and λ_T are the plasma oscillation and radiation wavelengths, respectively, i.e., $\lambda_e/\lambda_T\cong(V/c)\ll1$. Also the transverse field will be considered as a perturbation on the longitudinal field. Under these circumstances we can reduce (31) and (32) to two radiation equations for $\nabla\times\mathbf{E}_T$ with fairly simple sources dependent on the longitudinal field, i.e., on the presence of plasma oscillations. For T_0 constant, the density gradient equation becomes

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2(x)-c^2\nabla^2\right)\nabla\times\mathbf{E}_T$$
$$=(\nabla\omega_e^2)\times\left[\left(\frac{2V^2}{3\omega_e^2}\nabla^2-1\right)\mathbf{E}_L\right], \tag{33}$$

and with N constant, the temperature gradient equation becomes

$$[(\partial^2/\partial t^2)+\omega_e^2-c^2\nabla^2]\nabla\times\mathbf{E}_T=(\nabla V^2)\times(\nabla^2\mathbf{E}_L). \tag{34}$$

Thus if we specify the longitudinal field $\mathbf{E}_L$, say, for a wave packet of plasma oscillations, we may calculate the radiation produced from temperature or density gradients. The longitudinal field will first be calculated from (24) by setting $\mathbf{E}_T=0$, since as far as the propagation of plasma oscillations goes the radiation field will make only a small perturbation. From (33) and (34) we see that the component of $\mathbf{E}_L$ *across* the gradient is the radiative one.

4. PLASMA OSCILLATION SOLUTION IN GRADIENTS

If one sets the perturbation transverse field $\mathbf{E}_T=0$, the wave equation for plasma oscillations in a density gradient along the x direction $\mathbf{i}$ becomes, from (24),

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2(x)-V^2\nabla^2\right)\mathbf{E}_L=-\frac{2V^2}{3N}\mathbf{i}\frac{dN}{dx}\nabla\cdot\mathbf{E}_L-\frac{V^2}{3N}\frac{dN}{dx}\frac{\partial}{\partial x}\mathbf{E}_L. \quad (35)$$

If we now write the irrotational vector $\mathbf{E}_L$ as $\nabla\phi$ in (35) and drop terms $O(L^{-2})$, we find a wave equation for the scalar potential ϕ,

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2(x)-V^2\nabla^2\right)\nabla^2\phi+\frac{\partial\phi}{\partial x}\frac{d\omega_e^2}{dx}+\frac{V^2}{\omega_e^2}\frac{d\omega_e^2}{dx}\nabla^2\frac{\partial\phi}{\partial x}=0. \quad (36)$$

We shall examine the behavior of an infinite plane plasma wave with its propagation vector $\mathbf{k}$ in the xy plane.

$$\Psi(\omega,k_y,x)\exp(-i\omega t+ik_y y). \quad (37)$$

Equation (36) then becomes for Ψ,

$$-\frac{d^4\Psi}{dx^4}+\frac{1}{\omega_e^2}\frac{d\omega_e^2}{dx}\frac{d^3\Psi}{dx^3}+\frac{d^2\Psi}{dx^2}(k_y^2-k_x^2)+\frac{d\Psi}{dx}\left(\frac{1}{V^2}\frac{d\omega_e^2}{dx}-\frac{k_y^2}{\omega_e^2}\frac{d\omega_e^2}{dx}\right)+k_y^2k_x^2\Psi=0, \quad (38)$$

where

$$k_x^2(x)V^2=\omega^2-\omega_e^2(x)-k_y^2V^2. \quad (39)$$

This is now in a form suitable for solution by the WKB approximation since the characteristic length L of the density gradient is $\gg\lambda$, i.e., $k_x(x)$ is slowly varying. Thus writing

$$\Psi=\exp(S_0+S_1+\cdots),$$

and calculating S_0 and S_1 from (38) in the usual way, yields for an arbitrary wave packet defined by ψ_N,

$$\phi=(2\pi)^{-1}\int_{-\infty}^{\infty}d\omega dk_y\frac{\psi_N(\omega,k_y)\omega_e(x)}{k(x)k_x^{\frac{1}{2}}(x)}\times\exp\left[-i\omega t+ik_y y+i\int_0^x k_x(x')dx'\right] \quad (40)$$

from which the electric field $\mathbf{E}_L=\nabla\phi$ is easily calculated.

