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Quantum Theory of Transport in a Magnetic Field*

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A quantum-mechanical theory of transport of charge for an electron gas in a magnetic field is presented that takes account of the quantization of the electron orbits. A transport equation for the necessary elements of the density matrix is developed for arbitrary values of the magnetic field. The scattering is taken to be elastic and is treated only in the Born approximation. The effect of both the magnetic and electric fields on the collisions is taken into account. The influence of the latter has been neglected in previous theories. It is proven, however, to be important in high magnetic fields. It is established that the effect of a transverse electric field on the scattering can be described as the tendency of the electrons to "relax" to a distribution characteristic of thermal equilibrium in the presence of the electric field.

Previous theories of transport for large Hall angles are consistent with this theory. They can be obtained as a special solution, found by iteration, of this transport equation.

The special case of isotropic scattering has been considered in detail. In this case it is demonstrated that for small enough magnetic fields the usual classical result obtains.

I. INTRODUCTION

THE phenomenon of charge transport in magnetic fields so large that the quantization of the electron orbits becomes important has received recently considerable attention from both the experimental and theoretical viewpoints. The primary theoretical interest stems from the fact that this phenomenon provides a very simple situation for which the usual Boltzmann transport equation is not applicable and a quantum-mechanical transport theory is necessary. We shall be concerned here exclusively with the basic theory of this phenomenon. In particular we shall present a theory of the ohmic electric current that an electric field induces in a gas of noninteracting electrons in the presence of a magnetic field and a set of impurities that scatter elastically.

The first calculation on this topic was done by Titeica.[1] He calculated the transverse current for large Hall angles by tacitly assuming that it is produced only by the drift of the centers of the cyclotron orbits of the electrons in the magnetic field due to the electric field and the scattering by phonons. Such a semi-classical procedure neglects the possibility that the cyclotron motion itself may contribute to the current. Essentially the same method of calculation was followed by a number of different workers[2] in various applications to metals and semiconductors. The question of the validity of the method of calculation of the transverse current was, however, left untouched.

Lifshitz[3] and the author[4] have attempted to develop a transport theory for this phenomenon on the basis of quantum-mechanical principles. The quantum-mechanical generalization of the classical distribution function, which is necessary for the calculation of the current, is the density matrix. In the Landau[5] representation, i.e., in the stationary states of an electron in a magnetic field, it is realized that for the calculation of the current some off-diagonal elements of the density matrix are required. This is so because the Landau states do not have any average velocity perpendicular to the magnetic field, as they represent stationary

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[2] B. Davydov and I. Pomeranchuk, J. Phys. U.S.S.R. **2**, 147 (1940); G. E. Zilberman, J. Exptl. Theoret. Phys. U.S.S.R. **29**, 762 (1955) [translation: Soviet Phys. JETP **2**, 650 (1956)]; J. Appel, Z. Naturforsch. **11a**, 892 (1956).

[3] E. M. Lifshitz, J. Phys. Chem. Solids **4**, 11 (1958).

[4] P. N. Argyres, Phys. Rev. **109**, 1115 (1958); J. Phys. Chem. Solids **8**, 124 (1959); Westinghouse Research Report 6-94760-2R10 (unpublished).

[5] L. Landau, Z. Physik **64**, 629 (1930).

circular motions. It was then suggested[3] that the steady-state density matrix obeys a transport equation analogous to the classical Boltzmann equation. The collisions were there assumed to be described by some operator, which was not, however, established explicitly. In reference 4 the quantum-mechanical transport equation was established by use of time-dependent perturbation theory, and an explicit expression was given for the scattering operator in the Landau representation for elastic and inelastic collisions in the case of $\omega_0\tau \gg 1$, where ω_0 = cyclotron frequency and τ = relaxation time. These two theories gave, in general, expressions for the transverse current different from that assumed by Titeica.[1]

Recently Kubo *et al.*,[6] Adams and Holstein,[7] and Argyres and Roth[8] have worked out theories for the transverse current for the case $\omega_0\tau \gg 1$, by finding the steady-state density matrix directly from its equation of motion in powers of the scattering interaction. Their results justify rigorously the method of calculation of Titeica[1] and, at the same time, point out that it is correct only for nonoscillating electric fields. It is, therefore, of interest to investigate the reasons for the failure of the theories of Lifshitz[3] and the author.[4] This has been the motivation of the present work.

It is proposed that the latter calculations[3,4] are deficient in the following respect. The terms arising from the interference between the electric field and the impurities have been left out of consideration. In other words, in references 4, the terms of order (electric field) $\times$ (scattering interaction)^2 in the time development of the density matrix were neglected. For the off-diagonal elements of the density matrix these terms are of the same order as the other terms kept, as it can be seen from the proposed transport equations. One might say that, since the collisions take place in the presence of the electric field, they force the electrons to a "relaxation" distribution characteristic of thermodynamic equilibrium in its presence. This turns out to be the case in our problem. Although these interference terms are of higher order for the usual distribution function, for the off-diagonal elements of the density matrix they are as important as the usual terms.

These electric field-scattering interference terms can be taken into account within the framework of the calculation of reference 4. We shall not, however, pursue this point of view. Instead, we shall investigate the problem by a more rigorous approach, analogous to that of Kohn and Luttinger.[9] We demonstrate this in the following section. We, thus, derive a quantum transport equation for the matrix elements of the density operator necessary for the calculation of the current for arbitrary values of the magnetic field. The collisions are treated only in the Born approximation. It is shown in Sec. 3 that for $\omega_0\tau \gg 1$ this calculation gives results identical to those of other recent theories[6–8] for this case. In the last section the case of isotropic scattering is considered for arbitrary $\omega_0\tau$. It is proven that in the classical limit, i.e., high quantum numbers and $kT \gg \hbar\omega_0$, the usual result of the classical Boltzmann equation obtains.

II. QUANTUM TRANSPORT EQUATION

In this section we derive the basic transport equation for the density operator in the Landau representation.

For a magnetic field $\mathbf{H}$ in the z direction, i.e., $\mathbf{H} = (0,0,H)$, it is convenient to use the Landau gauge $\mathbf{A} = (0,Hx,0)$. The Hamiltonian of an electron (charge e, mass m) is then

$$H_0 = (1/2m)[p_x^2 + (p_y + m\omega_0 x)^2 + p_z^2], \tag{2.1}$$

where $\omega_0 = |e|H/mc$ is the cyclotron frequency. This particular gauge has the advantage that the canonical momenta p_y and p_z commute with H_0 and, therefore, the energy eigenfunctions can be taken to be states of definite momenta in the y and z directions. Such a choice defines the "Landau representation." The basis functions are then characterized by the oscillator quantum number $n = 0, 1, 2, \cdots$ and the two-dimensional wave vector $k(k_y,k_z)$ as follows:

$$\psi_{nk} = \phi_n(x-X)e^{ik_y y}e^{ik_z z}(L_y L_z)^{-\frac{1}{2}}, \tag{2.2}$$

where $\phi_n(x-X)$ are the harmonic oscillator wave functions, and $X = -\hbar k_y/m\omega_0$ is the center of the cyclotron orbit corresponding to this particular state (nk_yk_z). The energy eigenvalues are

$$\epsilon_{nk} = \hbar\omega_{nk} = (n+\tfrac{1}{2})\hbar\omega_0 + \hbar^2 k_z^2/2m. \tag{2.3}$$

We shall occasionally indicate a set of quantum numbers, e.g., (nk_yk_z), by a single Greek letter, e.g., ν. X_ν will then stand for $(-\hbar k_y/m\omega_0)$. Also $\nu\pm1$ will indicate the state $(n\pm1, k_yk_z)$ and (ν) the quantum number n.

In this representation the velocity components are given by the matrices

$$(n'k'|v_x+iv_y|nk) = 2i(\hbar\omega_0/2m)^{\frac{1}{2}}(n+1)^{\frac{1}{2}}\delta_{n',n+1}\delta_{k'k}, \tag{2.4}$$

$$(n'k'|v_z|nk) = (\hbar k_z/m)\delta_{n'n}\delta_{k'k}. \tag{2.5}$$

The current density $\mathbf{J}$ is given in general by the trace of the velocity operator and the steady-state density operator ρ_T. Because of Eqs. (2.4) and (2.5),

$$\begin{aligned} J_x + iJ_y &= (e/\Omega)\,\mathrm{Tr}\{\rho_T(v_x+iv_y)\} \\ &= e2i(\hbar\omega_0/2m)^{\frac{1}{2}}(2/\Omega) \\ &\qquad \times \sum_{nk}(n+1)^{\frac{1}{2}}(\rho_T)_{nk,n+1k}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} J_z &= (e/\Omega)\,\mathrm{Tr}\{\rho_T v_z\} \\ &= (e\hbar/m)(2/\Omega)\sum_{nk} k_z(\rho_T)_{nk,nk}, \end{aligned} \tag{2.7}$$

[6] Kubo, Hasegawa, and Hashitsume, J. Phys. Soc. Japan **14**, 56 (1959).
[7] E. N. Adams and T. D. Holstein, J. Phys. Chem. Solids **10**, 254 (1959).
[8] P. N. Argyres and L. M. Roth, J. Phys. Chem. Solids (to be published).
[9] W. Kohn and J. M. Luttinger, Phys. Rev. **108**, 590 (1957).

with the following normalization:

$$(1/\Omega)\ \mathrm{Tr}\{\rho_T\}=N=\text{electron concentration}. \tag{2.8}$$

Here Ω denotes the volume of the specimen.

