In other words, the amplitudes of the incoming and outgoing waves will now contain the factor $\ln(kr-kz)$ which diverges at infinity. In this case, therefore, we have difficulty in giving physical meaning to the wave function $\psi(\mathbf{x})$, and so we can calculate neither the cross section nor the renormalization constant Z_2.

The appearance of a logarithmically diverging amplitude may be due to the failure of the expansion in k/ϵ. As we have noted earlier, a calculation of the potential in terms of such an expansion leads, in the lowest-order approximation, to the ordinary Coulomb potential; the next-order approximation contains an $r^{-1}\ln r$ term. This $r^{-1}\ln r$ potential has a much longer range than the ordinary Coulomb potential $1/r$ and so would inevitably lead to an additional distortion of the incoming and outgoing waves. There remains the possibility that we can circumvent these difficulties by avoiding a perturbation-theoretical calculation in k/ϵ. Unfortunately, this general case is extremely difficult to solve.

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Detection and Generation of Gravitational Waves*

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Methods are proposed for measurement of the Riemann tensor and detection of gravitational waves. These make use of the fact that relative motion of mass points, or strains in a crystal, can be produced by second derivatives of the gravitational fields. The strains in a crystal may result in electric polarization in consequence of the piezoelectric effect. Measurement of voltages then enables certain components of the Riemann tensor to be determined. Mathematical analysis of the limitations is given. Arrangements are presented for search for gravitational radiation.

The generation of gravitational waves in the laboratory is discussed. New methods are proposed which employ electrically induced stresses in crystals. These give approximately a seventeen-order increase in radiation over a spinning rod of the same length as the crystal. At the same frequency the crystal gives radiation which is about thirty-nine orders greater than that of a spinning rod.

INTRODUCTION

THE question of gravitational radiation has always been a central issue in the General Theory of Relativity. Long ago, Einstein[1] and Eddington[2] studied the problem and predicted that very small amounts of energy would be radiated by a spinning rod or a double star. A great deal of theoretical work on the radiation problem has appeared, during the past four decades.

Experimental work along these lines now appears possible. Two avenues of approach will be considered.[3] First we should like to detect the presence of gravitational radiation incident on earth from either the sun or outside the solar system. Secondly it would be highly desirable to be able to generate and detect this radiation in a small laboratory.

Devices for detection of the radiation operate essentially by measuring the Fourier transform of the Riemann tensor. These will be discussed first. This will then be followed by proposals for generation of gravitational radiation which may give an increase of many orders over the gravitational radiation from a spinning rod.

DETECTION OF GRAVITATIONAL RADIATION

Suppose we have a system of masses which may interact with each other. We start with the action principle

$$\delta I=\delta\left[-cm\int ds+W\right]=0. \tag{1}$$

In (1) m is the rest mass and W is the part of the action function associated with forces arising from the motion of the mass relative to other masses with which it interacts. The line element ds is given by

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu. \tag{2}$$

For δW we assume a function given by

$$-c\delta W=\int F_\mu \delta x^\mu ds; \tag{3}$$

(3) identifies F_μ as the four-force. The Euler-Lagrange

* Supported by the National Science Foundation.
[1] A. Einstein, Sitzber. deut. Akad. Wiss. Berlin, Kl. Math. Physik u. Tech. (1916), p. 688; (1918), p. 154.
[2] A. S. Eddington, Proc. Roy. Soc. (London) **A102**, 268 (1923).
[3] A number of the results discussed here were given without proof in the author's Gravity Research Foundation Prize Essays, April 1958 and April 1959, and at the Royaumont Conference on the Relativistic Theories of Gravitation, Royaumont, France, June, 1959 (unpublished).

equations resulting from (1) are arranged[4] to obtain

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=\frac{F^\mu}{mc^2}. \tag{4}$$

$\Gamma^\mu{}_{\alpha\beta}$ is the Christoffel symbol of the second kind. (4) may be written in terms of the four-velocity $p^\mu=dx^\mu/ds$, as

$$\frac{\delta}{\delta s}\left(\frac{dx^\mu}{ds}\right)=\frac{\delta p^\mu}{\delta s}=\frac{F^\mu}{mc^2}. \tag{5}$$

The symbol $\delta/\delta s$ means the covariant derivative with respect to s.