In a way similar to the foregoing, the behavior of a wave packet of plasma oscillations in a temperature gradient can be calculated from (24), which becomes

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2-V^2(x)\nabla^2\right)\mathbf{E}_L=\frac{2V^2}{3T}\frac{dT}{dx}\nabla E_{Lx}+\frac{V^2}{3T}\frac{dT}{dx}\frac{\partial}{\partial x}\mathbf{E}_L,$$

and gives for the WKB solution,

$$\phi=(2\pi)^{-1}\int_{-\infty}^{\infty}d\omega dk_y\frac{\psi_T(\omega,k_y)}{k_x^{\frac{1}{2}}(x)V^2(x)k^2(x)}\times\exp\left[-i\omega t+ik_y y+i\int_0^x k_x(x')dx'\right]. \quad (41)$$

The plasma oscillations given by (40) and (41) are refracted out of the direction of increasing N or T, i.e.,

$$\mathbf{k}=\left\{\left[\frac{\omega^2-\omega_e^2(x)}{V^2(x)}-k_y^2\right]^{\frac{1}{2}},\ k_y\right\}.$$

Further, the electric vector rotates about an elongated ellipse whose major axis along the direction of propagation is of order $L\lambda^{-1}$ times the minor axis. The amplitude of the electric vector given by (40) decreases in the direction of decreasing N, the energy being transferred to the temperature fluctuation as the wave tends to propagate more like a sound wave.

The energies Y_N and Y_T carried by the wave packets (40) and (41) are obtained by integrating the flux $\mathbf{S}$ over t and y. Using (14) and (17), (19) becomes for our longitudinal field,

$$\mathbf{S}=-\frac{kT_0}{32\pi^2Ne^2}\left[\frac{7}{2}\frac{\partial\mathbf{E}_L}{\partial t}\cdot(\nabla\mathbf{E}_L^*)+2\left(\frac{\partial\mathbf{E}_L}{\partial t}\cdot\nabla\right)\mathbf{E}_L^*\right], \quad (42)$$

where we have neglected terms $O(L^{-1})$. Thus from (40) and (41), we find

$$Y_N=\int_{-\infty}^{\infty}S_x dy dt=\frac{11V^2}{48\pi}\int d\omega dk_y\omega\psi_N^2, \quad (43)$$

$$Y_T=\frac{11}{48\pi V^2\omega_e^2}\int d\omega dk_y\omega k^{-2}\psi_T^2. \quad (44)$$

The slow dependence of Y_T on x derives simply from the neglect of terms in L^{-1} in the approximate expression (42).

5. RADIATION BY PLASMA OSCILLATIONS

Basically there are two extreme cases of gradients which we are able to investigate from the point of view of radiation. The first are slowly varying gradients for which $L\gg\lambda$ and to which the radiation Eqs. (33) and (34) apply. The second, which for practical purposes will be considered as density discontinuities, are those for which $L\ll\lambda$. These can be treated by applying boundary conditions to the solutions of the homogeneous wave equation across a discontinuity.

Case 1, $L \gg \lambda$

We shall consider the density radiation equation first. We also consider a two-dimensional radiation problem by using a plasma wave packet localized with its propagation vector in the xy plane, but of infinite extent in the z direction. This will be incident on a region of infinite plane density variation $N(x)$. The source term in Eq. (33) then gives rise to a z component of $\nabla\times\mathbf{E}_T$, i.e., essentially the magnetic field. Thus writing $\Phi=(\nabla\times\mathbf{E}_T)_z=-\dot{H}_z/c$, we need to solve

$$\left(\frac{\partial^2}{\partial t^2}+\omega_e^2(x)-c^2\nabla^2\right)\Phi=\frac{d\omega_e^2}{dx}\left(\frac{2V^2}{3\omega_e^2}\nabla^2-1\right)E_{Ly}. \quad (45)$$

We shall take for E_{Ly} that given by the two-dimensional wave packet (40) with wave numbers $\mathbf{k}$ and frequencies ω. Those of the radiation field will be written $\mathbf{K}$ and ν. Then taking Fourier transforms over y and t,

$$\Phi=(2\pi)^{-1}\int_{\infty}^{\infty} d\nu dK_y\Lambda(x,\nu,K_y)\exp(-i\nu t+iK_y y), \quad (46)$$

