Electron Velocity Distributions in a Partially Ionized Gas*

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The Boltzmann method is applied to the problem of calculating the electron velocity distribution in a partially ionized gas. Inelastic collisions with neutral molecules as well as random two-body Coulomb interactions are included. The latter are treated by use of the Fokker-Planck equation. Application is made to a hydrogen plasma subjected to an externally applied electric field. Static solutions are obtained, by numerical means, as a function of the ionization degree, and the classical gas discharge parameter, E/p. It is shown that the evolution of the electron velocity distribution function from that characteristic of a poorly ionized gas to the Maxwellian distribution occurs over a very large range in ionization degree. Several applications are also made to energy relaxation phenomena, and the electrical conductivity is evaluated.

I. INTRODUCTION

IN the theoretical treatment of ionized gases, a problem of fundamental importance is that of determining the electron velocity distribution. A knowledge of it permits the calculation of ionization rates, electrical current, heat flow, and other important transport phenomena. A number of authors have dealt with this problem, and there are now well-developed techniques available for the completely ionized gas,[1] and for the poorly ionized gas[2] in which encounters between electrons and neutral molecules alone are considered.

The purpose of the present paper is to combine, and apply these methods to the intermediate case where both Coulomb and electron-neutral encounters are of importance. Different methods of treating this case have been formulated by Cahn[3] and by Hazeltine,[4] who have, however, neglected the inelastic collisions between electrons and neutrals. These collisions are important because they can have a pronounced effect upon the nature of the electron velocity distribution. In this paper we take account of inelastic collisions, and employ the Fokker-Planck equation to describe Coulomb interactions. Our main objective is to study the transition from the non-Maxwellian distribution characteristic of a poorly ionized gas to the Maxwellian distribution as the degree of ionization rises. The results we obtain[5] may be applied to various laboratory as well as astrophysical gas discharge phenomena.

II. FORMULATION OF THE BASIC EQUATIONS

Following standard procedure we define the electron velocity distribution function $F(\mathbf{v},t)$ such that Fd^3v gives the number of electrons whose velocities lie in the element d^3v located around the point $\mathbf{v}$ in velocity space. We restrict our treatment to plasmas whose macroscopic properties do not vary from point to point in space. Therefore, F satisfies the Boltzmann equation in the form

$$\partial F/\partial t-(e/m)\mathbf{E}\cdot\nabla_v F=(\partial F/\partial t)_c. \quad (1)$$

In this equation, $\nabla_v F$ denotes the gradient of F in velocity space, and $\mathbf{E}$ is an externally applied electric field. Magnetic fields are assumed to be absent. $(\partial F/\partial t)_c$ is a symbolic notation for the time rate of change in F due to collisions, and it consists of contributions from various types of encounters which we now proceed to examine in detail.

A. Coulomb Collisions

The mechanism of Coulomb interactions in a plasma has recently been clarified by a number of authors.[6] In these treatments it is shown that the interactions can be roughly divided into two kinds. The first associated with distances larger than the Debye length, λ, originally introduced in the theory of electrolytes[7] represents organized or collective plasma oscillations. The second kind associated with distances smaller than λ represents random interactions which are characteristic of the thermal motion of the individual particles, and seems best described by two-body encounters. In this paper it is assumed that the mechanism for exciting collective oscillations is absent. The random two-body encounter is therefore the sole mechanism for Coulomb interaction considered here.

Our choice of collision term is based upon the fact that distant encounters, resulting in small-angle scattering, are generally much more important in determining F than collisions which result in large momentum interchange. This fact allows us to expand the Boltzmann collision integral, ordinarily used in gas kinetics, in powers of the momentum interchange.[8] To the first approximation this results in the Fokker-

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[1] Cohen, Spitzer, and Routley, Phys. Rev. **80**, 230 (1950); L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953).
[2] T. Holstein, Phys. Rev. **70**, 367 (1946).
[3] J. H. Cahn, Phys. Rev. **75**, 293 (1949); J. H. Cahn, Phys. Rev. **75**, 838 (1949).
[4] W. R. Hazeltine, J. Math. Phys. **18**, 174 (1939).
[5] These results were first reported at the Ninth Annual Gaseous Electronics Conference, Pittsburgh, 1956 [H. Dreicer, Bull. Am. Phys. Soc. **2**, 85 (1957)].
[6] See, for example, D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[7] P. Debye and E. Huckel, Physik. Z. **24**, 185 (1923).
[8] J. Keilson and J. E. Storer, Quart. Appl. Math. **10**, 243 (1952).

Planck equation

$$\left(\frac{\partial F}{\partial t}\right)_{cc}=-\sum_i \frac{\partial}{\partial v_i}(\langle \Delta v_i\rangle F)+\tfrac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial v_i \partial v_j}(\langle \Delta v_i \Delta v_j\rangle F), \quad (2)$$

which has found important application in the theory of the Brownian motion. In this equation v_i and v_j are the components of the particle velocity in the i and j directions, and the indicated summations are carried out over the three Cartesian coordinates x, y, z in a fixed frame of reference. Every encounter between a test particle of velocity $\mathbf{v}$ and a field particle[9] of velocity $\mathbf{v}'$ results in an incremental change Δv_k in the velocity $\mathbf{v}$ along the k direction. The exact value of Δv_k depends upon the velocities $\mathbf{v}$, and $\mathbf{v}'$, upon the angle ϵ between the orbital plane and the plane determined by $\mathbf{v}$ and $\mathbf{v}'$, and upon the scattering angle, β, as measured in a coordinate frame moving with the center of mass. The average increments $\langle \Delta v_k\rangle$ and $\langle \Delta v_k \Delta v_j\rangle$ which a test particle of velocity $\mathbf{v}$ experiences in its motion through an ionized gas are given by

$$\langle \Delta v_k\rangle=\sum_f \int_{\mathbf{v}'} F_f(\mathbf{v}')d^3v' \int_\Omega \sigma g \Delta v_k d\Omega, \quad (3)$$

$$\langle \Delta v_k \Delta v_j\rangle=\sum_f \int_{\mathbf{v}'} F_f(\mathbf{v}')d^3v' \int \sigma g \Delta v_k \Delta v_j d\Omega, \quad (4)$$

where

$$d\Omega=\sin\beta d\beta d\epsilon,$$

$$g=|\mathbf{v}-\mathbf{v}'|,$$

$$\sigma=\frac{1}{4}\left(\frac{e_t^2 e_f^2}{4\pi\epsilon_0 m_0 g^2}\right)^2 \csc^4(\beta/2),$$

$$m_0=m_t m_f/(m_t+m_f),$$

and

$$4\pi\epsilon_0=[1/(9\pi)]10^{-9}\ \text{Coulomb-volt}^{-1}\text{-meter}^{-1}.$$

The subscripts t and f refer to test and field particle, respectively. The distribution function associated with the test particle will be denoted simply by F.

Chandrasekhar[10] has termed the quantity $\langle \Delta \mathbf{v}\rangle/v$ the coefficient of dynamical friction, since $\langle \Delta \mathbf{v}\rangle$ is the average deceleration experienced by a particle of velocity $\mathbf{v}$ in its motion through a plasma. It is important to note that $\langle \Delta v\rangle$ and Δv differ dimensionally by sec^{-1}, because the average involves the rate at which collisions occur. If dynamical friction were the sole result of Coulomb encounters, all particles would eventually assume the average velocity of the gas, and their average random energy as viewed in a frame moving with the gas would vanish. That this mechanism by itself provides an insufficient description, may therefore be seen from energy considerations. In Sec. V, A, it will become more apparent that the second derivative term in the Fokker-Planck equation causes a particle diffusion in velocity space, and thus describes the effect of random fluctuations about the average force. This provides the mechanism required for maintaining a nonzero mean square velocity.

Rosenbluth, MacDonald, and Judd[11] have shown that the average increments can be expressed in the simple form

$$\langle \Delta v_k\rangle=\partial H_t/\partial v_k, \quad (5)$$

$$\langle \Delta v_k \Delta v_j\rangle=\partial^2 G_t/\partial v_k \partial v_j, \quad (6)$$

where

$$H_t(\mathbf{v})=\sum_f \frac{m_t+m_f}{m_f}\Gamma_{tf}\int \frac{F_f(\mathbf{v}')d^3v'}{g}, \quad (7a)$$

$$G_t(\mathbf{v})=\sum_f \Gamma_{tf}\int F(\mathbf{v}')g d^3v', \quad (7b)$$

$$\Gamma_{tf}=4\pi\left(\frac{e_t^2 e_f^2}{4\pi\epsilon_0 m_0}\right)^2 \ln(\lambda/p_0), \quad (8)$$

and p_0 is the average impact parameter for a 90° Coulomb deflection.