The determination of the steady-state density matrix $(\rho_T)_{nk,n'k}$ is the task of the transport theory. An assembly of dynamically independent electrons is described by the one-electron density operator $\rho_T(t)$ that satisfies the equation of motion

$$i\hbar d\rho_T/dt=[H_0+V+F,\rho_T(t)]. \tag{2.9}$$

F denotes the interaction with the electric field $\mathbf{E}$, i.e.,

$$F=-e\mathbf{E}\cdot\mathbf{r}, \tag{2.10}$$

and V is the "scattering" interaction, which we take to be due to a set of identical impurities randomly distributed in space, i.e.,

$$V(\mathbf{r})=\sum_i v(\mathbf{r}-\mathbf{r}_i). \tag{2.11}$$

Before we turn on the electric field (at $t=0$) we assume the ensemble to be in thermal equilibrium, i.e.,

$$\rho_T(0)=f(H_0+V), \tag{2.12}$$

where $f(x)$ is a Boltzmann or a Fermi-Dirac function. The correlations brought about by the exclusion principle are rigorously taken into account in this manner for elastic collisions, as it was proved by Kohn and Luttinger.[9] It is easily seen then that at any later time

$$\rho_T(t)=f(H_0+V)+\rho(t), \tag{2.13}$$

where to the first order in the electric field

$$i\hbar d\rho(t)/dt=[H_0+V,\rho(t)]+[F,f(H_0+V)];\quad \rho(0)=0. \tag{2.14}$$

Following Kohn and Luttinger,[9] we introduce the Laplace transform of $\rho(t)$, namely

$$g(s)=s\int_0^\infty e^{-st}\rho(t)dt. \tag{2.15}$$

The inverse transform gives $\rho(t)$ in terms of $g(s)$, i.e.,

$$\rho(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}\frac{g(s)}{s}ds. \tag{2.16}$$

Letting $C=[f(H_0+V),F]$, Eq. (2.14) gives for $g(s)$

$$[H_0,g(s)]-i\hbar sg(s)=C+[g(s),V]. \tag{2.17}$$

In the Landau representation we have in general

$$(\epsilon_{\nu\nu'}-i\hbar s)g_{\nu\nu'}(s)=C_{\nu\nu'}+\sum_\mu[g_{\nu\mu}(s)V_{\mu\nu'}-V_{\nu\mu}g_{\mu\nu'}(s)], \tag{2.18}$$

where $\epsilon_{\nu\nu'}=\epsilon_\nu-\epsilon_{\nu'}$.

Up to this point everything is quite general. We shall now attempt to find a solution of this equation assuming that the strength of the scattering interaction V is small. We shall carry this out only in the lowest approximation.

To begin with, we observe that the known operator C can be expanded in powers of V as follows:

$$C=\sum_i C^{(i)}=\sum_i[f^{(i)},F], \tag{2.19}$$

where

$$f_{\mu\nu}{}^{(0)}=f(H_0)_{\mu\nu}=f_\mu\delta_{\mu\nu}, \tag{2.20}$$

$$f_{\mu\nu}{}^{(1)}=(f_{\mu\nu}/\epsilon_{\mu\nu})V_{\mu\nu}, \tag{2.21}$$

$$f_{\mu\nu}{}^{(2)}=\frac{1}{\epsilon_{\mu\nu}}\sum_\lambda\left(\frac{f_{\mu\lambda}}{\epsilon_{\mu\lambda}}-\frac{f_{\lambda\nu}}{\epsilon_{\lambda\nu}}\right)V_{\mu\lambda}V_{\lambda\nu}. \tag{2.22}$$

In these expressions we have put $f(\epsilon_\mu)=f_\mu$, $f_\mu-f_\nu=f_{\mu\nu}$. The terms with vanishing denominators have the limiting values obtained by letting these denominators approach zero smoothly, e.g., $f_{\mu\nu}/\epsilon_{\mu\nu}\rightarrow df(\epsilon_\mu)/d\epsilon_\mu=f'(\epsilon_\mu)$ for $\epsilon_\nu=\epsilon_\mu$.

The matrix elements of the scattering interaction V, Eq. (2.11), in the Landau representation present an irregular behavior, because of the random spatial distribution of the individual scattering centers. For such an arrangement the matrix element $V_{nk,n'k'}$ is a rapidly varying function of k' with a mean value equal to the diagonal matrix element of V. Its average value over an ensemble where all possible distributions of the scattering centers are equally probable is

$$\langle V_{nk,n'k'}\rangle=N_I w(\mathbf{O})\delta_{nn'}\delta_{kk'},$$

where N_I=concentration of scattering centers and $w(\mathbf{q})=\int v(\mathbf{r})\exp(-i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}$ is the Fourier transform of the scattering interaction of each center. Without loss of generality, we can take the diagonal element of V to be zero. If it is not, it can be absorbed into H_0. It just gives the shifting of the unperturbed energy levels to the first order in V. Since it is independent of the particular state in this case, it corresponds to a trivial uniform shifting of all levels. The product $V_{nk,n'k'}V_{mk',m'k''}$, however, as a function of k'' oscillates about its nonvanishing value for $k''=k$. For a very large system and a completely random arrangement it is equal to its ensemble average

$$\langle V_{nk,n'k'}V_{mk',m'k''}\rangle=\delta_{kk''}(N_I/\Omega)\sum_{q_x}|w(q_x,k-k')|^2\times J_{nk,n'k'}(q_x)J_{mk',m'k}{}^*(q_x), \tag{2.23}$$

where

$$J_{\mu\mu'}(q_x)=\int_{-\infty}^{+\infty}e^{iq_xx}\phi_\mu(x-X_\mu)\phi_{\mu'}(x-X_{\mu'})dx. \tag{2.24}$$

Similar behavior is exhibited by higher products. Thus, in this representation there is a sharp distinction between the diagonal-in-k and off-diagonal-in-k matrix elements of V, V^2, etc., which is due to the randomness of the distribution of the impurities. This singular property has been made the basis of an extensive study of the problem of approach to equilibrium by Van Hove.[10]

[10] L. Van Hove, Physica **21**, 517 (1955); **23**, 441 (1957).

This suggests that we treat the off-diagonal-in-k and diagonal-in-k matrix elements of the density operator differently. We shall see that by assuming that certain diagonal-in-k density matrix elements are larger than the others by an order of magnitude in V, we can get solutions to the general Eq. (2.18) in some form of a power series in the strength of V, which are consistent with this assumption. What makes this possible is the singular property of V discussed above.

Since the cases of transverse ($\mathbf{E}\perp\mathbf{H}$) and longitudinal ($\mathbf{E}\|\mathbf{H}$) electric fields turn out to require different initial assumptions, we shall consider them separately.

A. Transverse Electric Field

It is convenient to take the transverse electric field in the x direction, $\mathbf{E}=(E_x,0,0)$. For then $F=-eE_x x$ and the matrix elements of x in the Landau representation are regular. There is, of course, no fundamental difficulty with an electric field in the y direction. The calculation of certain commutators needs only some care.

We note, to begin with, that for this case we have, according to (2.19) and (2.20) and the easily found matrix elements of x,

$$C_{\nu\nu'}{}^{(0)}=f_{nn'}F_{nn'}\delta_{kk'}=(-eE_x)(\hbar/2m\omega_0)^{\frac{1}{2}}f_{nn'}\times[(n+1)^{\frac{1}{2}}\delta_{n',n+1}+n^{\frac{1}{2}}\delta_{n',n-1}]\delta_{kk'}. \quad (2.25)$$

We note that, although $x_{nn}=X_{nk}=-\hbar k_y/m\omega_0\neq 0$, $C_{\nu\nu'}{}^{(0)}$ has no diagonal matrix elements, since $f_{nn}=0$. Only the diagonal-in-k matrix elements for which $n'=n\pm 1$ do not vanish.

Thus, in Eq. (2.18) only the matrix elements for which $\nu'=\nu\pm 1$ are "driven" in the lowest order in V. We shall now assume that the matrix elements $g_{\nu\nu'}(s)$ for which $\nu'=\nu\pm 1$ are larger than the others by an order of magnitude in V. We can then find from Eq. (2.18) the "small" matrix elements of $g(s)$ in terms of the "large" ones, $g_{\nu,\nu\pm 1}(s)$, by iteration. We have for $\nu'\neq\nu\pm 1$ the "small" matrix elements in the lowest approximation

$$g_{\nu\nu'}=\frac{C_{\nu\nu'}{}^{(1)}+\sum_\mu(G_{\nu\mu}V_{\mu\nu'}-V_{\nu\mu}G_{\mu\nu'})}{\epsilon_{\nu\nu'}-i\hbar s}, \quad (2.26)$$

where, for economy, we have indicated the "large" matrix by $G_{\nu\nu'}$, i.e.,

$$G_{\nu\nu'}=g_{\nu\nu'}(\delta_{n',n+1}+\delta_{n',n-1})\delta_{k'k}. \quad (2.27)$$

In Eq. (2.26) we have neglected the sum over the "small" matrix elements of $g(s)$ in the commutator $[g(s),V]$, as it is of smaller order of magnitude in V. It is, however, important to realize that this proves to be an acceptable procedure, because the dependence of this sum on the volume of the specimen turns out to be the same as that of the sum of the fewer "large" matrix elements we have kept in (2.26). This is so because of the singular properties of the scattering interaction V. For a more detailed account of the nature of the approximation involved here the reader is referred to Appendix C.

Substituting the "small" elements (2.26) into Eq. (2.18) for $\nu'=\nu\pm 1$, somewhat rearranged, we obtain the following equation for the "large" matrix elements in the lowest approximation (dropping the common k):

$$G_{nn'}(s)=-i\omega_{nn'}\frac{G_{nn'}(s)}{s}+\left(\frac{i}{\hbar}\right)(C_{nn'}{}^{(0)}+C_{nn'}{}^{(2)})\frac{1}{s}+\hbar^{-2}\sum_\mu\left[V_{n\mu}C_{\mu n'}{}^{(1)}\frac{1}{s(s+i\omega_{\mu n'})}-C_{n\mu}{}^{(1)}V_{\mu n'}\frac{1}{s(s+i\omega_{n\mu})}\right]$$
$$+\hbar^{-2}\sum_\nu\sum_\mu\left[V_{n\nu}V_{\mu n'}\frac{G_{\nu\mu}(s)}{s(s+i\omega_{n\mu})}+V_{n\mu}V_{\nu n'}\frac{G_{\mu\nu}(s)}{s(s+i\omega_{\mu n'})}-V_{\nu\mu}V_{\mu n'}\frac{G_{n\nu}(s)}{s(s+i\omega_{n\mu})}-V_{n\mu}V_{\mu\nu}\frac{G_{\nu n'}(s)}{s(s+i\omega_{\mu n'})}\right]. \quad (2.28)$$

Here we have not written down the terms proportional to V, as their ensemble average is zero. We have also included in the sums a few terms [e.g., the terms $\mu=(n'\pm 1,k)$ in the first term of the first sum] which do not belong there. Their contribution, however, is vanishing small for a large specimen compared to the rest of the sum. Finally, although not explicitly indicated, all products $V_{n\mu}V_{\mu' n'}$ in (2.28) and in what follows should be replaced by their ensemble averages, given by Eq. (2.23). This is justified in Appendix D.