(45) becomes

$$\left(\frac{d^2}{dx^2}+K_x^2(x)\right)\Lambda=g(x), \quad (47)$$

where

$$K_x^2=\frac{\nu^2-\omega_e^2}{c^2}-K_y^2 \quad (48)$$

and

$$g(x)=\frac{1}{c^2}\frac{d\omega_e^2}{dx}\psi_N(\nu,K_y)\frac{iK_y\omega_e}{k_x^{\frac{1}{2}}k}\left(\frac{2V^2k^2}{3\omega_e^2}+1\right)\times\exp\left[i\int_0^x k_x(x')dx'\right] \quad (49)$$

with now

$$k_x^2V^2=\nu^2-\omega_e^2-K_y^2V^2 \quad (50)$$

and

$$k^2V^2=\nu^2-\omega_e^2.$$

The amplitude Λ of the radiation in (46) is determined by (47) in which the source $g(x)$ derives from those frequencies $\omega=\nu$ and wave numbers $k_y=K_y$ in the plasma oscillation packet. An approximate solution of (47) can now be generated from the two WKB solutions of the homogeneous equation,

$$\Lambda_{1,2}\cong K_x^{-\frac{1}{2}}(x)\exp\left[\pm i\int_0^x K_x(x')dx'\right], \quad (51)$$

as

$$\Lambda=\frac{\Lambda_1}{2i}\int_a^x \Lambda_2 g dx'-\frac{\Lambda_2}{2i}\int_b^x \Lambda_1 g dx'. \quad (52)$$

Suppose we now consider the density gradient region to be confined between two infinite planes, $|x|\lesssim L$, about the origin and the plasma to be of uniform density outside this region. Then for large $|x|>L$ the two terms of (52) correspond to waves traveling to the right and left, respectively. In the plasma to the left of the gradient region there should be no transverse wave traveling to the right, and similarly on the right of the gradient there should be no wave traveling to the left. Thus we take the constants $a=-\infty$, $b=\infty$. Equation (46) then breaks up into two radiation wave packets for large $|x|\gg L$, Φ_{N1} traveling to the right in the positive x region, and Φ_{N2} to the left in the region x negative, i.e.,

$$\Phi=\Phi_{N1}+\Phi_{N2},$$

with

$$\Phi_{N1,N2}=(4\pi c^2)^{-1}\int_{-\infty}^{\infty} d\nu dK_y\frac{\psi_N K_y}{K_x^{\frac{1}{2}}}G_{1,2}(\nu,K_y)\times\exp\left[-i\nu t+iK_y y\pm i\int_0^x K_x(x')dx'\right], \quad (53)$$

where

$$G_{1,2}=\int_{-\infty}^{\infty}\frac{dx\omega_e(x)}{K_x^{\frac{1}{2}}kk_x^{\frac{1}{2}}}\frac{d\omega_e^2}{dx}\left(\frac{2V^2k^2}{3\omega_e^2}+1\right)\times\exp\left[\mp i\int_0^x K_x(x')dx'+i\int_0^x k_x(x')dx'\right]. \quad (54)$$

The integrand of $G_{1,2}$ is a product of a rapidly oscillating function with a slowly varying wavelength λ, and a slowly varying function with a gradient scale L. Explicit calculation of the radiation field now depends on evaluating these integrals for a particular density gradient $N(x)$ and wave packet of plasma oscillations defined by ψ_N.

In a similar way the radiation field resulting from plasma oscillations traversing a temperature gradient follows from the wave equation (34). Writing Φ_T for the z component of $\nabla\times\mathbf{E}_T$, we again have 2 waves traveling to the right and left, respectively,

$$\Phi_T=\Phi_{T1}+\Phi_{T2},$$

where

$$\Phi_{T1,T2}=(4\pi c^2)^{-1}\int_{\infty}^{\infty} d\nu dK_y\frac{K_y\psi_T(\nu,K_y)}{K_x}F_{1,2}(\nu,K_y)\times\exp[-i\nu t+iK_y y\pm iK_x x] \quad (55)$$

and

$$F_{1,2}=\int_{-\infty}^{\infty}\frac{dx}{V^2k_x^{\frac{1}{2}}}\frac{dV^2}{dx}\exp\left[\mp iK_x x+i\int_0^x k_x(x')dx'\right]. \quad (56)$$

We shall require in the following discussion the radiated energies represented by the radiation wave packets (53) and (55). These are derived from the flux in a similar way to that for the plasma oscillations, namely, by using

$$T=\frac{c}{8\pi}\int_{-\infty}^{\infty}E_yH_z{}^*dydt. \tag{57}$$

We shall attach suffixes T_{N1}, T_{N2}, T_{T1}, T_{T2} for the energies of the two waves originating in the density or temperature gradients respectively. The electric and magnetic vectors $\mathbf{E}$ and $\mathbf{H}$ of the four radiation fields required in (57) can be obtained from Φ_{N1}, Φ_{N2}, Φ_{T1}, and Φ_{T2} using Maxwell's equations $\Phi=-\dot{H}_z/c$ and $\nabla\cdot\mathbf{E}=0$. Thus for example the components of the magnetic and electric fields of wave 1 for a density gradient are