The summation over f takes account of the various types of field particles which contribute to the H and G function. The relation between H, G, and F can be expressed in the following alternative forms

$$\nabla_v^2 H_t=\sum_k \frac{\partial^2 H_t}{\partial v_k^2}=-4\pi\sum_f\left(\frac{m_t+m_f}{m_f}\right)F_f(\mathbf{v}), \quad (9a)$$

$$\nabla_v^2 G_t=2\sum_f\left(\frac{m_f}{m_t+m_f}\right)H_f, \quad (9b)$$

$$\nabla_v^4 G_t=\sum_{k,j}\frac{\partial^4 G_t}{\partial v_k^2 \partial v_j^2}=-8\pi\sum_f F_f(\mathbf{v}). \quad (9c)$$

By substituting Eqs. (5) and (6) into Eq. (2) we obtain the Cartesian form of the Fokker-Planck equation in terms of the H and G functions:

$$\left(\frac{\partial F}{\partial t}\right)_{cc}=-\sum_k \frac{\partial}{\partial v_k}\left(F\frac{\partial H_t}{\partial v_k}\right)+\tfrac{1}{2}\sum_{k,j}\frac{\partial^2}{\partial v_k \partial v_j}\left(F\frac{\partial^2 G_t}{\partial v_k \partial v_j}\right). \quad (10)$$

With the help of tensor calculus, Eq. (10) is readily transformed to spherical coordinates where it takes the

[9] We follow the terminology introduced by S. Chandrasekhar, Astrophys. J. **93**, 285 (1941); and denote the particle whose motion we are following by the name test particle. All other particles are called field particles.

[10] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

[11] Rosenbluth, MacDonald, and Judd, Phys. Rev. **107**, 1 (1957).

form

$$\left(\frac{\partial F}{\partial t}\right)_{cc}=-\frac{1}{v^2}\frac{\partial}{\partial v}\left(Fv^2\frac{\partial H_t}{\partial v}\right)-\frac{1}{v^2}\frac{\partial}{\partial\mu}\left[F(1-\mu^2)\frac{\partial H_t}{\partial\mu}\right]$$
$$+\frac{1}{2v^2}\frac{\partial^2}{\partial v^2}\left(Fv^2\frac{\partial^2 G_t}{\partial v^2}\right)+\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\left[\frac{F}{v^4}(1-\mu^2)^2\frac{\partial^2 G_t}{\partial\mu^2}\right.$$
$$\left.+\frac{F}{v^3}(1-\mu^2)\frac{\partial G_t}{\partial v}-\frac{F}{v^4}\mu(1-\mu^2)\frac{\partial G_t}{\partial\mu}\right]$$
$$+\frac{1}{v^2}\frac{\partial^2}{\partial v\partial\mu}\left[F(1-\mu^2)\frac{\partial^2 G_t}{\partial\mu\partial v}-\frac{F}{v}(1-\mu^2)\frac{\partial G_t}{\partial\mu}\right]$$
$$+\frac{1}{2v^2}\frac{\partial}{\partial v}\left[\frac{2\mu F}{v}\frac{\partial G_t}{\partial\mu}-\frac{F}{v}(1-\mu^2)\frac{\partial^2 G_t}{\partial\mu^2}-2F\frac{\partial G_t}{\partial v}\right]$$
$$+\frac{1}{2}\frac{\partial}{\partial\mu}\left[\frac{F}{v^4}\mu(1-\mu^2)\frac{\partial^2 G_t}{\partial\mu^2}+\frac{2F\mu}{v^3}\frac{\partial G_t}{\partial v}\right.$$
$$\left.+\frac{2F}{v^3}(1-\mu^2)\frac{\partial^2 G_t}{\partial\mu\partial v}-\frac{2F}{v^4}\frac{\partial G_t}{\partial\mu}\right]. \quad (11)$$

In this coordinate system F is a function of v and μ, where $\mu=\cos\theta$, and θ is the angle subtended by v_z and v. All quantities are assumed to be invariant to rotation about the v_z axis.

B. Elastic and Inelastic Collisions Between Electrons and Molecules

The collision terms are well-known for these interactions,[2] and will not be re-derived here. For the elastic case we have, to first order in the small quantity m/M,

$$\left(\frac{\partial F}{\partial t}\right)_{cm}=Nv\int_{\Omega'}[F(\mu',v)-F(\mu,v)]\sigma_e(\beta,v)d\Omega'$$
$$+N\frac{m}{M}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\int_{\Omega'}(1-\cos\beta)\sigma_e(\beta,v)v^4F(\mu',v)d\Omega'\right], \quad (12)$$

where N=density of neutral molecules, m=electron mass, M=molecular mass, $\sigma_e(\beta,v)$=differential cross section for elastic scattering, through the angle β. $d\Omega'=\sin\beta d\beta d\alpha$, and $\mathbf{v}$=electron velocity after collision with a molecule. We also introduce $\mathbf{w}$, the electron velocity before collision with a molecule, and use it to define the angles μ' and α as follows: μ'=cosine of the angle subtended by $\mathbf{w}$ and $\mathbf{k}$, the unit vector in the z direction. α=azimuthal angle subtended by the vector $\mathbf{v}\times(\mathbf{k}\times\mathbf{v})$ and the plane determined by $\mathbf{v}$ and $\mathbf{w}$.

Figure 1 illustrates the geometry of the collision. The relation between the angles is easily seen to be

$$\mu'=\mu\cos\beta+(1-\mu^2)^{\frac{1}{2}}\sin\beta\cos\alpha.$$

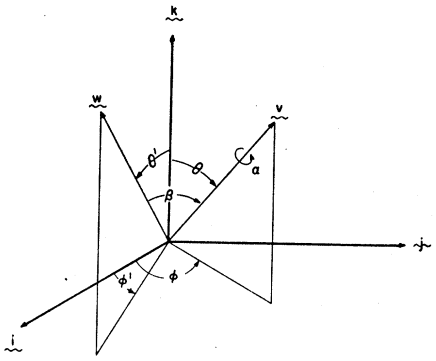

FIG. 1. Geometry of the collision between an electron and an infinitely massive molecule. The velocity vector $\mathbf{v}$ is the polar axis for the azimuthal angle α.

The inelastic collision term is given by

$$\left(\frac{\partial F}{\partial t}\right)_{cx}=\sum_h Nv\int_{\Omega'}\left[F(\mu',w)\frac{w^2}{v^2}\sigma_h(\beta,w)\right.$$
$$\left.-F(\mu,v)\sigma_h(\beta,v)\right]d\Omega', \quad (13)$$

where $w^2=v_h^2+v^2$, $\frac{1}{2}mv_h^2$=energy required to excite the hth excited state of the molecule, $\sigma_h(\beta,w)$=differential cross section for the excitation of the hth energy state, and $\sum_h$ denotes the summation over all excited states. Our basic equation consists of Eq. (1) supplemented by Eqs. (11), (12), and (13).

III. LORENTZ APPROXIMATION

The combined Boltzmann-Fokker-Planck equation is a nonlinear integropartial-differential equation for F in terms of the v and μ variables which cannot be solved exactly by known mathematical techniques. To achieve some simplification of our basic equation, we shall employ a perturbation method, originally due to Lorentz,[12] whose basic assumption is that collisions are instrumental in setting up a nearly spherically symmetric velocity distribution. Small deviations from spherical symmetry are then described accurately enough by the second coefficient, $F^1(v)$, in the spherical harmonic expansion of F:

$$F(\mu,v)=\sum_n F^n(v)P_n(\mu)\simeq F^0(v)+\mu F^1(v), \quad (14)$$

where the P_n are Legendre polynomials. In our formulation the perturbation requirement,

$$F^1\ll F^0,$$

is tantamount to the physical condition that the average velocity of the electron gas be small compared

[12] H. A. Lorentz, *The Theory of Electrons* (B. G. Teubner, Leipzig, 1909, and G. E. Stechert and Company, New York, 1923).

to the root mean square electron speed, i.e.,

$$\left|\int F^1\mu v d^3v\right| \ll \left[\int F^0 v^2 d^3v\right]^{\frac{1}{2}}.$$

Subject to this requirement we may, as is well known, eliminate μ by averaging over angles. The resulting two coupled equations for F^0 and F^1 involve the variables v and t only. We shall derive these equations by applying the Lorentz approximation to each of the collision terms separately.