With the help of the inverse Laplace transform, Eq. (2.16), we can find, from the previous equation for $G_{\nu\nu'}(s)$, the corresponding one for the density matrix $R_{\nu\nu'}(t)$, where

$$R_{\nu\nu'}=\rho_{\nu\nu'}(\delta_{n',n+1}+\delta_{n',n-1})\delta_{k'k}. \quad (2.29)$$

It will be noticed from Eq. (2.6) that this is exactly the matrix necessary for the calculation of the transverse current density. Since $\rho(0)=0$, we find for $n'=n\pm 1$ and $t>0$

$$dR_{nn'}(t)/dt$$
$$=-i\omega_{nn'}R_{nn'}(t)+(i/\hbar)(C_{nn'}{}^{(0)}+C_{nn'}{}^{(2)}+B_{nn'})$$
$$+\hbar^{-2}\sum_\nu\sum_\mu\left[V_{n\nu}V_{\mu n'}\int_0^t R_{\nu\mu}(t-\tau)e^{-i\omega_{n\mu}\tau}d\tau\right.$$
$$+V_{n\mu}V_{\nu n'}\int_0^t R_{\mu\nu}(t-\tau)e^{-i\omega_{\mu n'}\tau}d\tau$$
$$-V_{\nu\mu}V_{\mu n'}\int_0^t R_{n\nu}(t-\tau)e^{-i\omega_{n\mu}\tau}d\tau$$
$$\left.-V_{n\mu}V_{\mu\nu}\int_0^t R_{\nu n'}(t-\tau)e^{-i\omega_{n\mu'}\tau}d\tau\right]. \quad (2.30)$$

Here we have introduced the notation

$$B_{nn'}=\sum_\mu [C_{n\mu}{}^{(1)}V_{\mu n'}\theta(\epsilon_{n\mu})-V_{n\mu}C_{\mu n'}{}^{(1)}\theta(\epsilon_{\mu n'})], \quad (2.31)$$

$$\theta(\epsilon)=(i/\hbar)\int_0^t \exp(-i\epsilon\tau/\hbar)d\tau \underset{t\to\infty}{\longrightarrow} i\pi\delta(\epsilon)+P\left(\frac{1}{\epsilon}\right). \quad (2.32)$$

This is a transport equation that describes the approach of the density matrix $R_{nn'}(t)$ to the steady state. The first term on the right-hand side gives the rate of change due to the unperturbed motion of the system. The next three inhomogeneous terms describe the various effects of the electric field. The last four are recognized as describing the effects of the interaction with the impurities alone. They have the feature that they make the rate of change of the density matrix at a certain instant depend not on the instantaneous value of the density matrix, but on all its previous values up to that instant. We can, however, simplify these terms as follows. We expand formally $R(t-\tau)$ in powers of τ,

$$R(t-\tau)=R(t)-\tau dR(t)/dt+\tfrac{1}{2}\tau^2d^2R(t)/dt^2-\cdots. \quad (2.33)$$

Now, for, say, the first of the collision terms we have from Eq. (2.30)

$$dR(t)/dt=-i\omega_{\nu\mu}\left[R_{\nu\mu}(t)-\frac{C_{\nu\mu}{}^{(0)}}{\epsilon_{\nu\mu}}\right]+\cdots. \quad (2.34)$$

The neglected terms above are of orders V^2 and V^2R and, therefore, can be dropped, as they would give rise to terms in the transport equation of higher order in V compared to the other terms that have been kept. By successive differentiation and iteration of Eq. (2.34) all the terms of Eq. (2.33) are obtained. Upon substitution in Eq. (2.33) we get an infinite series for $R(t-\tau)$, which can be summed, however, to give in this approximation

$$R_{\nu\mu}(t-\tau)=R_{\nu\mu}(t)e^{i\omega_{\nu\mu}\tau}+\frac{C_{\nu\mu}{}^{(0)}}{\epsilon_{\nu\mu}}(1-e^{i\omega_{\nu\mu}\tau})+\cdots. \quad (2.35)$$

Using this expression in the transport equation (2.30) we can carry out the integrations over τ in terms of the function $\theta(\epsilon)$, Eq. (2.32), and thus find in the lowest approximation a transport equation for the desired density matrix elements ($n'=n\pm1$), which can be written as

$$dR_{nn'}/dt=(i/\hbar)[R,H_0]_{nn'}+(i/\hbar)C_{nn'}{}^{(0)} +(i/\hbar)(C_{nn'}{}^{(2)}+B_{nn'}+A_{nn'})+(S[R])_{nn'}, \quad (2.36)$$

where

$$A_{nn'}=\sum_{\mu\nu}\left[V_{n\mu}C_{\mu\nu}{}^{(0)}V_{\nu n'}\frac{\theta(\epsilon_{n\mu})-\theta(\epsilon_{n\nu})}{\epsilon_{\mu\nu}}+V_{n\nu}C_{\nu\mu}{}^{(0)}V_{\mu n'}\frac{\theta(\epsilon_{\mu n'})-\theta(\epsilon_{\nu n'})}{\epsilon_{\nu\mu}} -C_{n\mu}{}^{(0)}V_{\mu\nu}V_{\nu n'}\frac{\theta(\epsilon_{\mu\nu})-\theta(\epsilon_{n\nu})}{\epsilon_{n\mu}}-V_{n\nu}V_{\nu\mu}C_{\mu n'}{}^{(0)}\frac{\theta(\epsilon_{\nu\mu})-\theta(\epsilon_{\nu n'})}{\epsilon_{\mu n'}}\right] \quad (2.37)$$

$$(S[R])_{nn'}=(i/\hbar)\sum_{\mu\nu}[V_{n\mu}V_{\mu\nu}R_{\nu n'}\theta(\epsilon_{\mu\nu})+R_{n\nu}V_{\nu\mu}V_{\mu n'}\theta(\epsilon_{\nu\mu})-V_{n\mu}R_{\mu\nu}V_{\nu n'}\theta(\epsilon_{\nu n'})-V_{n\nu}R_{\nu\mu}V_{\mu n'}\theta(\epsilon_{n\nu})]. \quad (2.38)$$

$B_{nn'}$ is given by Eq. (2.31) and $C_{nn'}{}^{(2)}$ through Eqs. (2.22) and (2.19).

The transport equation (2.36) constitutes the main result of this section. $[R,H_0]$ and $C^{(0)}$ give the rates of change due to the "unperturbed" motion of the system and the action of the electric field alone, respectively. $C^{(2)}$, B, and A describe the interference effects between the electric field and the interaction with the impurities. As we shall see below, these are particularly important in the case of high magnetic fields. The effects of the interaction with the impurities alone are described by the operator S, given by (2.38).

The effects of the scattering interaction are given in (2.36) in the lowest order. Higher approximations are obtained by iterating (2.18) for the "small" matrix elements with the help of their lowest approximation (2.26). These in turn are used in (2.18) for the "large" matrix elements, giving rise, in the same manner as before, to a transport equation for the density matrix $R_{nn'}$ analogous to (2.36). The difference is that the effects of the scattering interaction, in both the scattering operator S and the operators C, B, A, that describe its interference with the electric field, will now be given in a power series in V, the lowest terms of which are those given above. Among these higher order terms are the ones that describe the broadening of the Landau levels due to the interaction with the impurities. These terms are quite important for the details (amplitude, phase) of the oscillations of the resistivity with the magnetic field and other points (see next section). We shall not examine these higher order terms of the transport equation here, thus restricting ourselves to the "Born approximation" for the various effects of the scattering interaction. (For higher approximations and a more detailed discussion of this point see Appendix C.)

For long times the real part of $\theta(\epsilon)$, according to (2.32), becomes equal to $P(1/\epsilon)$, the "principal part" of $(1/\epsilon)$. These terms can be interpreted generally as representing the shifting of the Landau levels due to the interaction with the impurities in the second order. Although they could be significant in some special

cases, they are not of an essential nature and we shall, for simplicity, neglect them in the following.

We can then rewrite the scattering operator S more simply. From Eqs. (2.38) and (2.32) we have for long times

$$(S[R])_{nn'}=(\pi/\hbar)\sum_{\mu\nu}\{V_{n\mu}R_{\mu\nu}V_{\nu n'}[\delta(\epsilon_{n\mu})+\delta(\epsilon_{\nu n'})]-(V_{n\mu}V_{\mu\nu}R_{\nu n'}+R_{n\mu}V_{\mu\nu}V_{\nu n'})\delta(\epsilon_{\mu\nu})\}. \quad (2.39)$$

The structure of this scattering operator is quite analogous to the corresponding one of the usual Boltzmann equation for zero magnetic field. The transition probabilities, that give there the net rate of increase of the occupation probabilities due to the collisions, are replaced here by matrices of the general form $(2\pi/\hbar)V_{n\mu}V_{\nu n'}\delta(\epsilon_{n\mu})$ that give in turn the rate of increase of the density matrix $R_{nn'}$ due to scattering in the Born approximation.

A similar simplification occurs for the driving terms of the transport equation that describe the effect of the electric field on the collisions, namely, $(i/\hbar)(C^{(2)}+B+A)$, which we shall denote collectively by D. From Eqs. (2.25) and (2.37) one finds $A_{nn'}$. Similarly, (2.19) and (2.21) give $C_{nn'}{}^{(1)}$. We note that the diagonal elements of x contribute to $C_{nn'}{}^{(1)}$, in contrast to the case of $C_{nn'}{}^{(0)}$. With the help of (2.31), $B_{nn'}$ can then be determined. We thus find for long times

$$D_{nn'}=-(\pi/\hbar)\sum_{\mu\nu}\left[V_{n\mu}F_{\mu\nu}\frac{f_{\mu\nu}}{\epsilon_{\mu\nu}}V_{\nu n'}[\delta(\epsilon_{n\mu})+\delta(\epsilon_{\nu n'})]-\left(V_{n\mu}V_{\mu\nu}F_{\nu n'}\frac{f_{\nu n'}}{\epsilon_{\nu n'}}+F_{n\mu}\frac{f_{n\mu}}{\epsilon_{n\mu}}V_{\mu\nu}V_{\nu n'}\right)\delta(\epsilon_{\mu\nu})\right]. \quad (2.40)$$

For the diagonal matrix elements of F it is essential to recall that $f_{n\mu}/\epsilon_{n\mu}\rightarrow f'(\epsilon_n)$ for $\epsilon_\mu=\epsilon_n$.