$$E_{1z}=H_{1x}=H_{1y}=0,$$

$$H_{1z}=-\frac{i}{4\pi c}\int_{-\infty}^{\infty}\frac{d\nu dK_y}{\nu}\bar{G}_1,$$

$$E_{1x}=\frac{i}{4\pi c^2}\int_{-\infty}^{\infty}\frac{d\nu dK_yK_y}{K^2}\bar{G}_1,$$

$$E_{1y}=-\frac{i}{4\pi c^2}\int_{-\infty}^{\infty}\frac{d\nu dK_yK_x}{K^2}\bar{G}_1,$$

where

$$\bar{G}_1=\frac{K_y\psi_N}{K_x^{\frac{1}{2}}}G_1\exp\left[-i\nu t+iK_yy+i\int_0^x K_x(x')dx'\right]. \tag{58}$$

Similar expressions are easily obtained for the other radiation waves. Using these in (57), the radiated energies become

$$T_{N1,2}=(32\pi c^2)^{-1}\int_{-\infty}^{\infty}\frac{d\nu dK_y}{K^2\nu}|G_{1,2}|^2\psi_N{}^2K_y{}^2 \tag{59}$$

from the density gradient, and

$$T_{T1,2}=(32\pi c^2)^{-1}\int_{-\infty}^{\infty}\frac{d\nu dK_y}{\nu K_xK^2}|F_{1,2}|^2\psi_T{}^2K_y{}^2 \tag{60}$$

from a temperature gradient. We next calculated the functions G and F for a particular physically plausible gradient.

Approximate Evaluation of F and G

The WKB solutions (53) and (55) obviously break down at the classical turning points represented by the zeros of K_x or k_x. Better solutions are obtainable but we shall restrict ourselves here to this simpler case $|K_x^{-2}dK_x/dx|\ll 1$. Further, the functions F and G contain exponentially decaying factors as soon as K_x becomes complex. Thus the condition we need to satisfy in order to obtain radiation capable of *propagating* is

$$\frac{\nu^2-\omega_e{}^2}{c^2}-K_y{}^2>0,$$

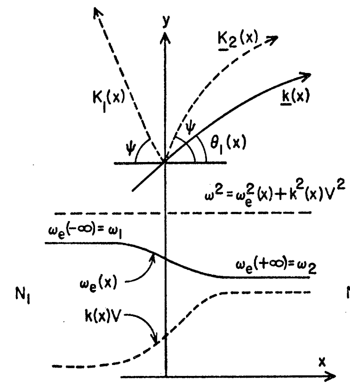

FIG. 1. Radiation by plasma oscillations traversing a slowly varying density gradient in the neighborhood $|x|\leqslant L$.

i.e., the angle $\psi=\arctan(K_y/K_x)$ which the two radiation waves make with the $\pm x$ axis at the point of emission (Fig. 1) must be less than $\pi/2$. The radiated waves have the same frequencies $\nu=\omega$ and wave number components $K_y=k_y$ as the plasma oscillations. Only the x component of the wave number changes so that the condition for radiation can also be written

$$\tan\theta_1=\frac{k_y}{k_x}<\frac{V}{(c^2-V^2)^{\frac{1}{2}}}\simeq\frac{V}{c}, \tag{61}$$

where θ_1 is the angle the plasma oscillation wave number $\mathbf{k}$ makes with the gradient direction; i.e., the plasma oscillations must propagate in a narrow "radiation cone" about the gradient direction in order to produce radiation capable of propagating in the plasma.

Now both the quantities F and G of (54) and (56) may be written in the form

$$G_{1,2}=\int_{-\infty}^{\infty}dx\,g(x)\frac{d\omega_e{}^2}{dx}\times\exp\left[\mp i\int_0^x K_x(x')dx'+i\int_0^x k_x(x')dx'\right], \tag{62}$$

$$F_{1,2}=\int_{-\infty}^{\infty}dx\,f(x)\frac{dV^2}{dx}\times\exp\left[\mp iK_xx+i\int_0^x k_x(x')dx'\right], \tag{63}$$

where g and f are slowly varying provided we avoid the zero of K_x or k_x. We next choose particular functions to represent the gradient regions in the plasma, namely,

$$\omega_e{}^2(x)=\omega_1{}^2+\frac{(\omega_2{}^2-\omega_1{}^2)}{L\pi^{\frac{1}{2}}}\int_{-\infty}^{x}dx\exp\left(-\frac{x^2}{L^2}\right), \tag{64}$$

$$V^2(x)=V_1{}^2+\frac{(V_2{}^2-V_1{}^2)}{L\pi^{\frac{1}{2}}}\int_{-\infty}^{x}dx\exp\left(-\frac{x^2}{L^2}\right), \tag{65}$$