A. Coulomb Collision Terms

The G (and H) potential is determined by the distribution function, and therefore must also be expanded in Legendre polynomials. A simple procedure consists of expanding the relative velocity g itself in Legendre polynomials:

$$g=\sum_{n=0}^{\infty} A_n(v,v')P_n(z),$$

where

$$z=\cos\beta=\mu\mu'+[1-\mu^2]^{\frac{1}{2}}[1-(\mu')^2]^{\frac{1}{2}}\cos(\phi-\phi').$$

We make use of the addition theorem and the orthogonality relations of spherical harmonics and find with the help of Eq. (14) that

$$G_t(v,\mu)=\sum_f\sum_{n=0}^{\infty}\frac{4\pi}{2n+1}\Gamma_{tf}P_n(\mu)\int_0^{\infty}A_nF_f^n(v')(v')^2dv'$$
$$\simeq G_t^0(v)+\mu G_t^1(v), \tag{15}$$

where

$$A_n(v,v')=\frac{2n+1}{2}\int_{-1}^{+1}[v^2+(v')^2-2vv'z]P_n(z)dz.$$

Further reduction results in

$$G_t^0(v)=\sum_f G_f^0(v)=4\pi\sum_f\Gamma_{tf}\left[v\int_0^v F_f^0(v')^2dv'+\frac{1}{3v}\int_0^v(v')^4F_f^0dv'+\int_v^{\infty}(v')^3F_f^0dv'+\frac{v^2}{3}\int_v^{\infty}(v')F_f^0dv'\right], \tag{16}$$

and

$$G_t^1(v)=\sum_f G_f^1(v)=\frac{4\pi}{15}\sum_f\Gamma_{tf}\left[\frac{1}{v^2}\int_0^v(v')^5F_f^1dv'-5\int_0^v F_f^1(v')^3dv'+v^3\int_v^{\infty}F_f^1dv'-5v\int_v^{\infty}(v')^2F_f^1dv'\right]. \tag{17}$$

Substitution of Eqs. (14), (16), and (17) into Eq. (11), and use of Eqs. (9b) and (9c) results after appropriate integrations over the angle variable μ in the following two equations:

$$\left(\frac{\partial F^0}{\partial t}\right)_{cc}=\sum_f\left[4\pi b\Gamma_{tf}F_f^0F^0+\frac{\partial F^0}{\partial v}\left(\frac{b}{v^2}\frac{\partial G_f^0}{\partial v}+\frac{2c}{v}\frac{\partial^2G_f^0}{\partial v^2}+c\frac{\partial^3G_f^0}{\partial v^3}\right)+\frac{1}{2}\frac{\partial^2G_f^0}{\partial v^2}\frac{\partial^2F^0}{\partial v^2}\right], \tag{18}$$

and

$$\left(\frac{\partial F^1}{\partial t}\right)_{cc}=\sum_f\left[\frac{1}{2}\frac{\partial^2G_f^0}{\partial v^2}\frac{\partial^2F^1}{\partial v^2}+\frac{\partial F^1}{\partial v}\left(c\frac{\partial^3G_f^0}{\partial v^3}+\frac{2c}{v}\frac{\partial^2G_f^0}{\partial v^2}+\frac{b}{v^2}\frac{\partial G_f^0}{\partial v}\right)-F^1\left(\frac{b}{2}\frac{\partial^4G_f^0}{\partial v^4}+\frac{2b}{v}\frac{\partial^3G_f^0}{\partial v^3}+\frac{1}{v^3}\frac{\partial G_f^0}{\partial v}\right)+\frac{1}{2}\frac{\partial^2F_f^0}{\partial v^2}\frac{\partial^2G_f^1}{\partial v^2}+\frac{\partial F^0}{\partial v}\left(c\frac{\partial^3G_f^1}{\partial v^3}+\frac{2c}{v}\frac{\partial^2G_f^1}{\partial v^2}+\frac{2b-1}{v^2}\frac{\partial G_f^1}{\partial v}+\frac{1-2b}{v^3}G_f^1\right)-F^0\left(\frac{b}{2}\frac{\partial^4G_f^1}{\partial v^4}+\frac{2b}{v}\frac{\partial^3G_f^1}{\partial v^3}-\frac{2b}{v^2}\frac{\partial^2G_f^1}{\partial v^2}-\frac{4}{v^3}\frac{\partial G_f^1}{\partial v}\right)\right], \tag{19}$$

where

$$b=m_t/m_f, \tag{20}$$

$$c=(m_f-m_t)/(2m_f). \tag{21}$$

Equation (18) is equivalent to an equation derived earlier by the author.[13] Chandrasekhar[14] has applied a linearized version of Eq. (18) to astrophysical problems by assuming that the velocity distribution of field particles is Maxwellian, and their temperature independent of time. Cohen, Spitzer, and Routley[1] have used the same approximation in their solution of Eq. (19). These calculations precede the introduction of the H and G functions.

B. Electron-Molecule Collision Terms

Substitution of the Lorentz expansion into the elastic and inelastic collision terms given in Eqs. (12) and (13) results after integration over angles in

$$\left(\frac{\partial F^0}{\partial t}\right)_{cm}+\left(\frac{\partial F^0}{\partial t}\right)_{cx}=\frac{1}{v^2}\frac{m}{M}\frac{\partial}{\partial v}(v^3F^0\nu_e)+\sum_h\left[\frac{w}{v}F^0(w)\nu_h(w)-F^0(v)\nu_h(v)\right], \tag{22}$$

$$\left(\frac{\partial F^1}{\partial t}\right)_{cm}=-F^1\nu_e(v), \tag{23}$$

[13] H. Dreicer, Massachusetts Institute of Technology, Ph.D. thesis, 1955 (unpublished). The main results of this thesis also appear in W. P. Allis, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 21.

[14] S. Chandrasekhar, Astrophys. J. 98, 54 (1943).

where ν_e, the frequency for momentum transfer, and ν_h are defined by

$$\nu_e(v)=2\pi Nv\int_0^\pi (1-\cos\beta)\,\sin\beta\sigma_e(\beta,v)d\beta, \quad (24)$$

$$\nu_h(v)=2\pi Nv\int_0^\pi \sin\beta\sigma_h(\beta,v)d\beta. \quad (25)$$

Equation (22) describes the effect on F^0 of energy transferred both elastically and inelastically to neutral molecules. The possibility of energy transfer from molecules to electrons has been ignored. Equation (23) describes the effect on F^1 (and therefore on the electron current) of the collisional friction force which neutral molecules exert on electrons. This mechanism is responsible for the electrical conductivity of a poorly ionized gas. The contribution from inelastic collisions to $\partial F^1/\partial t$ is smaller than $F^1\nu_e$, the contribution from elastic collisions with neutrals, by the factor $\sum_h \nu_h/\nu_e$. In the range of average electron energies considered in this paper (up to 10 ev) we shall find this factor to be 10^{-2} and less. For this reason we shall neglect the effect of inelastic collisions upon current flow.

C. Complete Form of the F^0 and F^1 Equations

The collision terms just derived, when supplemented by the terms originating from the spherical harmonic expansion of $(e/m)\mathbf{E}\cdot\nabla_v F$, give us the following F^0 and F^1 equations:

$$\frac{\partial F^0}{\partial t}-\frac{eE}{3mv^2}\frac{\partial}{\partial v}(v^2F^1)=\left(\frac{\partial F^0}{\partial t}\right)_{cc}+\left(\frac{\partial F^0}{\partial t}\right)_{cm}+\left(\frac{\partial F^0}{\partial t}\right)_{cx}, \quad (26)$$

$$\frac{\partial F^1}{\partial t}=\frac{e}{m}E\frac{\partial F^0}{\partial v}+\left(\frac{\partial F^1}{\partial t}\right)_{cc}+\left(\frac{\partial F^1}{\partial t}\right)_{cm}. \quad (27)$$

Simultaneous analytic solution of these equations is possible only if a number of simplifying assumptions are made. For example, in the theory of electrical breakdown, Coulomb terms may be completely ignored, and steady state solutions can be obtained provided tractable forms are chosen for ν_e and ν_h. In the opposite limit of a fully ionized gas, the steady state F^1 equation must be solved numerically even when F^0 is assumed to be Maxwellian. In our application of Eqs. (26) and (27) to the partially ionized gas we shall neglect $(\partial F^1/\partial t)_{cc}$, and assume the positive ions and molecules to be infinitely massive. The physical limitations imposed by these approximations will be made clear by several minor applications into which we enter next.

IV. ILLUSTRATIVE EXAMPLES

A. Energy Relaxation in a Fully Ionized Gas

In this section we apply Eq. (18) to the problem of calculating the rate of energy transfer between particles of a highly ionized gas.