Upon examination of expressions (2.39) and (2.40), it is obvious that the driving term $D_{nn'}$ can be written in terms of the scattering operator S as follows:

$$D_{nn'}=-(S[f(H_0+F)])_{nn'}, \quad (2.41)$$

where $f(H_0+F)$ is to be taken to the first order in the electric field. This is so, because $F_{\mu\nu}(f_{\mu\nu}/\epsilon_{\mu\nu})=[f(H_0+F)-f(H_0)]_{\mu\nu}$ to terms linear in F and $S[f(H_0)]=0$, as it can easily be verified. Thus, one may describe the effect of the transverse electric field on the scattering by saying that the electrons "relax" through collisions to a distribution appropriate to a state of thermal equilibrium in the presence of the electric field.

Therefore, for the transverse case the transport equation for the "large" matrix elements of the density operator can be written, to the first order in the electric field and for long times,

$$dR_{nn'}/dt=(i/\hbar)[R,H_0]_{nn'}+(i/\hbar)[f(H_0),F]_{nn'}+(S[R-f(H_0+F)])_{nn'}, \quad (2.42)$$

where S is given through Eq. (2.39). For the steady state we have, of course, $dR/dt=0$. Clearly $n'=n\pm1$.

It is important to observe that the transport equation (2.42) has a solution $R_{nk,n'k}$ that is independent of k_y, i.e., representing a uniform distribution of electrons in space. This, of course, is just the solution of interest in calculating the usual conductivity. This can be seen as follows. According to (2.23) and (2.24), the product $V_{n\mu}V_{\mu'n'}$, that comes into the expressions for the scattering operator S and the driving term D, is easily seen to depend on k_y and k_y' only through the combination $(k_y'-k_y)$, since the range of integration in (2.24) is infinite. For the diagonal elements of F, the additional dependence of D on k_y and k_y' is again of the same form, as only $F_{\mu\mu}-F_{nn}\propto(k_y'-k_y)$ comes in. With the ansatz for a k_y-independent solution $R_{nn'}$, we may sum over k_y'. This eliminates any dependence on k_y of the operators S and D, since both are functions of $(k_y'-k_y)$ and the range of integration of k_y' is infinite. The other terms of the transport equation are clearly k_y-independent. It is thus possible to obtain a solution $R_{nn'}$ that does not depend on k_y.

It is also of interest to point out that the predicted conductivity tensor σ_{ij} satisfies the reciprocity relations which have been proven[11] to follow from a generalization of Onsager's relations in irreversible processes. In their general form these are

$$\sigma_{ij}(\mathbf{H})=\sigma_{ji}(-\mathbf{H}), \quad (2.43)$$

i.e., the symmetric part of the tensor is an even function of the magnetic field, whereas the antisymmetric part is an odd function of $\mathbf{H}$. Clearly we cannot establish these relations for our system in this general form, as we can only find σ_{xx}, σ_{yx} in the coordinate system we have chosen. If, however, the scattering interaction is symmetric enough, e.g., if $v(x,-y,z)=v(x,y,z)$, the calculated σ_{yx} will be then an element of the antisymmetric part of the conductivity tensor, and as such, according to (2.43), it must satisfy the relation $\sigma_{yx}(-\mathbf{H})=-\sigma_{yx}(\mathbf{H})$. On the other hand $\sigma_{xx}(-\mathbf{H})=\sigma_{xx}(\mathbf{H})$. These somewhat restricted Onsager's relations can be established for our system on the basis of this formalism as follows. Upon reversal of the magnetic field, i.e., for $\mathbf{H}=(0,0,-H)$, the new Hamiltonian $\bar{H}_0$ is obtained from H_0, Eq. (2.1), by changing the sign of $m\omega_0x$. The new eigenfunctions, which will be denoted by $\bar{\mu}$, differ from the old ones only in the sign of $X=-\hbar k_y/m\omega_0$ in the argument of ϕ_n. This is clearly so, as a reversal of $\mathbf{H}$ is equivalent to a change of sign of the electric charge of the carrier. It is easy to see, by a mere change of variables in (2.23), that $\sum_{k_y'}V_{\bar{n}\bar{\mu}}V_{\bar{\mu}'\bar{n}'}$ is equal to $\sum_{k_y'}V_{n\mu}V_{\mu'n'}$ because of the symmetry of $v(\mathbf{r})$. Thus $\bar{S}=S$, i.e., the scattering operator S is an even function of $\mathbf{H}$. The same holds true for the part of the operator D that comes from the off-diagonal matrix elements of F, as $x_{nn'}(n\neq n')$ do not depend on the sign of $\mathbf{H}$ and are k_y-independent.

[11] P. Mazur and S. R. DeGroot, Phys. Rev. 94, 224 (1954).

The diagonal matrix elements of F do change sign upon reversal of $\mathbf{H}$ and contribute to operator $\bar{D}$ a term $L\sum_{k_y'}(k_y-k_y')V_{\bar{n}\bar{\mu}}V_{\bar{\mu}\bar{n}'}$. This is again easily proven to be equal to the corresponding part of D that comes from the diagonal elements of F, namely $L\sum_{k_y'}(k_y'-k_y)\times V_{n\mu}V_{\mu n'}$. That is, D is also an even function of $\mathbf{H}$. Clearly, since only the off-diagonal elements of F contribute to $C_{nn'}{}^{(0)}$, this term of the transport equation is an even function of $\mathbf{H}$ too. Thus, the density matrix $R_{nn'}$ as determined from Eq. (2.42) does not change sign upon reversal of $\mathbf{H}$. Now it is easily verifiable that whereas the matrix elements of $v_x=\bar{v}_x=p_x/m$ are even functions of $\mathbf{H}$, the elements of $\bar{v}_y=(p_y-m\omega_0x)/m$ in the $\bar{\mu}$ basis are opposite to those of $v_y=(p_y+m\omega_0x)/m$ in the μ basis. From the relation $\mathbf{J}\sim\mathrm{Tr}\{\rho\mathbf{v}\}$, we thus have the desired relations.

There is a large class of scattering interactions $v(\mathbf{r})$ for which the scattering operator S simplifies somewhat. It is shown in Appendix A that interactions for which $|w(\mathbf{q})|^2$ can be expanded in powers of $(q_x^2+q_y^2)$ have the following property:

$$\sum_{k_y'} V_{nk,mk'}V_{m'k',n'k}\neq 0 \quad \text{only if} \quad m'-m=n'-n. \tag{2.44}$$

For the spatially uniform current distributions we have then

$$\begin{aligned}(S[\rho])_{n,n\pm1} &= (2\pi/\hbar)\textstyle\sum_\mu\{\rho_{\mu,\mu\pm1}V_{n\mu}V_{\mu\pm1,n\pm1}\delta(\epsilon_{\mu n})\\ &\quad-\rho_{n,n\pm1}[\tfrac{1}{2}|V_{n\mu}|^2\delta(\epsilon_{n\mu})\\ &\qquad+\tfrac{1}{2}|V_{n\pm1,\mu}|^2\delta(\epsilon_{n\pm1,\mu})]\}.\end{aligned} \tag{2.45}$$

This is identical to the collision operator of reference 4. It was established there by a time-dependent perturbation method for large magnetic field, i.e., $\omega_0\tau\gg1$. The selection rule (2.44) makes this restriction unnecessary, however. A similar simplification occurs for the operator D.

The determination of the steady-state density matrix from the transport equation is not a simple matter. The difficulty stems from the complicated nature of the scattering operator S, which connects a particular matrix element $R_{nn'}$ in general to all other elements. A systematic way, however, of solving this system of equations consists in starting from the lowest Landau state, $n=0$, and proceeding to higher ones. The case of the "quantum limit," i.e., when all carriers occupy the state associated with the quantum number $n=0$, is easily tractable. We shall not exhibit this here, however.

B. Longitudinal Electric Field

When the electric field is in the z direction, $\mathbf{E}=(0,0,E_z)$, we have $F=-eE_zz$. The driving term representing the acceleration due to the electric field alone is now

$$\begin{aligned}C_{\nu\nu'}{}^{(0)} &= -eE_z[f(H_0),z]_{\nu\nu'}\\ &= -eE_z(\hbar/i)(\hbar k_z/m)f'(\epsilon_\nu)\delta_{\nu\nu'}.\end{aligned} \tag{2.46}$$

Thus, only the diagonal matrix elements ρ_{nn} are "driven" in this case, and they are, according to (2.7), the ones that give rise to the longitudinal current.

We shall now develop a transport equation for the diagonal density matrix elements $\rho_{\nu\nu}\equiv\rho_\nu$. In contrast to the transverse case, we assume now that the diagonal matrix elements ρ_ν are larger than the others. They will turn out to be of order V^{-2}. Proceeding as in the previous case, we find from Eq. (2.18) for the "small" matrix elements $(\nu'\neq\nu)$ in the lowest approximation,

$$g_{\nu\nu'}(s)=\frac{(g_\nu-g_{\nu'})V_{\nu\nu'}}{\epsilon_{\nu\nu'}-i\hbar s}. \tag{2.47}$$

Substituting this into Eq. (2.18) for the diagonal elements, we find

$$g_n(s)=(i/\hbar)C_{nn}{}^{(0)}\frac{1}{s}+\hbar^{-2}\sum_\mu 2|V_{n\mu}|^2\frac{g_\mu(s)-g_n(s)}{s^2+\omega_{n\mu}{}^2}. \tag{2.48}$$

In terms of the density matrix, this equation gives, with the help of (2.16), the following transport equation for the diagonal matrix elements:

$$\begin{aligned}d\rho_n(t)/dt &= (i/\hbar)C_{nn}{}^{(0)}+\hbar^{-2}\sum_\mu 2|V_{n\mu}|^2\\ &\quad\times\int_0^t[\rho_\mu(t-\tau)-\rho_n(t-\tau)]\cos\omega_{n\mu}\tau d\tau.\end{aligned} \tag{2.49}$$

The collision terms can be simplified as before. Expanding $\rho(t-\tau)$ in powers of τ and neglecting terms of order $V^2\rho$ we find in the lowest approximation and for long times, using (2.32),

$$d\rho_n/dt=(i/\hbar)C_{nn}{}^{(0)}+\textstyle\sum_\mu[W_{\mu n}\rho_\mu(t)-W_{n\mu}\rho_n(t)], \tag{2.50}$$

where

$$W_{n\mu}=W_{\mu n}=(2\pi/\hbar)|V_{n\mu}|^2\delta(\epsilon_n-\epsilon_\mu) \tag{2.51}$$

is the Born approximation for the rate of transition probability from Landau state n to μ. The structure of the scattering operator here is quite transparent and identical to the one of reference 4. Clearly the terms describing the effect of the electric field on the collisions are of higher order in this case. For the steady-state $d\rho_n/dt=0$.