Following essentially the method of Synge and Schild,[5] we now obtain from (5) an equation similar to the equation of geodesic deviation. We introduce a parameter v such that each world line corresponds to a given value of v. Taking the covariant derivative of (5) with respect to v gives

$$\frac{\delta^2p^\mu}{\delta v\delta s}=\frac{\delta F^\mu/m}{c^2\delta v}. \tag{6}$$

Employing the commutation law for covariant differentiation enables us to express (6) in the form

$$\frac{\delta^2p^\mu}{\delta v\delta s}=\frac{\delta^2p^\mu}{\delta s\delta v}-R^\mu{}_{\alpha\beta\gamma}p^\alpha p^\beta\frac{\partial x^\gamma}{\partial v}. \tag{7}$$

In (7), $\partial x^\gamma/\partial v$ is a unit vector normal to the world lines, and the four-velocity p^μ is a unit vector tangent to the world lines. The vector n^γ, defined by

$$n^\gamma=\frac{\partial x^\gamma}{\partial v}dv, \tag{8}$$

is an infinitesimal vector joining points with the same value of s on neighboring world lines with values of v differing by dv. The covariant derivative of $\partial x^\gamma/\partial v$ with respect to s can be written in the forms

$$\frac{\delta}{\delta s}\left(\frac{\partial x^\gamma}{\partial v}\right)=\frac{\delta}{\delta v}\left(\frac{\partial x^\gamma}{\partial s}\right)=\frac{\delta p^\gamma}{\delta v}. \tag{9}$$

Employing (6), (7), (8), and (9) then gives

$$\frac{\delta^2n^\mu}{\delta s^2}+R^\mu{}_{\alpha\beta\gamma}p^\alpha n^\beta p^\gamma=\frac{1}{c^2}\frac{\delta F^\mu/m}{\delta v}dv. \tag{10}$$

MASS QUADRUPOLE DETECTOR

In order to discuss the detector[6] of Fig. 1, we imagine the two world lines are those of the two masses. Let n be given by

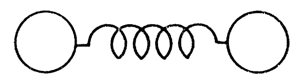

FIG. 1. Harmonic oscillator driven by gravitational waves.

$$n^\gamma=r^\gamma+\xi^\gamma, \tag{11}$$

with r^γ defined by

$$\delta r^\gamma/\delta s=0 \quad \text{for all } s,$$
$$r^\gamma\rightarrow n^\gamma, \tag{12}$$

in the limit of large internal damping and all components of $R^\mu{}_{\alpha\beta\delta}=0$. Equation (10) becomes

$$\frac{\delta^2\xi^\mu}{\delta s^2}+R^\mu{}_{\alpha\beta\gamma}p^\alpha p^\gamma[r^\beta+\xi^\beta]=\frac{f^\mu}{mc^2}. \tag{13}$$

In (13) we have denoted by f^μ the differences in (nongravitational) forces at the two masses. For f^μ we assume a restoring force $-k_\alpha{}^\mu\xi^\alpha$ and a damping force $-cD_\alpha{}^\mu(\delta\xi^\alpha/\delta s)$; $k_\alpha{}^\mu$ and $D_\alpha{}^\mu$ are tensors associated with the spring. (13) then becomes

$$\frac{\delta^2\xi^\mu}{\delta s^2}+\frac{D_\alpha{}^\mu}{cm}\frac{\delta\xi^\alpha}{\delta s}+\frac{k_\alpha{}^\mu\xi^\alpha}{mc^2}=-R^\mu{}_{\alpha\beta\gamma}p^\alpha p^\gamma[r^\beta+\xi^\beta]. \tag{14}$$

We now let time run in the direction of the tangents to the world lines. The center of mass of the oscillator is a freely falling platform. We use coordinates in which the Christoffel symbols vanish and write (14) in the approximate form (assuming $\xi\ll r$)

$$\frac{d^2\xi^\mu}{dt^2}+\frac{D_\alpha{}^\mu}{m}\frac{d\xi^\alpha}{dt}+\frac{k_\alpha{}^\mu\xi^\alpha}{m}=-c^2R^\mu{}_{0\alpha0}r^\alpha. \tag{15}$$

In (15) we see that the driving force for the harmonic oscillator is the Riemann tensor. Measurement of displacement amplitude or power absorbed enables one to calculate certain components of the Riemann tensor.[7]

Suppose now that sinusoidal (weak-field approximation) gravitational waves are incident, with angular frequency ω. An orthogonal comoving coordinate system is employed, with the oscillator oriented in the direction of the x^1 axis. $k_\alpha{}^\mu$ and $D_\alpha{}^\mu$ are imagined to have one component only, $k_1{}^1=k$, and $D_1{}^1=D$. Taking the

[4] See for example, W. Pauli, *Theory of Relativity* (Pergamon Press, New York, 1958), p. 41.