1. Electron-Ion Encounters

The rate of energy transfer between particles of equal masses greatly exceeds the rate of energy transfer between particles of widely differing masses. For this reason the electrons and ions of a plasma approach Maxwellian distributions characteristic of the temperatures T_e and T_i long before these temperatures approach each other. Thus only a small error is committed in assigning Maxwellian distributions to the ions and electrons. With this choice, self collisions do not contribute to Eq. (18), and the G^0 function for electron-ion encounters is given by

$$G^0(v)=n\Gamma_{ei}\left(\frac{2kT_i}{M}\right)^{\frac{1}{2}}\left\{\mathcal{E}_2(x)\left[x+\frac{1}{2x}\right]+\frac{1}{2}\frac{d\mathcal{E}_2}{dx}\right\}, \quad (28)$$

where

$$\mathcal{E}_2(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt, \quad x=\left(\frac{Mv^2}{2kT_i}\right)^{\frac{1}{2}},$$

and Γ_{ei} is defined by Eq. (8) with the ions playing the role of the field particles. The energy equation is derived with the help of Eq. (28) by integrating Eq. (18) appropriately over all velocities. The result

$$\frac{dT_e}{dt}=-\frac{dT_i}{dt}=\frac{8}{3\sqrt{\pi}}n\Gamma_{ei}\frac{m}{M}\frac{T_e-T_i}{(2kT_i/M+2kT_e/m)^{\frac{3}{2}}}, \quad (29)$$

agrees with a different derivation due to Spitzer.[15] Equation (29) may be integrated conveniently with the help of the following definitions

$$\xi(t)=[T_e(t)/T_A]-1, \quad T_A=[T_e(t)+T_i(t)]/2.$$

In terms of these, we obtain

$$\frac{d\xi}{dt}=-\frac{1}{\tau_{ei}}\left\{\xi\Big/\left[1+\xi\left(\frac{M-m}{M+m}\right)\right]^{\frac{3}{2}}\right\}, \quad (30)$$

where

$$\frac{1}{\tau_{ei}}=\frac{16}{3\sqrt{\pi}}n\Gamma_{ei}\frac{m}{M}\left(\frac{m_0}{2kT_A}\right)^{\frac{3}{2}}=1.2\times10^{-8}n\frac{Z_i^2}{W}\left(\frac{3}{2}\frac{kT_A}{e}\right)^{-\frac{3}{2}}\ln\frac{\lambda}{P_0}\ \text{sec}^{-1}. \quad (31)$$

In the last expression the density, n, is to be expressed in cm^{-3}, and the average energy, $\frac{3}{2}(kT_A/e)$, in electron volts. W is the atomic weight of the ion, and Z_i its

[15] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

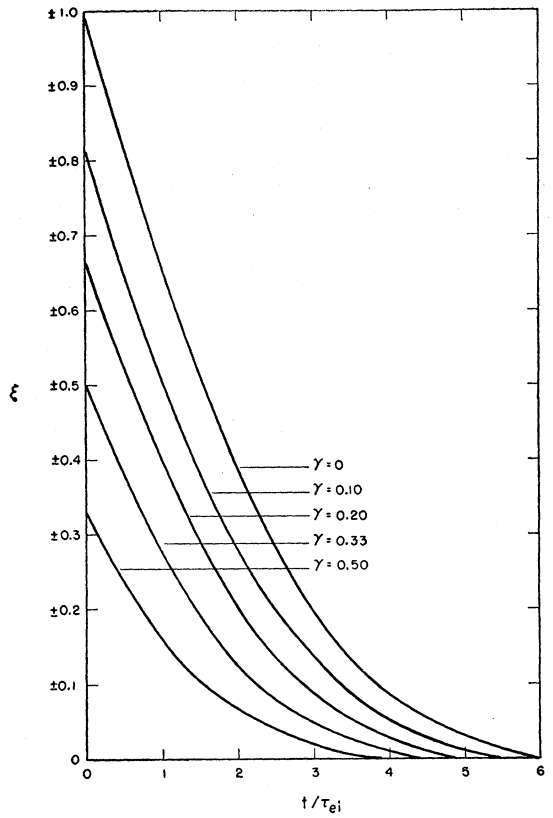

FIG. 2. The variation with normalized time of the temperature deviation ratio, ξ, characteristic of electron-ion energy exchange. The curves are parametric in the ratio of initial temperatures.

state of ionization. The relaxation time is a function of the average temperature, T_A, the reduced mass, m_0, and the total mass of the colliding particles. With the neglect of m compared to M, we finally have

$$\frac{t}{\tau_{ei}}=\tfrac{2}{3}[1+\xi(0)]^{\frac{3}{2}}-\tfrac{2}{3}[1+\xi(t)]^{\frac{3}{2}}+2[1+\xi(0)]^{\frac{1}{2}}-2[1+\xi(t)]^{\frac{1}{2}}+\ln[\xi(0)/\xi(t)]. \quad (32)$$

The temperature deviation ratio ξ is shown in Fig. 2 as a function of t/τ_{ei}. Positive and negative values of ξ correspond to $T_e(0)>T_i(0)$ and $T_e(0)<T_i(0)$, respectively. The curves shown are parametric in the initial temperature ratio γ which is defined to be $T_i(0)/T_e(0)$ when ξ is positive, and $T_e(0)/T_i(0)$ when ξ is negative.

2. *Encounters Between Identical Particles*

Again we assume that the particles are distributed according to a Maxwellian distribution. We then focus our attention on a single test particle whose initial energy is $\frac{1}{2}m_f v^2(0)$, and calculate the time required for its energy to approach the average energy, $\frac{3}{2}kT_f$, of the gas. The G^0 function for self collision is given by

$$G^0(v)=n\Gamma_{ff}\left(\frac{2kT_f}{m_f}\right)^{\frac{1}{2}}\left\{\mathcal{E}_2(y)\left[y+\frac{1}{2y}\right]+\frac{1}{2}\frac{d\mathcal{E}_2}{dy}\right\}, \quad (33)$$

where

$$y=(m_f v^2/2kT_f)^{\frac{1}{2}}.$$

The energy equation is again obtained by averaging Eq. (18) over all velocities, with the result

$$\frac{d}{dt}\left(\frac{m_f v^2}{2}\right)=-\frac{1}{\tau}\frac{\frac{1}{2}m_f v^2-\frac{3}{2}kT_f}{[(m_f v^2/3kT_f)+1]^{\frac{3}{2}}}. \quad (34)$$

The relaxation rate is given by

$$\frac{1}{\tau_{ee}}=\frac{8}{3\sqrt{\pi}}n\Gamma_{ee}\left(\frac{2m}{3kT_e}\right)^{\frac{3}{2}}, \quad (35)$$

for electron-electron encounters, and by

$$\frac{1}{\tau_{ii}}=Z_i^4\left(\frac{m}{M}\right)^{\frac{1}{2}}\frac{1}{\tau_{ee}}, \quad (36)$$

for ion-ion encounters, provided the temperatures involved are equal.

Equation (34) can be integrated exactly for the temperature deviation ratio ξ,

$$\xi(t)=(m_f v^2/3kT_f)-1,$$

in terms of t/τ, and several examples of the solution are shown in Fig. 3 with the ratio $\frac{1}{2}m_f v^2(0)/\frac{3}{2}kT_f$ treated as a parameter. Comparison of Eq. (35) and (36) with Eq. (31) shows that for energy relaxation we have

$$\frac{\tau_{ee}}{\tau_{ei}}=2Z_i^2\frac{m}{M}\left(\frac{T_e}{T_A}\right)^{\frac{3}{2}}, \quad (35a)$$

and

$$\frac{\tau_{ii}}{\tau_{ei}}=\frac{1}{Z_i^2}\left(\frac{m}{M}\right)^{\frac{1}{2}}\left(\frac{T_i}{T_A}\right)^{\frac{3}{2}}. \quad (35b)$$

These results help to justify the use of two Maxwellian distributions characterized by different temperatures in the calculation of the electron-ion relaxation rate. In our later treatment of the partially ionized gas we ignore the energy transfer between electrons and ions by taking the ions to be infinitely massive. We justify this approximation in Sec. V, D, where we show that this transfer of energy is negligible compared to the electron energy lost inelastically in the electronic excitation of neutrals.