For some scattering interactions the collision operator simplifies considerably. For a solution of the form

$$\rho_n=k_z\chi(\epsilon_{nk}), \tag{2.52}$$

the scattering operator can be simply described by a momentum relaxation time, namely,

$$\textstyle\sum_\mu W_{\mu n}(\rho_\mu-\rho_n)=-\rho_n/\tau_{nk}, \tag{2.53}$$

where

$$\tau_{nk}{}^{-1}=\textstyle\sum_\mu W_{n\mu}(1-k_z'/k_z). \tag{2.54}$$

For the solution (2.52) to exist, it is sufficient that the relaxation time τ_{nk} as given by (2.54) turn out to depend on the state only through its energy ϵ_{nk}. The

solution of the transport equation is trivial in this case.

Such solutions have been considered in reference 4. We shall not discuss the longitudinal case here any further.

III. CURRENT DENSITY FOR LARGE HALL ANGLES

We shall be concerned here with the solution of the transport equation (2.42) and the calculation of the transverse current for the special case of large Hall angles. It is well known that classically in the absence of any scattering, a transverse electric field induces a steady current in a direction perpendicular to both **E** and **H**, which we shall denote as the y direction. The quantization of the electron orbits by the magnetic field does not affect this result, as we shall see presently. If a weak scattering is introduced into the model, a small component of the current develops in the direction of the electric field. The current then flows on the transverse plane in a direction that makes with the direction of the electric field an angle, the so-called Hall angle, slightly less than 90°. More quantitatively, "weak" scattering is understood here to mean $\omega_0\tau \gg 1$, where τ is a measure of the relaxation time. This region is fairly frequently encountered experimentally, and it is almost always the case for semiconductors in magnetic fields such that $\hbar\omega_0 \gtrsim kT$, where characteristic quantum effects can be observed.

This case can be treated simply on the basis of the transport equation (2.42). We note that for $V=0$ a steady-state density matrix exists and is

$$R_{nn'}{}^{(0)} = C_{nn'}{}^{(0)}/\epsilon_{nn'} = f_{nn'}F_{nn'}/\epsilon_{nn'}. \tag{3.1}$$

We can thus seek a solution of the transport equation in powers of V,

$$R = R^{(0)} + R^{(1)} + R^{(2)} + \cdots. \tag{3.2}$$

By iteration we obtain for the desired matrix elements

$$R_{nn'}{}^{(2)} = \frac{-i}{\omega_{nn'}}[D_{nn'} + (S[R^{(0)}])_{nn'}]. \tag{3.3}$$

From the structure of the operators D and S, Eqs. (2.39) and (2.40), and the expression (3.1) for $R^{(0)}$ it is clear that the terms in $D+S[R^{(0)}]$ arising from the off-diagonal matrix elements of F and $R^{(0)}$ cancel each other. The only nonvanishing terms are the contributions in $D_{nn'}$ of the diagonal elements of the matrix $F_{n\nu}f_{n\nu}/\epsilon_{n\nu}$, namely $f'(\epsilon_n)F_{nn} = -eE_x f'(\epsilon_n)X_n = eE_x(\hbar k_y/m\omega_0)f'(\epsilon_n)$. Thus

$$R_{nn'}{}^{(2)} = (-eE_x)\frac{i\pi}{\hbar\omega_0}\sum_\mu V_{n\mu}V_{\mu n'}(X_n - X_\mu)f'(\epsilon_\mu) \times[\delta(\epsilon_{n\mu}) + \delta(\epsilon_{\mu n'})]. \tag{3.4}$$

As we shall see below $R_{n,\,n+1}{}^{(2)}$ for symmetric enough scattering interactions contributes only to J_x. On the other hand $R_{n,\,n+1}{}^{(0)}$, according to (3.1) and (2.25), is real and gives rise only to the Hall current J_y. We thus find, using (2.8),

$$J_y = (-e^2E_x/m\omega_0)(2/\Omega)\sum_{nk}(n+1) \times[f(\epsilon_{n+1}) - f(\epsilon_n)] = -NecE_x/H, \tag{3.5}$$

$$J_x = (-e^2E_x)(2/\Omega)\sum_{\mu\nu}\delta(\epsilon_{\nu\mu}) \times f'(\epsilon_\nu)(X_\nu - X_\mu)(\pi/\hbar)(\hbar/m\omega_0)^{\frac{1}{2}} \times\{[2(\mu+1)]^{\frac{1}{2}}V_{\mu\nu}V_{\nu,\mu+1} + [2(\mu)]^{\frac{1}{2}}V_{\mu-1,\nu}V_{\nu\mu}\}. \tag{3.6}$$

Clearly Eq. (3.5) gives the unperturbed current in the y direction. The collisions do not affect it in the second order, and it is identical to its classical value. The quantization of the levels has no effect upon it. We are neglecting, as we mentioned above, the changes from the shifting of the energy levels due to the interaction with the impurities.

The component of the current in the direction of the electric field, J_x, is brought about by the collisions. The physical meaning of the various terms in (3.6) is not obvious in this form. This equation can, however, be transformed into a physically more meaningful expression with the help of the well-known properties of the Hermite polynomials. It is found[8] that if $v(-x,y,z) = v(x,y,z)$

$$J_x = (-e^2E_x)(1/\Omega)\sum_{\mu\nu} f'(\epsilon_\nu)(X_\nu - X_\mu)^2 W_{\mu\nu}, \tag{3.7}$$

where

$$W_{\mu\nu} = (2\pi/\hbar)|V_{\mu\nu}|^2\delta(\epsilon_\mu - \epsilon_\nu) \tag{3.8}$$

is the transition probability rate from state μ to ν in the Born approximation. Since X_μ denotes the center of the cyclotron orbit associated with the state μ, Eq. (3.7) describes the current in the direction of the electric field as due entirely to the net drift of the centers of the cyclotron orbits of the electrons. The detailed motion around these centers does not contribute anything to the net current.

Expression (3.7) is essentially the same that Titeica[1] and others[2] first assumed in their work. It has been recently derived by Kubo *et al.*,[6] Adams and Holstein,[7] and Argyres and Roth[8] by different methods. Adams and Holstein[7] have made extensive applications of Eq. (3.7) for different scattering mechanisms. The interested reader is referred to this work for detailed results on the phenomenon of oscillatory magnetoresistance and its dependence on the magnetic field in the "quantum limit." In references 6 and 8 it was also proved that for inelastic collisions

$$J_x = (e^2E_x/kT)(1/\Omega)\sum_{\mu\nu} f(\epsilon_\mu)(X_\mu - X_\nu)^2 W_{\mu\nu} \times[1 - f(\epsilon_\mu)], \tag{3.9}$$

where the transition probability is given by

$$W_{\mu\nu} = (2\pi/\hbar)\sum_{NN'} P(N)|V_{\mu N,\nu N'}|^2 \times\delta(\epsilon_\mu + E_N - \epsilon_\nu - E_{N'}). \tag{3.10}$$

Here, N and E_N stand for the stationary states and eigenenergies, respectively, of the scattering system and $P(N)$ is the probability for it to be in state N.

It was also pointed out in these last two references that the electronic motions around their cyclotron centers are uncorrelated, and therefore contribute nothing to the current, only for a static electric field. Thus the description of the irreversible current in terms of the drift of the cyclotron centers alone is not valid for oscillating electric fields. This can easily be seen in the framework of this calculation, as the denominators in the expression for $D_{nn'}$ will be shifted by the frequency of the oscillating field and will not cancel the corresponding terms of $(S[R^{(0)}])_{nn'}$.

It must finally be noticed that expression (3.7) for the irreversible current in the case of elastic collisions does not give in general convergent results. This is due primarily to the neglect of the broadening of the Landau states we mentioned earlier and the fact that there are packets of electrons with energies such that $\omega_0\tau(\epsilon)\lesssim 1$ (for an example, see next section). In practical applications, however, this divergence can be qualitatively circumvented, as Adams and Holstein[7] have done.

IV. CURRENT DENSITY FOR ISOTROPIC SCATTERING

It is interesting to consider some special scattering mechanisms for which the solution of the transport equation (2.42) can be obtained for arbitrary values of $\omega_0\tau$. One can thus see in a fairly simple way the transition from the classical case, i.e., when the quantum of the electronic energy in a magnetic field, $\hbar\omega_0$, is much smaller than the energy of the electron and kT, to the quantum-mechanical one, where the quantization of the orbits induces characteristic changes in the galvanomagnetic properties of the system. The classical limit can be treated independently, as is well known, on the basis of the usual Boltzmann equation for the distribution function. The magnetic field is treated there simply as an additional driving force. Its possible effect on the relaxation time is not considered, although the final results are taken to be valid for arbitrary values of $\omega_0\tau$. The neglect of the influence of the magnetic field on the scattering has not been justified, as far as we are aware. It will become clear from this quantum-mechanical calculation that in the classical limit such a procedure is correct for the scattering mechanisms considered here. It becomes also possible to study the first quantum-mechanical deviations from the classical result. It is, for example, conceivable that for some scattering mechanisms the quantization of the electron orbits might bring about, apart from the characteristic oscillatory behavior, a smooth dependence of the relaxation time on the magnetic field, much the same way as it induces a nonvanishing diamagnetism in an electron gas.