i.e., either the density changes from N_1 to N_2 or the temperature from T_1 to T_2 in a region $|x|\lesssim L$ about the origin. Such gradients might, for example, describe the density or temperature transition through a shock wave. The integrals (62) and (63) are complicated and we shall here only approximately evaluate them. Consider first (62). Expanding the slowly varying wave numbers $K_x(x)$ and $k_x(x)$, we can write for the exponentials in (62) approximately

$$\exp[ix(k_x(0)\mp K_x(0))+\tfrac{1}{2}ix^2(k_x'(0)\mp K_x'(0))], \quad (66)$$

which is valid for $|x|<L$. The integrand vanishes rapidly for larger x. Similarly for the functions $f(x)$ and $g(x)$:

$$f(x)\cong f_0+xf_0', \qquad g(x)\cong g_0+xg_0', \quad (67)$$

with

$$f_0=V^{-2}k_x^{-\frac{1}{2}}|_{x=0},$$

$$g_0=\frac{\omega_e}{K_x^{\frac{1}{2}}kk_x^{\frac{1}{2}}}\left(\frac{2V^2k^2}{3\omega_e^2}+1\right)\Bigg|_{x=0}. \quad (68)$$

Thus (62) becomes

$$G_{1,2}\cong\frac{(\omega_2^2-\omega_1^2)}{L\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}dx(g_0+xg_0')\times\exp\left[-\frac{x^2}{L^2}+iax+ibx^2\right], \quad (69)$$

where $a=k_x(0)\mp K_x(0)$ and $b=(k_x'(0)\mp K_x'(0))/2$, and the suffixes 1 and 2 on G correspond to the $-$ or $+$ values, respectively, in a or b. The above Fourier integral becomes simply

$$G_{1,2}\cong\frac{(\omega_2^2-\omega_1^2)}{(1-iL^2b)^{\frac{1}{2}}}\left[g_0-\frac{g_0'iaL^2}{2(ibL^2-1)}\right]\exp\left[-\frac{a^2L^2}{4(1-ibL^2)}\right],$$

i.e.,

$$|G_{1,2}|^2\cong\frac{(\omega_2^2-\omega_1^2)^2}{L^2|b|}\left(g_0-\frac{ag_0'}{2b}\right)^2\exp\left(-\frac{a^2}{2b^2L^2}\right), \quad (70)$$

since $b=O(\lambda^{-1}L^{-1})$, i.e., $L^{-2}\ll b^2$. Further since $K_x\ll k_x$ and $K_x'\ll k_x'$, we have $a\cong k_x(0)$ and $b\cong k_x'(0)/2$ so that $|G_1|^2\cong|G_2|^2$ and the energies of the two radiated wave packets from (59) become

$$T_{N1}\cong T_{N2}\cong\frac{V^2\omega_e^2}{8c^2}\left|\frac{d\omega_e^2}{dx}\right|\int_{-\infty}^{\infty}\frac{d\nu dK_y\,K_y^2\psi_N^2A^2}{K^2\nu K_xk^2}\times\exp\left[-\frac{8\pi k_x^4V^4}{(\omega_2^2-\omega_1^2)^2}\right]\Bigg|_{x=0}, \quad (71)$$

where

$$A=\left(1+\frac{2k^2V^2}{3\omega_e^2}\right)\left\{\frac{3}{2}+\frac{k_x^2V^2}{2K_x^2c^2}+\frac{k_x^2}{k^2}-\frac{k_x^2V^2(\omega_e^2+2k^2V^2)}{\omega_e^2(2k^2V^2+3\omega_e^2)}\right\},$$

and we have also used

$$k_x'(0)=-\frac{1}{2k_xV^2}\frac{d\omega_e^2}{dx}\Bigg|_0=-\frac{(\omega_2^2-\omega_1^2)}{2k_xV^2L\pi^{\frac{1}{2}}}.$$

In a similar way to the above, the functions $F_{1,2}$ may be approximated and yield for the radiated energies in a temperature gradient

$$T_{T1}\cong T_{T2}\cong\frac{1}{8c^2}\left|\frac{dV^2}{dx}\right|\int_{-\infty}^{\infty}\frac{d\nu dK_yK_y^2\psi_T^2}{\nu K_xK^2k^2V^2}\left(\frac{3}{2}+\frac{2k_x^2V^2}{\omega_e^2}\right)^2\times\exp\left[-\frac{8\pi V^4}{(V_2^2-V_1^2)^2}\right]\Bigg|_{x=0}. \quad (72)$$