[5] J. L. Synge and A. Schild, *Tensor Calculus* (The University of Toronto Press, Toronto, 1952), Chap. 3. See also F. A. E. Pirani, Helv. Physica Acta Suppl. IV, p. 198.

[6] An arrangement somewhat similar to this was independently suggested by H. Bondi at the Royaumont Conference, Royaumont, France, June, 1959 (unpublished).

[7] Measurement of the Riemann tensor by comparing accelerations of free test particles has been considered by F. A. E. Pirani, Proceedings of the Chapel Hill Conference on the Role of Gravitation in Physics, page 61, Astia Document No. AD 118180 and Acta Phys. Polon. XV **6**, 389 (1956). While free particles are convenient for some thought experiments, interacting particles appear to be essential, in practice, at low energies. The correspondence between voltage in a piezoelectric crystal and some components of the Riemann tensor, which is discussed in the next section, may provide a basis for consideration of the measurement problem in quantized General Relativity. In principle a very small crystal may be used since the acoustic resonance vibrations have a wavelength which is about five orders smaller than that of the gravitational wave which excites them.

Fourier transform of (15) leads to

$$\xi^\mu(\omega)=\frac{-mc^2R^\mu{}_{0\alpha 0}(\omega)r^\alpha}{(-\omega^2m+i\omega D\delta_1{}^\mu+k\delta_1{}^\mu)}. \tag{16}$$

(16) is a maximum at resonance, $-\omega^2m+k=0$. The total dissipation $D=D_{\text{ex}}+D_{\text{in}}$ where D_{ex} is the external dissipation and D_{in} is the internal dissipation associated with irreversible processes within the antenna. The power which can be delivered to auxiliary apparatus with D_{ex} is

$$\tfrac{1}{2}\omega^2D_{\text{ex}}\xi^2=\frac{m^2c^4(R^\mu{}_{0\alpha 0}r^\alpha)^2D_{\text{ex}}}{2(D_{\text{ex}}+D_{\text{in}})^2}. \tag{17}$$

(17) is a maximum when $D_{\text{ex}}=D_{\text{in}}$ and the maximum power P_M is given by

$$P_M=m^2c^4(R^\mu{}_{0\alpha 0}r^\alpha)^2/(8D_{\text{in}}). \tag{18}$$

The sinusoidal gravitational waves are now assumed to be radiated by a linear mass quadrupole oscillator. The transformation laws indicate that to a good approximation $R^\mu{}_{0\alpha 0}$ as seen in a frame fixed in the center of mass of the radiator is the same as that seen in a frame fixed in the center of mass of the detector, for small velocities. Using the known[8,9] solution for the linear mass quadrupole oscillator, the mean squared value of $R^\mu{}_{0\alpha 0}r^\alpha$ is calculated and averaged over all possible orientations of the receiving antenna. Let t_{0r} be the radiated power per unit area averaged over a sphere, for the linear mass quadrupole oscillator. The total radiated power P is given by

$$P=4\pi r^2t_{0r}=GI_0{}^2\omega^6/(60\pi c^5), \tag{19}$$

where I_0 is the amplitude of the quadrupole tensor. (19) and the known expressions for the fields then give, for the mean squared value of $R^\mu{}_{0\alpha 0}r^\alpha$, in a direction normal to the quadrupole radiator axis.

$$[R^\mu{}_{0\alpha 0}r^\alpha]^2{}_{\text{Av}}=[4\pi\beta^2|r|^2G/c^5]t_{0r}. \tag{20}$$

In (20), β is the propagation vector of the gravitational wave. Employing (18) and (20) gives

$$P_M=\pi m^2\beta^2|r|^2Gt_{0r}/(2cD_{\text{in}}). \tag{21}$$

The influence of the internal dissipation D_{in} will now be considered. First we assume that no irreversible processes take place within the antenna itself and that D_{in} is due entirely to radiation damping of the detector. The known solution for a linear mass quadrupole oscillator enables us to calculate the radiation resistance of the detector, D_{in}, as

$$D_{\text{in}}=2G\omega^4m^2|r|^2/(15c^5). \tag{22}$$

(21) and (22) give, in terms of the wavelength λ,

$$P_{M\text{ (radiation damping only)}}=(15\lambda^2/16\pi)t_{0r}. \tag{23}$$

The implication of (23) is that the average absorption cross section S_A for a detector which is damped only by its own reradiation is