We consider now some particular scattering mechanisms. They have the common feature that in the absence of any magnetic field they scatter isotropically. Thus the Fourier transform of the scattering interaction, $w(\mathbf{q})$, is a constant independent of $\mathbf{q}$, e.g., if the interaction with a scattering center is infinitely localized, i.e., $v(\mathbf{r})=a\delta(\mathbf{r})$, $w(\mathbf{q})=a$. The needed matrix elements of the interaction with a random distribution of such centers are, according to (2.23),

$$\langle V_{nk,mk'}V_{m'k',n'k}\rangle = G(1/\Omega)\sum_{q_x} J_{nk,mk'}(q_x)J_{m'k',n'k}^*(q_x), \quad (4.1)$$

where $G=a^2N_I$. Another isotropic scattering mechanism is the interaction with acoustical phonons (in the usual Debye approximation and in temperature ranges such that the collisions are effectively elastic). It is easy to see[4] that in such a case expression (4.1) is valid for semiconductors and semimetals with $G=C^2kT/\mu s^2$, where C = electron-lattice interaction constant in the deformation potential approximation, μ = mass density, and s = velocity of sound.

We can now seek a solution of the transport equation for the desired matrix elements, namely

$$0=i\omega_0\rho_{n,n+1}+(i/\hbar)C_{n,n+1}^{(0)} + D_{n,n+1}+(S[R])_{n,n+1}, \quad (4.2)$$

such that $\rho_{n,n+1}$ are independent of k_y.

For such solutions and scattering interactions satisfying the condition (4.1) the scattering operator S, given by (2.39), can be described in terms of a scalar relaxation time. Since ϵ_{nk} is also independent of k_y, we have [see Appendix B, (B.7)]

$$\sum_{k_y'} V_{nk,mk'}V_{m'k',n'k}=G(1/2\pi L_z)(m\omega_0/\hbar)\delta_{nn'}\delta_{mm'}. \quad (4.3)$$

Thus, since in our case $n'=n+1$, the first two terms of the scattering operator vanish and the other two reduce simply to multiplicative constants. More explicitly,

$$(S[R])_{n,n+1}=-\rho_{n,n+1}/\tau_n, \quad (4.4)$$

where

$$\tau_n^{-1}=\tau_1^{-1}(n)+\tau_1^{-1}(n+1) = (\pi/\hbar)\sum_\mu [|V_{n\mu}|^2\delta(\epsilon_{\mu n}) + |V_{n+1,\mu}|^2\delta(\epsilon_{\mu,n+1})]. \quad (4.5)$$

Thus for isotropic scattering mechanisms, the decay of $\rho_{n,n+1}$ due to collisions is described by an effective transition probability rate equal to one half the probability rate of transition from state (nk) to all other states plus one-half the corresponding probability rate of transition from state $(n+1, k)$. This is quite analogous to the circumstance in zero magnetic field, where the scattering can be described by a relaxation time equal to the "broadening" relaxation time. With the help of (4.3) we can write

$$\tau_1^{-1}(\epsilon_{nk})=\tfrac{1}{4}K\hbar\omega_0\sum_{n'}[\epsilon_{nk}-(n'+\tfrac{1}{2})\hbar\omega_0]^{-\frac{1}{2}}, \quad (4.6)$$

where $K=G(2^{\frac{1}{2}}m^{\frac{3}{2}}/\pi\hbar^4)$. That is, $\tau_1^{-1}(\epsilon)$ is proportional to the density of states in the magnetic field. Thus the effect of the magnetic field on the collisions is described in this case in a direct fashion in terms of its effect on the energy spectrum. Since the density of states, as a function of the magnetic field, only "oscillates" about its value for vanishing field, there is no smooth dependence of the relaxation time on the magnetic field. According to (4.6) the relaxation time presents sharp discontinuities at energies $\epsilon=(n'+\frac{1}{2})\hbar\omega_0$, the collision rate becoming infinite for electrons with these energies. This is a result of the neglect of the collision broadening terms. For the details of the oscillations (amplitude and phase) of the current these are quite important.

The term $D_{n,n+1}$ is given in general by (2.40). The terms from the off-diagonal matrix elements of F can easily be evaluated with the help of (4.3). The diagonal elements of F give rise, however, to terms which, again with help of the selection rule (4.3), involve the quantities

$$\sum_{k_y'} k_y' V_{nk,mk'} V_{mk',n+1k} = -G(1/2\pi L_z)(m\omega_0/\hbar)^{\frac{3}{2}}2^{-\frac{1}{2}}(n+1)^{\frac{1}{2}}. \quad (4.7)$$

This last equality is proved in Appendix B, (B.5). We thus find that the term describing the interference between the electric field and the scattering can also be given in terms of the relaxation time τ_n, Eq. (4.5), as follows:

$$D_{n,n+1}=(eE_x\hbar)(2m\hbar\omega_0)^{-\frac{1}{2}}(n+1)^{\frac{1}{2}} \times\left[f_{n+1}'\tau_1^{-1}(n+1)+f_n'\tau_1^{-1}(n)-\frac{f_{n+1,n}}{\hbar\omega_0}\tau_n^{-1}\right]. \quad (4.8)$$

It is clear that in the classical limit, i.e., for high quantum numbers, $\epsilon_n\gg\hbar\omega_0$, and $kT\gg\hbar\omega_0$, this term vanishes. For higher $\hbar\omega_0$, however, it is appreciable and leads to important modifications in the final expression for the current density.

With the help of the expressions (4.4) and (4.8), the transport equation (4.2) becomes a simple algebraic equation. $C_{n,n+1}^{(0)}$ is given by (2.25). The sought solution is thus

$$\rho_{n,n+1}=(-eE_x\hbar)(2m\hbar\omega_0)^{-\frac{1}{2}}(n+1)^{\frac{1}{2}} \times\left[\frac{f_{n+1}-f_n}{\hbar\omega_0}+\frac{f_n'\tau_1^{-1}(n)+f_{n+1}'\tau_1^{-1}(n+1)}{i\omega_0-\tau_n^{-1}}\right], \quad (4.9)$$

which is indeed independent of k_y, in accordance with our ansatz.

The steady-state density matrix having been determined, the current density is obtained from Eq. (2.6). It is thus found that an electric field in the x direction induces a transverse current density with components

$$J_x=(-e^2E_x/m)(2/\Omega)\sum_{nk}\frac{(n+1)\hbar\omega_0[f_n'\tau_1^{-1}(n)+f_{n+1}'\tau_1^{-1}(n+1)]}{\omega_0^2+\tau_n^{-2}}, \quad (4.10)$$

$$J_y=(-e^2E_x/m)(1/\omega_0)(2/\Omega)\sum_{nk}(n+1)\hbar\omega_0\left[\frac{f_{n+1}-f_n}{\hbar\omega_0}-\frac{\tau_n^{-1}[f_n'\tau_1^{-1}(n)+f_{n+1}'\tau_1^{-1}(n+1)]}{\omega_0^2+\tau_n^{-2}}\right]. \quad (4.11)$$

It is proposed that these expressions are valid in the whole range from classical to quantum limits, subject, of course, to the basic approximations of the theory.

To prove that in the classical limit these equations reduce to the usual ones obtained from the Boltzmann equation, it is helpful to carry out the summation over k in terms of an integral over the total energy ϵ of the electron. It is found that

$$J_x=(-e^2E_x/m)\int_{(\hbar\omega_0/2)}^{\infty}\frac{\alpha(\epsilon)[f'(\epsilon)\tau_1^{-1}(\epsilon)+f'(\epsilon+\hbar\omega_0)\tau_1^{-1}(\epsilon+\hbar\omega_0)]d\epsilon}{\omega_0^2+\tau^{-2}(\epsilon)}, \quad (4.12)$$

$$J_y=(-e^2E_x/m)(1/\omega_0)\int_{(\hbar\omega_0/2)}^{\infty}\alpha(\epsilon)\left[\frac{f(\epsilon+\hbar\omega_0)-f(\epsilon)}{\hbar\omega_0}-\frac{\tau^{-1}(\epsilon)[f'(\epsilon)\tau_1^{-1}(\epsilon)+f'(\epsilon+\hbar\omega_0)\tau_1^{-1}(\epsilon+\hbar\omega_0)]}{\omega_0^2+\tau^{-2}(\epsilon)}\right]d\epsilon, \quad (4.13)$$

where

$$\alpha(\epsilon)=(2m/\hbar^2)^{\frac{3}{2}}(2\pi)^{-2}\hbar\omega_0\sum_n(n+1)\hbar\omega_0 \times[\epsilon-(n+\tfrac{1}{2})\hbar\omega_0]^{-\frac{1}{2}}. \quad (4.14)$$

In the definition of $\alpha(\epsilon)$ the summation goes over all non-negative integers for which the summand is positive.

In the classical limit, i.e., $\epsilon\gg\hbar\omega_0$ and $kT\gg\hbar\omega_0$, we have clearly $f'(\epsilon+\hbar\omega_0)\to f'(\epsilon)$, $[f(\epsilon+\hbar\omega_0)-f(\epsilon)]/\hbar\omega_0\to f'(\epsilon)$, $\tau_1^{-1}(\epsilon)+\tau_1^{-1}(\epsilon+\hbar\omega_0)=\tau^{-1}(\epsilon)\to\tau_0^{-1}(\epsilon)$, and $\alpha(\epsilon)\to\alpha_0(\epsilon)$, where

$$\tau_0^{-1}(\epsilon)=K\epsilon^{\frac{1}{2}},\quad \alpha_0(\epsilon)=(2\pi)^{-2}(2m/\hbar^2)^{\frac{3}{2}}(4/3)\epsilon^{\frac{3}{2}}. \quad (4.15)$$

$\tau_0(\epsilon)$ is just the relaxation time for zero magnetic field. The transverse current in the classical region is then

$$J_x=(-e^2E_x/m)\int_0^{\infty}\frac{\alpha_0(\epsilon)\tau_0^{-1}(\epsilon)f'(\epsilon)}{\omega_0^2+\tau_0^{-2}(\epsilon)}d\epsilon, \quad (4.16)$$

$$J_y=(-e^2E_x/m)\omega_0\int_0^\infty \frac{\alpha_0(\epsilon)f'(\epsilon)}{\omega_0^2+\tau_0^{-2}(\epsilon)}d\epsilon, \qquad (4.17)$$

which is identical to the result of the classical Boltzmann equation for the distribution function.