Radiation Efficiency for Slowly Varying Gradients

The radiation efficiencies for these plasma waves propagating in the xy plane and encountering a region of density or temperature variation will be defined as $R_N=(T_{N1}+T_{N2})/Y_N$ and $R_T=(T_{T1}+T_{T2})/Y_T$, i.e., the fractional energy radiated away in traversing the infinite plane inhomogeneity considered here. Taking ψ_N and ψ_T to be highly peaked functions about some particular values ν and K_y for which we use the same notation, the above efficiencies become from (43), (44), (71), and (72),

$$R_N\cong\frac{12\pi}{11c^2}\left|\frac{d\omega_e^2}{dx}\right|\left(\frac{\omega_e}{\omega}\right)^2\frac{\sin^2\theta_1}{K^2K_x}A^2\times\exp\left[-\frac{8\pi k_x^4V^4}{(\omega_2^2-\omega_1^2)^2}\right]\Bigg|_{x=0}, \quad (73)$$

and

$$R_T\cong\frac{12\pi}{11c^2}\left|\frac{dV^2}{dx}\right|\left(\frac{\omega_e}{\omega}\right)^2\frac{\sin^2\theta_1}{K_x}\left(\frac{k}{K}\right)^2\left(\frac{3}{2}+\frac{2k_x^2V^2}{\omega_e^2}\right)^2\times\exp\left[-\frac{8\pi V^4}{(V_2^2-V_1^2)^2}\right]\Bigg|_{x=0}, \quad (74)$$

where $\sin\theta_1=K_y/k$. It should be emphasized that these expressions are restricted both to large $L\gg\lambda$ and to angles θ_1 of the plasma oscillation across the gradient which are sufficiently smaller than V/c that we avoid the singularity at $K_x=0$, i.e., the classical turning points of the WKB solutions. Oscillations propagating at an angle $\theta_1>V/c$ do not produce radiation capable of propagating.

We shall next consider radiation by plasma oscillations incident on density discontinuities, i.e., $L=0$. The radiation Eq. (33) will not be applicable to this case since it was derived neglecting terms of order λ^2/L^2 and above.

Case 2, $L=0$

Consider a sharp density transition from N_1 to N_2 inside the plasma (e.g., a shock front) and long wavelength plasma oscillations incident on it ($\lambda \gg L$), i.e., the opposite extreme to the case above. Then we may again calculate a radiation efficiency for such an encounter by treating the sharp transition as a discontinuity and applying boundary conditions to the uniform plasma solutions on either side. The boundary conditions are obtained from the exact Eqs. (3)–(5), together with Maxwell's equations by considering a sharp but continuous density variation from N_1 to N_2 and then performing the usual limiting processes.[6] In crossing the density "discontinuity," we shall hold the temperature of the electron component of the plasma constant. Thus, for example, such a discontinuity might correspond to a plasma shock wave in which the characteristic length over which the electrons are heated by the shocked ions is somewhat larger than the shock thickness.

Writing $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{H}_1$, and $\mathbf{H}_2$ for the total electric and magnetic fields on the sides 1 and 2 of the boundary, we find the boundary conditions at $x=0$,

$$\mathbf{H}_2-\mathbf{H}_1=0, \tag{75}$$

$$\mathbf{i}\times(\mathbf{E}_2-\mathbf{E}_1)=0, \tag{76}$$

$$\mathbf{i}\cdot(\dot{\mathbf{E}}_2-\dot{\mathbf{E}}_1)=4\pi e\mathbf{i}(N_2\mathbf{c}_2-N_1\mathbf{c}_1), \tag{77}$$

$$\dot{p}_{xj}(2)-\dot{p}_{xj}(1)=0, \tag{78}$$

where $\mathbf{i}$ is a unit vector in the x direction pointing towards the side 2 of the discontinuity. p_{xj} are the x components of the pressure tensor fluctuation. Equation (77) reduces to

$$\mathbf{i}\cdot(\nabla\times\mathbf{H}_2-\nabla\times\mathbf{H}_1)=0. \tag{79}$$

It should be noted that these do not reduce to Field's[2] boundary conditions for a plasma vacuum boundary. He neglected the surface charge in the condition on the normal component of electric field. Now consider a plasma wave $\mathbf{e}_1(\mathbf{k}_1)$ incident on the side 1 of the boundary. The boundary conditions are then satisfied by a reflected and transmitted plasma wave $\mathbf{e}_2(\mathbf{k}_2)$ and $\mathbf{e}_3(\mathbf{k}_3)$, together with two radiation waves $\boldsymbol{\varepsilon}_1(\mathbf{K}_1)$ and $\boldsymbol{\varepsilon}_2(\mathbf{K}_2)$ (see Fig. 2). These waves have wave numbers $\mathbf{k}_1$, $\mathbf{k}_2$, etc. We shall further consider them as plane waves, $\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$ propagating in the xy plane with a frequency ω. Thus we have five dispersion relations for these waves,

$$\omega^2=\omega_1^2+k_1^2V^2=\omega_1^2+k_2^2V^2=\omega_1^2+K_1^2c^2$$
$$=\omega_2^2+k_3^2V^2=\omega_2^2+K_2^2c^2, \tag{80}$$

[6] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, New York, 1941).