B. The Electrical Conductivity of a Partially Ionized Gas

We consider next the dependence of electrical conductivity upon ionization degree. It is clear that the resistance is controlled by collisions with neutral mole-

cules when the ionization degree is sufficiently low. On the other hand scattering in the Coulomb field of the positive ions dominates the resistance in a highly ionized gas. The ionization degree for which these encounters have equal effects upon the conductivity depends upon the electron temperature and the collision cross-sections involved. Mutual encounters between electrons alter the momentum of the electron gas only indirectly by influencing the rate of electron-ion encounters, and these self-collisions will be ignored here. We also ignore the small contributions, amounting to m/M of the total momentum interchange, which arise from the recoil motion of molecules and ions by taking these to be infinitely massive. Equation (19) then yields the simple form

$$(\partial F^1/\partial t)_{cc} = -n\Gamma_{ei}F^1/v^3, \tag{37}$$

where use has been made of $b=0$, $c=0.5$, $G^0=nv\Gamma_{ei}$, $G^1=0$. Substitution of Eqs. (23) and (37) into (27) yields the static solution

$$F^1 = \frac{(e/m)E}{[\nu_e(v)+(n\Gamma_{ei}/v^3)]}\frac{\partial F^0}{\partial v},$$

and the electrical conductivity, σ, is given by

$$\sigma = \frac{4\pi e^2}{3m}\int_0^\infty \frac{v^6(\partial F^0/\partial v)}{\nu_e(v)v^3+n\Gamma_{ei}}dv. \tag{38}$$

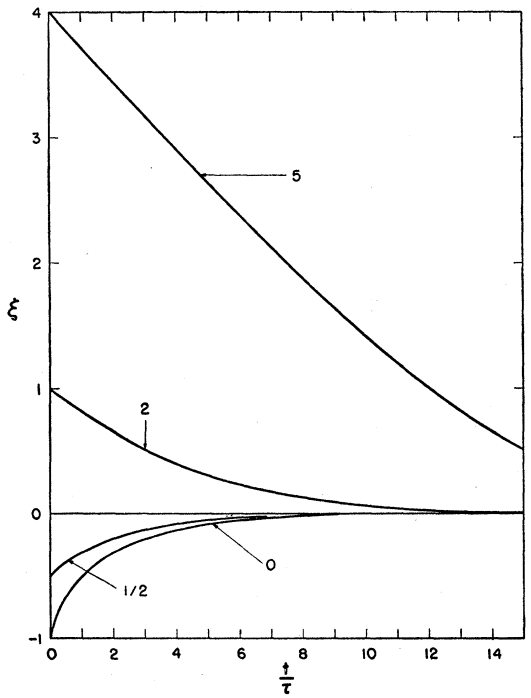

FIG. 3. The variation with normalized time of the temperature deviation ratio, ξ, characteristic of self-collision in a plasma. The curves are parametric in the ratio of the initial particle energy to the average particle energy in the gas.

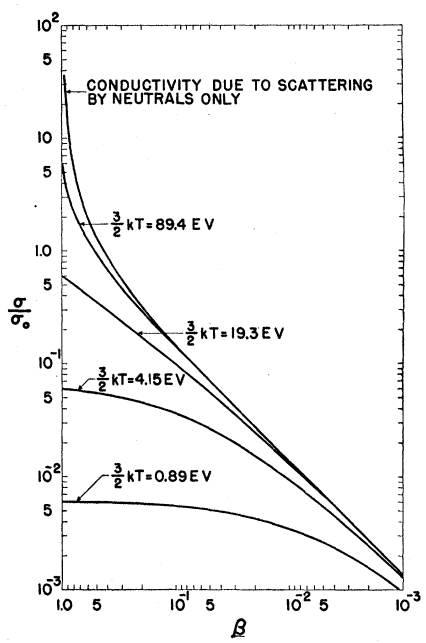

FIG. 4. The variation with ionization degree of the normalized electrical conductivity of a hydrogen plasma. These curves are parametric in the average electron energy. The uppermost curve describes the residual conductivity when Coulomb encounters are ignored.

With a known F^0 we can obtain σ after a single numerical integration. As an illustration we have chosen the case of a hydrogen plasma, for which ν_e is independent of v above several electron volts. It is given by the following alternative forms[16]:

$$\nu_e = 1.67\times10^{-7}N \text{ sec}^{-1} = g_0N$$
$$= 5.9\times10^{9}p \text{ sec}^{-1}, \tag{39}$$

where the neutral gas density N is measured in units of cm^{-3}, and p is the neutral gas pressure in mm-Hg at 0°C. The results of numerical integration for a Maxwellian distribution of electrons are shown in Fig. 4 where σ, normalized to

$$\sigma_0 = 4e^2/(3\sqrt{\pi}mg_0),$$

is plotted against the single stage ionization degree $\beta=n/(n+N)$ with average electron energy as a parameter. The conductivity controlled by scattering with neutrals only, shown as a separate curve in this figure, is asymptotic to all other curves as β tends to zero, and diverges as β tends to unity. In the temperature range of several electron volts, deviations from this curve first become important at a transition ionization degree, which we denote by β_M, of about 10^{-1}. The electron-ion collision rate decreases with increasing temperature and causes β_M to increase with increasing temperature. As we have stated earlier, we shall neglect $(\partial F^1/\partial t)_{cc}$ in our study of partially ionized

[16] W. P. Allis and S. C. Brown, Phys. Rev. 87, 419 (1952).

gases. Our treatment is therefore restricted to plasmas whose ionization degree does not exceed β_M. Nevertheless, as we shall see, mutual electron interaction first comes into play at a much smaller β and, of course, these terms are retained in Eq. (26).

V. MUTUAL ELECTRON INTERACTION

A. Physical Description

The neglect of the Coulomb terms in Eq. (27) greatly simplifies this equation. For the steady state solution we now obtain

$$F^1=(eE/m\nu_e)(\partial F^0/\partial v),$$

and substitution of this result into Eq. (26) yields

$$\frac{1}{3}\left(\frac{eE}{m}\right)^2\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^2}{\nu_e}\frac{\partial F^0}{\partial v}\right)+4\pi\Gamma_{ee}(F^0)^2+\frac{1}{v^2}\frac{\partial G^0}{\partial v}\frac{\partial F^0}{\partial v}$$
$$+\frac{1}{2}\frac{\partial^2 G^0}{\partial v^2}\frac{\partial^2 F^0}{\partial v^2}+\sum_h\left[\frac{w}{v}F^0(w)\nu_h(w)-F^0(v)\nu_h(v)\right]=0. \quad (40)$$

Here we have ignored the energy exchange resulting from elastic electron-neutral and electron-ion encounters compared to the energy exchanged by mutual electron interaction and inelastic collision. Equation (40) may also be written in terms of the average velocity increments which are the result of electron-electron encounters. It then takes the form,

$$\frac{1}{v^2}\frac{d}{dv}\left\{\left[\frac{1}{3\nu_e}\left(\frac{eE}{m}\right)^2+\tfrac{1}{2}\langle(\Delta v)^2\rangle^0\right]v^2\frac{dF^0}{dv}-\tfrac{1}{2}\langle\Delta v\rangle^0 v^2F^0\right\}$$
$$+\sum_h\left[\frac{w}{v}F^0(w)\nu_h(w)-F^0(v)\nu_h(v)\right]=0, \quad (41)$$

where Δv is the component of the incremental velocity change along the velocity of the test electron. In this equation the coefficient of dF^0/dv plays the role of a diffusion constant in velocity space. Both terms in this coefficient describe a radial outward flow in velocity space. The first is due to the Joule heating of the electrons by the electric field. The second is caused by mutual electron encounters. The average velocity increments $\langle\Delta v\rangle^0$ and $\langle(\Delta v)^2\rangle^0$ involve only the spherically symmetric part, F^0, of the distribution function. Since Coulomb interactions by themselves cannot alter the total energy in the electron gas, the average rate at which fluctuations [proportional to $\langle(\Delta v)^2\rangle$] deposit energy in the electrons must be exactly balanced by the rate at which dynamical friction (proportional to $\langle\Delta v\rangle$) removes energy from the electrons. To show this we multiply the Fokker-Planck equation by $\frac{1}{2}mv^2$ and integrate over all velocities. The result takes the form

$$\frac{d}{dt}(\tfrac{1}{2}m\langle v^2\rangle_{Av})=\tfrac{1}{2}m\langle\mathbf{v}\cdot\langle\Delta\mathbf{v}\rangle\rangle_{Av}+\tfrac{1}{2}m\langle\langle(\Delta v)^2\rangle\rangle_{Av},$$

where the average is taken over all velocities. By the conservation of energy this expression must vanish. Equation (40) [or (41)] of course satisfies this criterion also, for if we multiply each of its terms by $\frac{1}{2}mv^2$, and integrate over all velocities we find for the power conservation law

$$\frac{(eE)^2}{m}\int_0^\infty\frac{F^0}{\nu_e}4\pi v^2dv=-\sum_h\int_0^\infty\left[\frac{w}{v}F^0(w)\nu_h(w)-F^0(v)\nu_h(v)\right]\frac{mv^2}{2}4\pi v^2dv, \quad (42)$$

which does not involve mutual electron encounters directly. This equation states that the rate of energy input from the electric field is exactly balanced by the rate of inelastic energy loss. The G function does not appear implicitly in Eq. (42) since electron-electron encounters only influence the distribution of energy among the electrons, and do not alter the total energy of the electron gas. Indeed, as the ionization degree of the gas rises the increasingly frequent encounters between electrons tend to arrange the electron velocities according to a Maxwellian distribution characteristic of the temperature which satisfies this power balance statement.