V. SUMMARY AND DISCUSSION

Using a method similar to that of Kohn and Luttinger,[9] we have developed a transport equation for the elements of the quantum-mechanical density operator that are necessary for the calculation of the current induced by an electric field in a gas of non-interacting electrons in the presence of a magnetic field and a set of randomly distributed impurities that scatter elastically. The effects of the quantization of the electron orbits are taken into account from the start, by working in the Landau representation. In the lowest Born approximation for the collisions, this transport equation is given by Eq. (2.36) for a transverse electric field and by Eq. (2.50) for a longitudinal one.

The problem of electrical conduction by an ensemble of "Bloch" electrons with a band energy spectrum has been discussed by Kohn and Luttinger[9] and it is in some ways similar to the problem under consideration here. The main difference lies in the fact that for the case of the "Bloch" electrons it has not been proved possible to find a single transport equation for all the density matrix elements necessary for the calculation of the current. Instead, only the diagonal elements satisfy such a transport equation and the remaining "inter-band" matrix elements are found from the diagonal ones. For the case of the "Landau" electrons, however, it has been shown above that all the density matrix elements required for the calculation of the induced current (separately for the transverse and longitudinal cases) obey a transport equation. This is a consequence of the nature of the matrix elements of the coordinate and velocity operators in the Landau representation.

It is worth noting that the method of discussion of the transport problem we have followed above is based on the study of the long time behavior of an isolated system. It is, however, easily proved that this is in a certain sense mathematically equivalent to the long time behavior of an open system. More specifically, we may assume that the system under consideration interacts with the rest of the universe and that this interaction can be described[12] by a relaxation time $(1/\gamma)$. That is, the equation of motion (2.9) for the density matrix $\rho_T(t)$ should be supplemented by adding a term $-i\hbar\gamma[\rho_T(t)-f(H_0+V)]$ in the right-hand side, or, equivalently, adding a term $-i\hbar\gamma\rho(t)$ in the right-hand side of Eq. (2.14) for $\rho(t)=\rho_T(t)-f(H_0+V)$. Proceeding as before, we see that the fundamental equation (2.18) changes only in that the energy differences $\epsilon_{\nu\nu'}$ acquire a small imaginary part to become $(\epsilon_{\nu\nu'}-i\hbar\gamma)$. The only consequence of this for the final transport equation (2.36) is that the function

$$\theta(\epsilon)=(i/\hbar)\int_0^t \exp(-i\epsilon\tau/\hbar)d\tau \underset{t\to\infty}{\longrightarrow} i\pi\delta(\epsilon)+P(1/\epsilon),$$

[Eq. (2.32)], is replaced everywhere by

$$\theta(\epsilon-i\hbar\gamma) \underset{t\to\infty}{\longrightarrow} (1/\epsilon-i\hbar\gamma).$$

For very long relaxation times $1/\gamma$, however, these expressions are identical, since $\lim_{\gamma\to 0^+}(1/\epsilon-i\hbar\gamma)=i\pi\delta(\epsilon)+P(1/\epsilon)$.

We have restricted ourselves to the lowest Born approximation for the collisions, although higher approximations are easily obtained by iterating (2.18) for $\nu'\neq\nu\pm 1$ with the help of (2.26). This restriction entails the omission, among other things, of the effects of broadening of the "unperturbed" states due to interaction with the impurities. This is, however, quite important for the detailed description of the amplitude and phase of the oscillations that the resistivity presents as a function of the magnetic field.

The effect of the magnetic field on the collisions is automatically taken into consideration. The effects arising from the interference between the electric field and the interaction with the impurities, which were neglected in previous theories,[3,4] have also been taken into account. They are particularly important for high magnetic fields. In the simplifying approximation of neglecting the shifting of the unperturbed states due to the scattering interaction, it was proven that the effect of a transverse electric field on the collisions can be described by saying that the collisions force the electrons to a distribution appropriate to a state of thermal equilibrium in the presence of the electric field.

It was demonstrated that for large Hall angles the induced current can easily be calculated from the transport equation. It turned out that the component of the current in the direction of the electric field can be described by the net drift only of the centers of the cyclotron orbits of the electrons, as it was assumed by Titeica[1] and proved recently by others.[6–8]

For the case of completely isotropic scattering, it was shown that the effects of the collisions can be described simply by a relaxation time τ and the current can be calculated for all values of $\omega_0\tau$. For magnetic fields small enough for the magnetic quantum of energy, $\hbar\omega_0$, to be much smaller than both the energy of the electrons and kT, it was shown that the usual result of the classical Boltzmann equation obtains. This indicates that in the lowest Born approximation the magnetic field has no *classical* effect on the collisions with isotropic scatterers.

All results are applicable to both classical and quantum statistics, since we have been concerned only with elastic collisions.

[12] M. Lax, Phys. Rev. **109**, 1921 (1958).

Finally, the phenomena of cyclotron resonance absorption, Faraday rotation, etc., can be studied quantum mechanically along similar lines.

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APPENDIX A

We establish here the selection rule (2.43). Assuming that

$$|w(q_x,q_y,q_z)|^2=\sum_\nu C_\nu(q_x^2+q_y^2)^\nu, \tag{A.1}$$

we can write, according to (2.23) and (2.24), $(a\equiv\hbar/m\omega_0)$,

$$Q\equiv\sum_{k_y'} V_{nk,mk'}V_{m'k',n'k}\propto\int\int\int_{-\infty}^{+\infty}dxdydk_y'$$
$$\times\phi_n(x+ak_y)\phi_m(x+ak_y')\phi_{m'}(y+ak_y')\phi_{n'}(y+ak_y)$$
$$\times\int_{-\infty}^{+\infty}|w(q_x,k-k')|^2e^{iq_x(x-y)}dq_x. \tag{A.2}$$

We can clearly write $(1/i)[\partial/\partial(x-y)]$ instead of q_x in $|w(q_x,k-k')|^2$ in the last integral. Integration over q_x can then be carried out to give $2\pi\delta(x-y)$. By successive integrations by parts over x we have for each of the terms (A.1)

$$Q\propto\int\int_{-\infty}^{+\infty}dydk_y'\phi_{m'}(y+ak_y')\phi_{n'}(y+ak_y)$$
$$\times\left[(k_y-k_y')^2-\frac{d^2}{dy^2}\right]^\nu\phi_n(y+ak_y)\phi_m(y+ak_y'). \tag{A.3}$$

Introducing new variables $s=y+ak_y$ and $t=y+ak_y'$, we note that $d/dy=(d/ds)+(d/dt)$, $(k_y-k_y')=a^{-1}(s-t)$, and

$$a^{-2}(s-t)^2-\left(\frac{d}{ds}+\frac{d}{dt}\right)^2$$
$$=[\alpha_-(s)-\alpha_+(t)][\alpha_+(s)-\alpha_-(t)], \tag{A.4}$$

where α_+, α_- are the raising and lowering operators for the oscillator eigenfunctions, i.e.,

$$\alpha_\pm(s)\phi_n(s)=\left(a^{-1}s\mp\frac{d}{ds}\right)\phi_n(s)\propto\phi_{n\pm1}(s). \tag{A.5}$$

We then have

$$Q\propto\int\int_{-\infty}^{+\infty}dsdt\,\phi_{m'}(t)\phi_{n'}(s)[\alpha_-(s)-\alpha_+(t)]^\nu$$
$$\times[\alpha_+(s)-\alpha_-(t)]^\nu\phi_n(s)\phi_m(t). \tag{A.6}$$

We note that for each term (A.6) is a product of the general form

$$\int_{-\infty}^{+\infty}ds\phi_{n'}(s)\beta\phi_n(s)\int_{-\infty}^{+\infty}dt\phi_{m'}(t)\gamma\phi_m(t), \tag{A.7}$$

where always $\beta\phi_n(s)\propto\phi_{n+r}(s)$ and $\gamma\phi_m(t)\propto\phi_{m+r}(t)$, with the same integer r. It is thus clear that all terms, and, therefore, their sum $\propto Q$, vanish unless $n'-n=m'-m$.

APPENDIX B

We evaluate here the expression

$$I_1=\sum_{k_y'}k_y'V_{nk,mk'}V_{mk',\,n+1k} \tag{B.1}$$

for isotropic scattering interactions. According to their basic property (4.1) we can write, making use of the definitions of J's, Eq. (2.24), rearranging the order of integrations, and putting $s=\hbar k_y'/m\omega_0$,

$$I_1=G(1/\Omega)[L_xL_y/(2\pi)^2](m\omega_0/\hbar)^2$$
$$\times\int_{-\infty}^{+\infty}sds\int_{-\infty}^{+\infty}\phi_n(x-X_n)\phi_m(x+s)dx$$
$$\times\int_{\infty}^{+\infty}dy\,\phi_{n+1}(y-X_{n+1})\phi_m(y+s)$$
$$\times\int_{-\infty}^{+\infty}e^{iq_x(x-y)}dq_x. \tag{B.2}$$

Since the last integral equals $2\pi\delta(x-y)$, the integral over y can be carried out immediately to give

$$I_1=G(1/2\pi L_z)(m\omega_0/\hbar)^2$$
$$\times\int_{-\infty}^{+\infty}\phi_n(x-X_n)\phi_{n+1}(x-X_{n+1})dx$$
$$\times\int_{-\infty}^{+\infty}s\phi_m(x+s)\phi_m(x+s)ds. \tag{B.3}$$

The last integral is clearly equal to $(-x)$, if use is made of the known properties of the oscillator eigenfunctions. Since now $X_{n+1}=X_n=-\hbar k_y/m\omega_0$ and $\phi_n(x)$ is orthogonal to $\phi_{n+1}(x)$ we can write

$$I_1=-G(1/2\pi L_z)(m\omega_0/\hbar)^2\int_{-\infty}^{+\infty}x\phi_n(x)\phi_{n+1}(x)dx. \tag{B.4}$$

The integral above is equal to $(2m\omega_0/\hbar)^{-\frac{1}{2}}(n+1)^{\frac{1}{2}}$ and thus finally

$$I_1=-G(1/2\pi L_z)(m\omega_0/\hbar)^{\frac{3}{2}}2^{-\frac{1}{2}}(n+1)^{\frac{1}{2}}. \tag{B.5}$$

This proves Eq. (4.7) we used in the text.