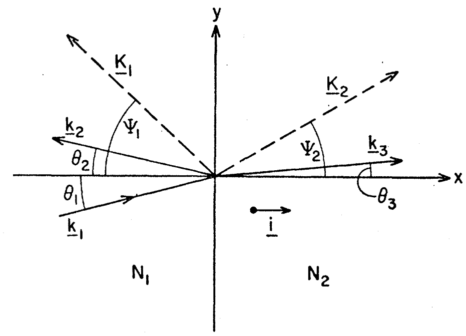

FIG. 2. Radiation by a plasma wave incident on a discontinuity in density.

where

$$\omega_{1,2}^2=4\pi e^2N_{1,2}/m \tag{81}$$

are the plasma frequencies on either side of the discontinuity.

Now in simplifying the boundary conditions (75)–(79), we note that $\mathbf{E}_1=\mathbf{e}_1+\mathbf{e}_2+\boldsymbol{\varepsilon}_1$ and $\mathbf{E}_2=\mathbf{e}_3+\boldsymbol{\varepsilon}_2$. Further we use Maxwell's Eqs. (17) and (18) to express the electron fluid velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ and $\mathbf{H}_1$ and $\mathbf{H}_2$ in terms of the electric fields, together with (15) for the pressure tensor p_{ij}. They then reduce to the four relations:

$$\epsilon_1K_1-\epsilon_2K_2=0, \tag{82}$$

$$e_2\sin\theta_2-e_3\sin\theta_3+\epsilon_1\cos\psi_1+\epsilon_2\cos\psi_2=-e_1\sin\theta_1 \tag{83}$$

$$-e_2\cos\theta_2-e_3\cos\theta_3+\epsilon_1(\sin^2\psi_1-\cos^2\psi_1)\omega_1^2/2\omega^2\sin\psi_1$$
$$-\epsilon_2(\sin^2\psi_2-\cos^2\psi_2)\omega_2^2/2\omega^2\sin\psi_2=-e_1\cos\theta_1 \tag{84}$$

$$e_2k_2(1+2\cos^2\theta_2)-e_3k_3(1+2\cos^2\theta_3)$$
$$-\epsilon_1 2K_1\cos\psi_1\sin\psi_1(\omega_1^2/\omega^2)$$
$$-\epsilon_2 2K_2\cos\psi_2\sin\psi_2(\omega_2^2/\omega^2)$$
$$=-k_1e_1(1+2\cos^2\theta_1). \tag{85}$$

We have used a further relation between the angles and wave numbers, namely the equality of phase for the various waves along the boundary, i.e.,

$$k_1\sin\theta_1=k_2\sin\theta_2=k_3\sin\theta_3=K_1\sin\psi_1=K_2\sin\psi_2. \tag{86}$$

The angles in these equations are between the wave numbers and the x axis as shown in Fig. 2. Thus, for example, from (80) and (86), $k_1=k_2$ and $\theta_1=\theta_2$.

The solution of Eqs. (82)–(85) for the amplitudes ϵ_1 and ϵ_2 of the radiated waves are then simply

$$\epsilon_1=K_2\epsilon_2/K_1=[\epsilon_1]/D, \tag{87}$$

where the determinants $[\epsilon_1]$ and D reduce to

$$[\epsilon_1]=6e_1\left(\frac{V}{c}\right)\sin\theta_1\cos\theta_1(\omega_2^2-\omega_1^2)/V^2, \tag{88}$$

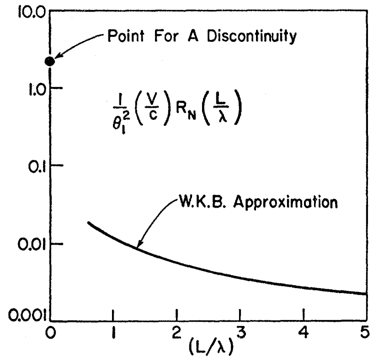

FIG. 3. Radiation efficiency $R_N(L)$ for long-wavelength plasma oscillations incident at small θ_1 on a density change from N_1 to $N_1/2$.