B. Approximate Inelastic Collision Term

Only a few of the cross sections, σ_h, appearing in the inelastic population and depopulation collision terms are known. We have therefore resorted to a model, already used by Allis and Brown,[16] whose features are the following:

(a) Electrons lose all of their energy in each inelastic collision.

(b) The sum over all inelastic collision rates, $\sum_h\nu_h(v)$, is replaced by the total frequency for excitation, $\nu_x(v)$. This quantity is expressed in terms of the experimentally defined excitation probability, P_x, by the relation

$$\nu_x(v)=vpP_x,$$

where p is the neutral gas pressure.

With the use of this model and the energy variable, u, defined by

$$u=mv^2/2e,$$

our final equation becomes

$$\tfrac{2}{3}u^{\frac{3}{2}}u_c\nu_e\frac{d^2F^0}{du^2}+\frac{2}{3}\frac{d}{du}(u^{\frac{3}{2}}u_c\nu_e)\frac{dF^0}{du}+4\pi\Gamma_{ee}\Big\{u^{\frac{1}{2}}(F^0)^2$$
$$+\left[\int_0^u F^0(t)t^{\frac{1}{2}}dt+u^{\frac{1}{2}}\int_u^\infty F^0(t)dt\right]\frac{dF^0}{du}$$
$$+\left[\tfrac{2}{3}\int_0^u F^0(t)t^{\frac{3}{2}}dt+\tfrac{2}{3}u^{\frac{3}{2}}\int_u^\infty F^0(t)dt\right]\frac{d^2F^0}{du^2}\Big\}$$
$$+\delta(u)\int_{u_x}^\infty F^0(t)\nu_x(t)t^{\frac{1}{2}}dt-F^0(u)\nu_x(u)u^{\frac{1}{2}}=0, \quad (43)$$

where u_x is the energy of the lowest electronic excitation level, and the parameter

$$u_c = eE^2/m\nu_e^2 \propto (E/p)^2 \propto (E/N)^2$$

is the energy gained by an electron from the electric field in the time interval between its elastic collisions with neutrals. The Dirac delta function $\delta(u)$ appears as a consequence of our inelastic collision model. Integration of the inelastic population term over all energies yields

$$\int_0^\infty \delta(u) \int_{u_x}^\infty F^0(t)\nu_x(t)t^{\frac{1}{2}}dtdu = \int_{u_x}^\infty F^0(t)\nu_x(t)t^{\frac{1}{2}}dt,$$

which is just the average inelastic excitation rate.

Equation (43) involves the electron and neutral gas densities, n and N, as well as the electric field, E. If we wish to study F^0 as a function of n/N, keeping $(n+N)$ constant, then we must permit u_c to vary if E is fixed. On the other hand if u_c (or E/p) is held constant then the total number density of particles, $(n+N)$, must be allowed to vary as n/N changes. The problem considered is time independent, and for each set of the parameters, n, N, E, we are considering a different steady state. Whether a plasma corresponding to a chosen set of parameters can actually exist in nature or not, depends upon the plasma distribution in real space, upon recombination, and upon a host of interactions involving excited molecules and radiation, which are not included in this simplified treatment.

C. Numerical Method of Solution for Hydrogen

Equation (43) is a nonlinear integro-differential equation in the variable u. The customary scheme of linearizing such an equation makes use of a method of successive approximations, and results in an ordinary second order differential equation at each stage of approximation. Its coefficients include the Coulomb collision integrals evaluated in terms of a distribution function obtained during the previous stage of the calculation. In general this ordinary differential equation cannot be evaluated in terms of known functions, and numerical techniques must be resorted to. If one considers that equations of the type (43) are not too involved for modern digital computers, then it becomes a matter of practicality to handle the nonlinear problem directly in this way. This approach was chosen, and applied to a hydrogen plasma.

The elastic collision rate, ν_e, for this gas has already been listed in Eq. (39). The total inelastic collision frequency in hydrogen can be represented by[16]

$$\nu_x(u) = \nu_e[h_0 u - h_1 - (h_2/u)], \tag{44}$$

where $h_0 = 8.7\times10^{-3}$ volt^{-1}, $h_1 = 76\times10^{-3}$, and ν_x vanishes below the lowest excitation potential, $u_x = 8.9$ volts. The frequency of ionization can be represented by[16]

$$\nu_i(u) = \nu_e h_i(u - u_i), \tag{45}$$

where $h_i = 9.2\times10^{-3}$ volt^{-1}, and ν_i vanishes below the ionization potential, $u_i = 16.2$ volts.

Equation (43) is equivalent to the following set of simultaneous equations:

$$\begin{gathered} R(u)dF_5/du + S(u)F_5 + T(u)F_1 = 0, \\ dF_1/du = F_5, \quad dF_3/du = u^{\frac{1}{2}}F_1, \\ dF_2/du = F_1, \quad dF_4/du = u^{\frac{3}{2}}F_1, \end{gathered} \tag{46}$$

where

$$F_1(u) = 2\pi\left(\frac{2e}{m}\right)^{\frac{1}{2}} F^0(u) = AF^0(u),$$

$$F_2(u) = \int_0^u F_1(t)dt,$$

$$F_3(u) = \int_0^u F_1(t)t^{\frac{1}{2}}dt,$$

$$F_4(u) = \int_0^u F_1(t)t^{\frac{3}{2}}dt,$$

and

$$R(u) = u^{\frac{3}{2}} + \frac{4\pi\Gamma_{ee}}{Au_c g_0}\frac{n}{N}\{u^{\frac{3}{2}}[F_2(\infty) - F_2(u)] + F_4(u)\},$$

$$S(u) = \tfrac{3}{2}u^{\frac{1}{2}} + \frac{6\pi\Gamma_{ee}}{Au_c g_0}\frac{n}{N}\{F_3(u) + u^{\frac{1}{2}}[F_2(\infty) - F_2(u)]\},$$

$$T(u) = \frac{6\pi\Gamma_{ee}}{Au_c g_0}\frac{n}{N}u^{\frac{1}{2}}F_1(u) + \frac{3}{2}\left[\frac{\delta(u)}{F_1(u)}\int_{u_x}^\infty F_1(t)\frac{\nu_x(t)}{\nu_e}t^{\frac{1}{2}}dt - u^{\frac{1}{2}}\frac{\nu_x(u)}{\nu_e}\right].$$

The procedure for simultaneous solution of these equations may be summarized as follows:

(1) The parameters u_c and n/N are chosen.

(2) $u = \infty$ is approximated by a finite number, which is denoted u_∞. In practice it has been chosen in the range $3u_i$ to $6u_i$.

(3) Best initial guesses are made at $F_1^{(1)}(u_\infty)$, $F_2^{(1)}(u_\infty)$, $F_4^{(1)}(u_\infty)$, and $F_5^{(1)}(u_\infty)$. The superscripts denote the number of the iteration cycle.

(4) $F_3(u_\infty)$ is assigned the value unity at the start of each iteration, because F_1 must satisfy normalization.

(5) Using a standard Runge-Kutta routine, the equations in (46) are integrated from u_∞ to zero, and the resulting solution $F_1^{(1)}$ is utilized in the evaluation of $F_2^{(2)}(u_\infty)$, $F_3^{(2)}(u_\infty)$, and $F_4^{(2)}(u_\infty)$.

(6) In addition the computer evaluates

(a) the average inelastic collision rate,

$$\langle\nu_x\rangle^{(1)} = \int_{u_x}^{u_\infty} \nu_x(u)F_1^{(1)}(u)u^{\frac{1}{2}}du; \tag{47}$$

(b) the average ionization rate,

$$\langle \nu_i \rangle^{(1)} = \int_{u_i}^{u_\infty} \nu_i(u) F_1^{(1)}(u) u^{\frac{1}{2}} du; \tag{48}$$

and (c) the power output

$$\langle u\nu_x \rangle^{(1)} = \int_{u_x}^{u_\infty} \nu_x(u) F_1^{(1)}(u) u^{\frac{1}{2}} du. \tag{49}$$

This last quantity according to Eq. (42) must equal the power input, $\nu_e u_c$. Thus it is determined by our choice of u_c (or E/p).