For

$$I_2=\sum_{ky'} V_{nk,mk'}V_{m'k',n'k}, \tag{B.6}$$

we get in a similar fashion

$$I_2=G(1/2\pi L_z)(m\omega_0/\hbar)\times\int_{-\infty}^{+\infty}\phi_n(x-X_n)\phi_{n'}(x-X_{n'})dx\times\int_{-\infty}^{+\infty}\phi_m(x+s)\phi_{m'}(x+s)ds=G(1/2\pi L_z)(m\omega_0/\hbar)\delta_{nn'}\delta_{mm'}, \tag{B.7}$$

since $X_n=X_{n'}$. This proves Eq. (4.3). It is a special case of the selection rule established in the previous Appendix.

APPENDIX C

Since, perhaps, the nature of the approximation made in deriving the basic transport equation (2.28) is not quite clear, a more detailed account of this point is given below.

Let us write the matrix $g_{\nu\nu'}$ as the sum of the matrix $G_{\nu\nu'}$ and another one, $J_{\nu\nu'}$, i.e.,

$$g=G+J. \tag{C.1}$$

From the exact Eq. (2.18) we have for $\nu'\neq\nu\pm1$

$$(\epsilon_{\nu\nu'}-i\hbar s)J_{\nu\nu'}=C_{\nu\nu'}+[G,V]_{\nu\nu'}+[J,V]_{\nu\nu'}. \tag{C.2}$$

We now observe that a solution of this equation is obtained in the following form:

$$J=\sum_{n=1}^{\infty}J^{(n)}, \tag{C.3}$$

where

$$(\epsilon_{\nu\nu'}-i\hbar s)J_{\nu\nu'}{}^{(1)}=C_{\nu\nu'}{}^{(1)}+[G,V]_{\nu\nu'}, \tag{C.4}$$

$$(\epsilon_{\nu\nu'}-i\hbar s)J_{\nu\nu'}{}^{(n)}=C_{\nu\nu'}{}^{(n)}+[J^{(n-1)},V]_{\nu\nu'}\cdot(n\geqslant 2). \tag{C.5}$$

It is important to realize that this is *not* an expansion in powers of the strength of the scattering interaction. By repeated use of (C.4) and (C.5) all $J^{(n)}$ are ultimately given in terms of $C^{(1)}$, $C^{(2)}$, etc., G, and V. For example,

$$J^{(2)}=\omega^{-1}C^{(2)}+\omega^{-1}[\omega^{-1}C^{(1)},V]+\omega^{-1}[\omega^{-1}[G,V],V], \tag{C.6}$$

where we have introduced for convenience the operator ω^{-1} to denote the energy denominators, i.e.,

$$(\omega^{-1}A)_{\nu\nu'}\equiv\frac{A_{\nu\nu'}}{\epsilon_{\nu\nu'}-i\hbar s}. \tag{C.7}$$

Again from the exact Eq. (2.18) we have for $\nu'=\nu\pm1$

$$(\epsilon_{\nu\nu'}-i\hbar s)G_{\nu\nu'}=C_{\nu\nu'}+[G,V]_{\nu\nu'}+[J,V]_{nn'}=C_{\nu\nu'}{}^{(0)}+[G,V]_{\nu\nu'}+\sum_{n=1}^{\infty}C_{\nu\nu'}{}^{(n)}+\sum_{n=1}^{\infty}[J^{(n)},V]_{\nu\nu'}. \tag{C.8}$$

If we now substitute in (C.8) the final expression for $J^{(n)}$, analogous to (C.6) for $J^{(2)}$, we obtain a transport equation for the operator G of the following form:

$$-[H_0,G]+i\hbar sG+(\sum_{n=1}^{\infty}S^{(n)})G+C^{(0)}+\sum_{n=1}^{\infty}D^{(n)}=0. \tag{C.9}$$

Here $S^{(n)}$ are operators of order n in the strength of the scattering interaction V. $D^{(n)}$ are similarly given operators of order n in V. We have explicitly

$$S^{(1)}G=[V,G], \tag{C.10}$$

$$S^{(2)}G=[V,\omega^{-1}[V,G]], \tag{C.11}$$

and in general

$$S^{(n)}G=[V,\omega^{-1}S^{(n-1)}G].\quad(n\geqslant 2). \tag{C.12}$$

Similarly

$$D^{(1)}=C^{(1)} \tag{C.13}$$

$$D^{(2)}=C^{(2)}+[\omega^{-1}C^{(1)},V], \tag{C.14}$$

$$D^{(n)}=C^{(n)}+[\omega^{-1}D^{(n-1)},V].\quad(n\geqslant 2). \tag{C.15}$$

Thus we have obtained a general transport equation for G, where the "collision" operator, $S=\sum_{n=1}^{\infty}S^{(n)}$, and the operator describing the effects of interference between the electric field and the scattering, $D=\sum_{n=1}^{\infty}D^{(n)}$, are given in the form of a power series in the strength of the scattering interaction V.

Now we make the approximation that for sufficiently small V, we may neglect $S^{(3)}$, etc., compared to $S^{(1)}+S^{(2)}$ (and similarly for D), i.e., we approximate only the "collision" and "driving" operators by their leading terms in their power series expansion. This gives exactly Eq. (2.28). Note that this procedure does not imply any assumption about the relative magnitudes of the relaxation time and the cyclotron period. It only calculates the effects of the scattering interaction in the first Born approximation.

APPENDIX D

We prove here the ensemble average theorem of Kohn and Luttinger,[9] for the problem at hand. Since this proof follows closely the method of KL,[9] we shall only indicate here the main steps of the calculation.

It is sufficient to prove that

$$\lim_{N\to\infty}\frac{\langle M^2\rangle-\langle M\rangle^2}{\langle M\rangle^2}=0, \tag{D.1}$$

where in this case

$$M=\frac{1}{\nu}\sum_{k'(A)} V_{nk,mk'}V_{m'k',n'k}. \tag{D.2}$$

Here $k'(A)$ indicates that k' lies in a small area A in k'-space (two-dimensional) over which $G(k')$, $\epsilon(k')$, etc., vary very little and ν is the number of states in A, i.e.,

$$\nu=AL_yL_z/(2\pi)^2. \tag{D.3}$$

Clearly ν is of order $\Omega^{\frac{2}{3}}$, or, equivalently, of order $N^{\frac{2}{3}}$, since $N/\Omega=N_I$ remains finite in the limit $N\to\infty$.

Now, by direct calculation we find, dropping the irrelevant oscillator indices n, m, etc.,

$$\begin{aligned}\langle M\rangle&=\frac{1}{\nu}\sum_{k'(a)}\langle V_{kk'}V_{k'k}\rangle\\ &=\frac{1}{\nu}\sum_{k'(A)}N\left[\frac{1}{\Omega^2}\sum_{q_x}b_{kk'}(q_x)b_{k'k}(-q_x)\right]\\ &=N\left[\frac{1}{\Omega^2}\sum_{q_x}b_{kk'}(q_x)b_{k'k}(-q_x)\right],\end{aligned} \tag{D.4}$$

where

$$b_{kk'}(q_x)=w(q_x,k-k')J_{kk'}(q_x). \tag{D.5}$$

Therefore,

$$\langle M\rangle^2=N^2\left[\frac{1}{\Omega^2}\sum_{q_x}b_{kk'}(q_x)b_{k'k}(-q_x)\right]^2. \tag{D.6}$$

Similarly we find by a straightforward calculation

$$\begin{aligned}&\langle V_{kk'}V_{k'k}V_{kk''}V_{k''k}\rangle\\ &=\frac{N}{\Omega^4}\sum_{q_x}\sum_{q_x'}\sum_{q_x''}b_{kk'}(q_x)b_{k'k}(q_x')b_{kk''}(q_x'')\\ &\quad\times b_{k''k}(-q_x-q_x'-q_x'')+\frac{N(N-1)}{\Omega^4}\\ &\quad\sum_{q_x}\sum_{q_x'}[b_{kk'}(q_x)b_{k'k}(-q_x)b_{kk''}(q_x')b_{k''k}(-q_x')\\ &\quad+b_{kk'}(q_x)b_{k'k}(q_x')b_{kk''}(-q_x')b_{k''k}(-q_x)\delta_{k'k''}\\ &\quad+b_{kk'}(q_x)b_{k'k}(q_x')b_{kk''}(-q_x)b_{k''k}(-q_x')\\ &\quad\times\delta_{2k,\,k'+k''}].\end{aligned} \tag{D.7}$$

From this we can write down directly the quantity

$$\langle M^2\rangle=\frac{1}{\nu^2}\sum_{k'(A)}\sum_{k''(A)}\langle V_{kk'}V_{k'k}V_{kk''}V_{k''k}\rangle. \tag{D.8}$$

Since, however, because of (D.3)

$$(N^2/\nu^2)\sum_{k'(A)}\sum_{k''(A)}\delta_{k'k''}=N^2/\nu=O(N^{4/3}), \tag{D.9}$$

and similarly,

$$(N^2/\nu^2)\sum_{k'(A)}\sum_{k''(A)}\delta_{2k,\,k'+k''}\leqslant N^2/\nu=O(N^{4/3}), \tag{D.10}$$

it is clear that the dominant term of $\langle M^2\rangle$ for large N is given by the first term in the square brackets of (D.7), all other terms, including the first term of (D.7), being of relative order $(N^{4/3}/N^2)=N^{-\frac{2}{3}}$ or smaller. Thus

$$\langle M^2\rangle=N^2\left[\frac{1}{\Omega^2}\sum_{q_x}b_{kk'}(q_x)b_{k'k}(-q_x)\right]^2\times[1+O(N^{-2/3})]. \tag{D.11}$$

Upon comparison of (D.11) and (D.6) it is perspicuous that the condition (D.1) is satisfied. This proves the validity of the ensemble average theorem for this problem.