and

$$D=(k_3K_1\cos\psi_2\cos\theta_1+k_1K_2\cos\psi_1\cos\theta_3)$$
$$\times(3-2c^2K_1^2\sin^2\theta_1/\omega^2)+k_1K_1\cos\psi_2\cos\theta_3$$
$$\times(3-2c^2K_2^2\sin^2\theta_1/\omega^2)+k_3K_2\cos\psi_1\cos\theta_1$$
$$\times(3-2c^2K_1^2\sin^2\theta_3/\omega^2)$$
$$-\frac{3(\omega_2^2-\omega_1^2)^2}{2\omega^2c^2}\left[1-2\sin^2\theta_1\left(\frac{c}{V}\right)^2\frac{\omega^2}{\omega^2-\omega_2^2}\right]. \qquad (89)$$

Next, consider the relations (80) and (86) between θ_1, ψ_1 and ψ_2. We see that as θ_1 increases, ψ_1 and ψ_2 increase. Since ψ_1 and ψ_2 cannot increase beyond $\pi/2$, the maximum value of $\sin\theta_1$ for which the first radiation wave exists is given by

$$\sin\theta_1<K_1/k_1=V/c, \qquad (90)$$

and for the second radiation wave

$$\sin\theta_1<\frac{K_2}{k_1}=\frac{V}{c}\left(\frac{\omega^2-\omega_2^2}{\omega^2-\omega_1^2}\right)^{\frac{1}{2}}. \qquad (91)$$

For both radiated waves to be propagated, the plasma oscillations must be incident within a very narrow cone around the normal to the density discontinuity.

Radiation Efficiency for a Discontinuity

We also require to calculate the fraction R_N of the energy Y of a wave packet of plasma oscillations which is emitted as radiation at the discontinuity. We shall restrict ourselves here to angles θ_1 sufficiently less than V/c so that both radiation waves contribute. Further, since $(V/c)\ll 1$, we shall neglect terms of relative magnitude $(V/c)^2$ in the derivation of Eq. (92). Using the expressions (19) and (42) for the energy fluxes, the components of the fluxes for the five waves at the discontinuity become

$$S_{1x}=\frac{11\omega k_1e_1^2(kT)}{64\pi^2N_1e^2}$$

for the plasma waves, and

$$S_{1x}^{\mathrm{rad}}=-\frac{c^2K_1\epsilon_1^2\cos\psi_1}{8\pi\omega},$$
$$S_{2x}^{\mathrm{rad}}=\frac{c^2K_2\epsilon_2^2\cos\psi_2}{8\pi\omega}$$

for the radiation. Thus the fractional energy of the incident plasma wave radiated becomes, for a discontinuity,

$$R_N(\theta_1)=\frac{2\omega_1^2}{11\omega^2k_1}\left(\frac{mc^2}{kT}\right)$$
$$\times\left[K_1\left(\frac{\epsilon_1}{e_1}\right)^2\cos\psi_1+K_2\left(\frac{\epsilon_2}{e_1}\right)^2\cos\psi_2\right], \qquad (92)$$

where ϵ_1 and ϵ_2 are given by (87)–(89).

6. DISCUSSION

The expressions (73) and (92) for the radiation efficiency in slowly varying gradients or at a discontinuity are in general complicated. We shall here give numerical results for a simple limiting case of plasma waves propagating almost directly along the gradient, i.e., $\theta_1\ll(V/c)$. We further consider the special case of *long-wavelength* plasma waves incident from the dense side N_1 with $N_1=2N_2$. Thus

$$k_1V\ll\omega_1\cong\omega \quad\text{and}\quad k_2V\cong\omega_1/\sqrt{2}\cong\omega_2,$$

i.e., on the tenuous side, plasma waves with a short wavelength just above the Debye radius are transmitted. Under these limiting cases we find for a discontinuity,

$$\left.\frac{1}{\theta_1^2}R_N(\theta_1,L=0)\right|_{\theta_1\to 0}=\frac{24}{11}\left(\frac{c}{V}\right),$$

and for our exponential gradient (64) of scale L the WKB result (73) becomes

$$\left.\frac{1}{\theta_1^2}R_N(\theta_1,L)\right|_{\theta_1\to 0}=1.1\ 10^{-2}\left(\frac{c}{V}\right)\left(\frac{\lambda(x=0)}{L}\right).$$

These expressions are plotted in Fig. 3. We note that in going from a discontinuity to a gradient for which L is a few wavelengths, the radiation efficiency drops by two orders of magnitude which makes interpolation across the region $L\cong\lambda$ only very rough. Once in the slowly varying gradient region $L\gg\lambda$, however, the radiation efficiency varies simply as L^{-1}.

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