(7) A second Runge-Kutta cycle proceeds with the new starting conditions $F_2^{(2)}(u_\infty)$, $F_4^{(2)}(u_\infty)$, or some weighted average between these and $F_2^{(1)}(u_\infty)$, $F_4^{(1)}(u_\infty)$. The starting slope, $F_5^{(1)}(u_\infty)$, and value of the function $F_1^{(1)}(u_\infty)$, assumed in step 3 are not altered, and steps 5 to 7 are repeated n times until successive iteration cycles converge to a final set of $F_1^{(n)}(u)$, $F_2^{(n)}(u)$, $F_3^{(n)}(u)$, $F_4^{(n)}(u)$, $\langle \nu_x \rangle^{(n)}$, $\langle \nu_i \rangle^{(n)}$, and $\langle \nu_x u \rangle^{(n)}$. A solution is accepted when successive iterations produce results differing by less than 1% in these quantities.

Moreover, a solution is accepted only if normalization, and the power balance described in steps 4 and 6(c) converge to unity and $u_c \nu_e$, respectively. In practice it is found that with reasonable starting values for $F_1^{(1)}(u_\infty)$, $F_2^{(1)}(u_\infty)$, $F_4^{(1)}(u_\infty)$, and $F_5^{(1)}(u_\infty)$ convergence is obtained in less than 10 iteration cycles. A poor choice of $F_5^{(1)}(u_\infty)$ and $F_1^{(1)}(u_\infty)$ is reflected in the violation of normalization and power conservation after the very first iteration cycle. Succeeding cycles then result in increasingly diverging values of $F_1(u)$. It

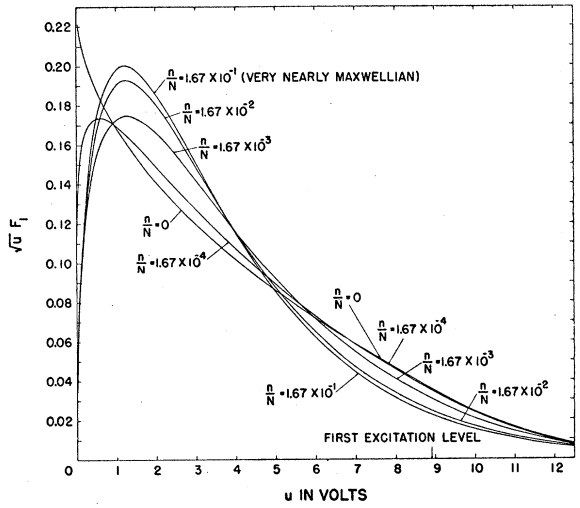

FIG. 5. Variation with electron energy (in ev) of the number of electrons per unit energy interval in a hydrogen plasma. In this illustration E/p was held constant at the value 28.3 volts cm^{-1} (mm Hg)$^{-1}$, and the density ratio was varied from zero to 0.167.

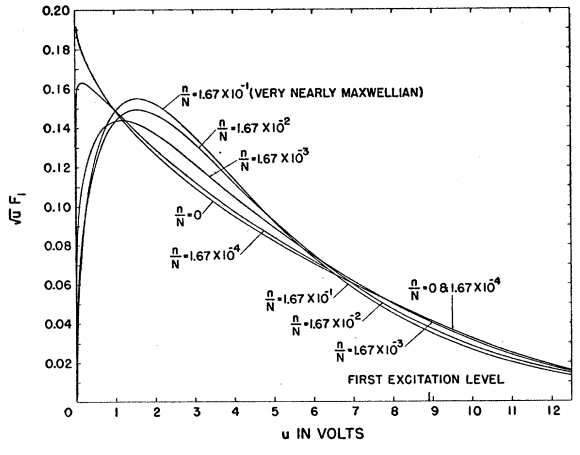

FIG. 6. Variation with electron energy (in ev) of the number of electrons per unit energy interval in a hydrogen plasma. In this illustration E/p was held constant at the value 48.9 volts cm^{-1} (mm Hg)$^{-1}$, and the density ratio was varied from zero to 0.167.

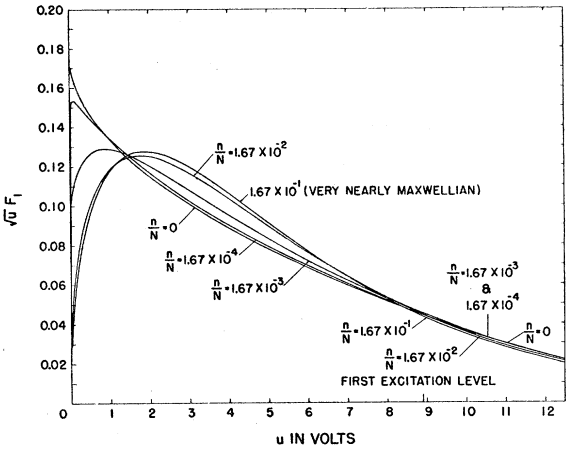

FIG. 7. Variation with electron energy (in ev) of the number of electrons per unit energy interval in a hydrogen plasma. In this illustration E/p was held constant at the value 70.7 volts cm^{-1} (mm Hg)$^{-1}$, and the density ratio was varied from zero to 0.167.

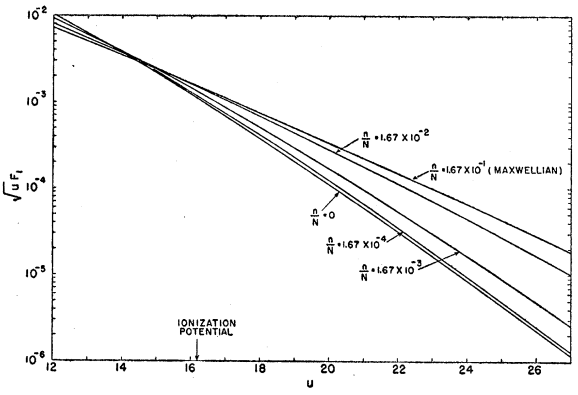

FIG. 8. Variation with electron energy (in ev) of the number of electrons per unit energy interval in a hydrogen plasma. In this illustration E/p was held constant at the value 28.3 volts cm^{-1} (mm Hg)$^{-1}$, and the density ratio was varied from zero to 0.167.

appears that highly accurate solutions can be obtained in the high-energy tail of the distribution by requiring convergence in the functions $\langle \nu_x \rangle$, $\langle \nu_i \rangle$, and $\langle \nu_x u \rangle$, which receive their contribution in this energy region.

D. Discussion of Solutions

Three numerical solutions corresponding to different values of u_c are illustrated in Figs. 5 to 7. There we show the evolution of $(\sqrt{u})F_1$ as n/N is varied by many orders of magnitude. Figure 8 is the extension of Fig. 5 to higher energies. Figures 6 and 7 extend to higher energies in a very similar manner. At very low energies we find that the density of electrons per unit energy interval, $(\sqrt{u})F_1$, decreases as the ionization degree increases. In the neighborhood of 1 ev the distributions cross each other, and the situation is reversed. At still higher energies, the distributions cross each other two more times until in the very-high-energy tail we find the population increasing with ionization degree. The average electron energy, and the normalized collision rates $\langle \nu_x \rangle / \nu_e$ and $\langle \nu_i \rangle / \nu_e$ are pictured in Figs. 9 to 13.

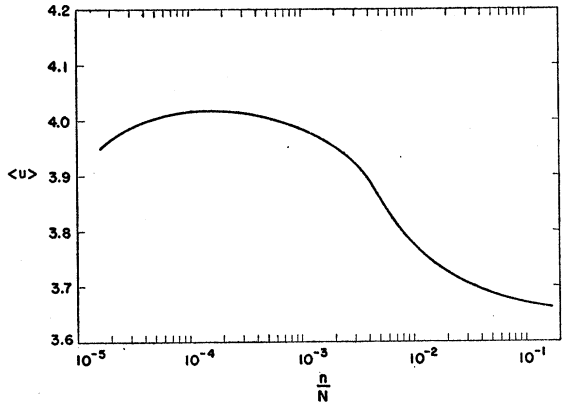

FIG. 9. The variation of average electron energy (in ev) with the density ratio n/N, for $E/p=28.3$ volts cm^{-1} (mm Hg)$^{-1}$. The curve was calculated for a hydrogen plasma.

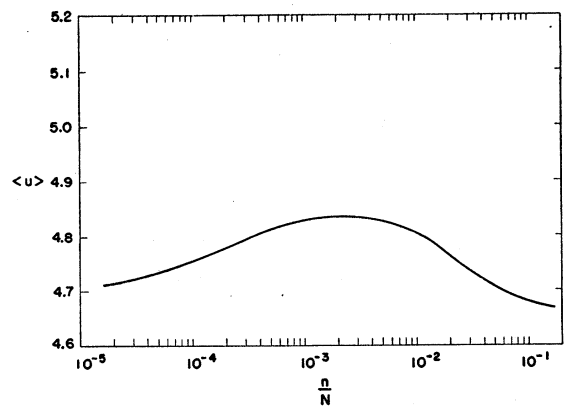

FIG. 10. The variation of average electron energy (in ev) with the density ratio n/N, for $E/p=48.9$ volts cm^{-1} (mm Hg)$^{-1}$. The curve was calculated for a hydrogen plasma.

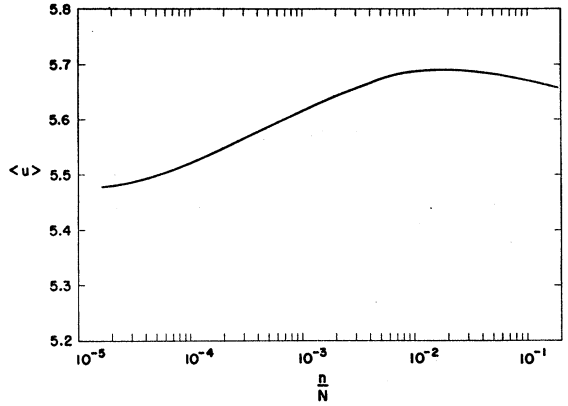

FIG. 11. The variation of average electron energy (in ev) with the density ratio n/N, for $E/p=70.7$ volts cm^{-1} (mm Hg)$^{-1}$. The curve was calculated for a hydrogen plasma.

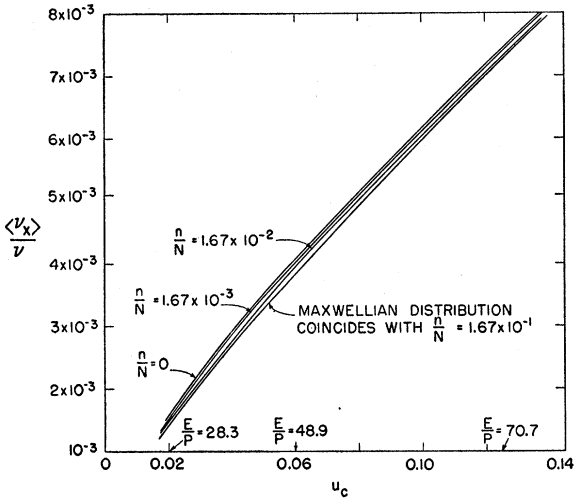

FIG. 12. The ratio of the average inelastic collision rate to the elastic collision rate is shown for a hydrogen plasma as a function of the energy parameter u_c expressed in ev. The density ratio n/N plays the role of a parameter.

For comparison these quantities were also calculated using a Maxwellian distribution of electrons, and covering a similar range of u_c. In this case, the original Eqs. (47), (48), and (49) become

$$u_c\nu_e=\frac{2}{\sqrt{\pi}(kT/e)^{\frac{3}{2}}}\times\int_{u_x}^{\infty}\nu_x(u)\exp\left[-u\Big/\left(\frac{kT}{e}\right)\right]u^{\frac{1}{2}}du, \quad (50)$$

$$\langle\nu_x\rangle=\frac{2}{\sqrt{\pi}(kT/e)^{\frac{3}{2}}}\times\int_{u_x}^{\infty}\nu_x(u)\exp\left[-u\Big/\left(\frac{kT}{e}\right)\right]u^{\frac{1}{2}}du, \quad (51)$$

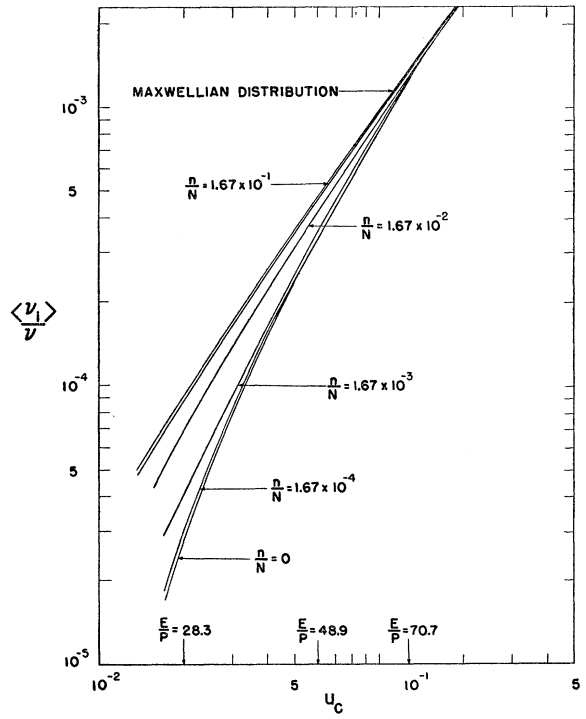

FIG. 13. The ratio of the average ionization rate to the elastic collision rate is shown for a hydrogen plasma as a function of the energy parameter u_c expressed in ev. The density ratio n/N plays the role of a parameter.

$$\langle \nu_i \rangle = \frac{2}{\sqrt{\pi}(kT/e)^{\frac{3}{2}}} \times \int_{u_i}^{\infty} \nu_i(u) \exp\left[-u \Big/ \left(\frac{kT}{e}\right)\right] u^{\frac{1}{2}} du. \quad (52)$$

For each assigned value of u_c, Eq. (50) yields a value of T, and Eqs. (51) and (52) are then used to obtain the Maxwellian collision rates.

For the range of u_c illustrated in Fig. (12), the change in ionization degree gave rise to remarkably little change in the inelastic collision rate. At this point, we may also note that the power output (or input), $u_c\nu_e$, in each case studied exceeds by far the power which electrons can transfer in elastic collisions with the positive ions. [See Eq. (39).] This result justifies the complete neglect of electron-ion collision terms in the F^0 equation provided the ionization degree lies below β_M, i.e., in the region where the Joule heating is controlled by encounters with neutral molecules. The distribution F_1 converges towards a Maxwellian distribution with rising ionization degree, and for the range of u_c studied in this paper becomes identical with one when the density ratio n/N has reached the value 0.167. The distribution probably becomes Maxwellian for a somewhat smaller n/N, lying somewhere between 0.167 and 1.67×10^{-2}, but no attempt was made to locate the exact value.

Some of the effects of electron interaction become apparent from the variation of average energy with n/N. As the ionization degree increases, we see that the average energy first rises and then falls. The rise is due to the fact that the slow electrons which arise from inelastic encounters exchange energy with other electrons, and are quickly redistributed over the body of the distribution. If we fix E/p and N, then this rise in average energy is accomplished by the addition to the distribution of electrons whose average energy exceeds the average energy corresponding to $n/N=0$. The subsequent fall in average energy as n/N increases even further, reflects the fact that Coulomb encounters have decreased the population in the neighborhood of the first excitation potential and have redistributed these electrons primarily into lower energy regions. A small number of electrons are also redistributed into the very-high-energy tail, giving rise to rather important variations in the ionization rate at low E/p.

Ultimately n/N becomes large enough so that the rate at which electrons exchange energy with each other very much exceeds the rate of energy gain from the field, and a Maxwell distribution prevails. The evolution of the distribution function from that characteristic of the Lorentz gas to the equilibrium distribution occurs roughly over four orders of magnitude in ionization degree. The *onset* of this transition may be approximately predicted by equating the electron-electron energy relaxation rate given in Eq. (34) with $u_c\nu_e$:

$$(kT/\tau_{ee}) \simeq u_c\nu_e.$$

This equation yields the transition density ratio in terms of temperature and E/p as follows:

$$(n/N)_t \simeq 10^{-7}(E/p)^2(kT/e)^{\frac{1}{2}}, \quad (53)$$

where kT/e and E/p are to be expressed in ev and volt-cm^{-1}/(mm-Hg), respectively. Since the transition takes place over a large range of n/N, we cannot employ Eq. (53) to predict the existence of a Maxwell distribution. As we see from Fig. 13, indiscriminate use of the Maxwell distribution in the transition region can lead to significant error in the ionization rate, especially at low E/p.

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