$$S_A=(15/16\pi)\lambda^2. \tag{24}$$

We see from (24) that under these conditions the average absorption cross section is roughly a wavelength squared, and is independent of the constant of gravitation. Unfortunately the condition that the internal damping be only due to radiation cannot be attained in practice because other irreversible phenomena within the antenna are many orders greater than the radiation damping. In order to make this clear, we calculate the quality factor, denoted by the symbol Q, which is defined by

$$Q=\omega(\text{maximum stored energy})/(\text{power dissipated}).$$

The Q associated with radiation damping, denoted by Q_R, is

$$Q_R=15c^5/(2G\omega^3m|r|^2). \tag{25}$$

For an antenna at $\omega=2\pi\times 10^7$, a reasonable value of $mr^2=10$ g cm^2 and (25) gives $Q_R\sim 10^{34}$. A practical antenna might be expected to have a $Q\sim 10^6$.

We therefore must deal with systems limited by internal damping orders larger than gravitational radiation damping, and under these conditions the average absorbed power will not be independent of the kind of antenna. For an antenna orientation arranged for maximum response, we write

$$(R^\mu{}_{0\alpha 0}r^\alpha)^2=15\pi G\beta^2|r|^2c^{-5}t_{0r}. \tag{26}$$

(26) and (18) lead to power absorbed, P_A, given by

$$P_A=(15/8)\pi Gm^2\beta^2|r|^2(cD_{\text{in}})^{-1}t_{0r} = (15/8)\pi GmQ_{\text{in}}\beta^2|r|^2(\omega c)^{-1}t_{0r}. \tag{27}$$

In (27), Q_{in} is the Q associated with internal irreversible processes, $Q_{\text{in}}=\omega m/D_{\text{in}}$. The cross section, S, implied by (27) is

$$S=(15/8)\pi GmQ_{\text{in}}\beta^2|r|^2\omega^{-1}c^{-1}. \tag{28}$$

For a continuous spectrum the absorbed power is

$$P_A=T^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{m^2c^4D_{\text{ex}}\omega\omega' R^\mu{}_{0\alpha 0}(\omega)R^\mu{}_{0\beta 0}(\omega')r^\alpha r^\beta e^{i(\omega-\omega')t}}{2(-\omega^2m+i\omega D+K)(-\omega'^2m-i\omega' D+K)}d\omega d\omega' dt\approx\pi^2Gm\beta^2|r|^2c^{-1}t_{0r}(\omega_0). \tag{29}$$

In (29), $t_{0r}(\omega_0)$ is the power spectrum of t_{0r} in the vicinity of the resonant frequency ω_0.

In order to further discuss these results we must consider the excitation of a continuous medium by a gravitational wave. This is necessary in order to be able to account for the interaction of the mass of the spring with the wave and to account for the effects of

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[8] N. Rosen and H. Shamir, Revs. Modern Phys. **29**, 429 (1957).
[9] W. B. Bonnor, Phil. Trans. Roy. Soc. London **A251**, 233 (1959).

the finite velocity of propagation of the elastic forces of the spring.

INTERACTION OF A CRYSTAL WITH A GRAVITATIONAL WAVE

The starting point for our discussion is expression (10). The infinitesimal vector n^μ is from a reference point in the crystal to a neighboring point. The mass m is imagined to belong to an infinitesimal volume surrounding the neighboring point. On the right side of (10) we must now include both elastic forces and dissipative forces. We write n^μ as

$$n^\mu = r^\mu + \epsilon_\alpha{}^\mu r^\alpha. \tag{30}$$

r^μ is defined by the conditions

$$\delta r^\mu/\delta s = 0 \quad \text{for all } s; \tag{31}$$
$$r^\mu \to n^\mu,$$

in the limit of large internal damping and flat space. We may now write Eq. (10) in the form

$$r^\mu \frac{\delta^2 \epsilon_{\mu\nu}}{\delta s^2} + r^\mu B_\nu{}^\alpha \frac{\delta \epsilon_{(\alpha\mu)}}{\delta s} + r^\mu Y^{\alpha\beta} \frac{\delta^2 \epsilon_{(\mu\nu)}}{\delta x^\alpha \delta x^\beta} + R_{\nu\alpha\mu\beta}[r^\mu + \epsilon_\gamma{}^\mu r^\gamma] p^\alpha p^\beta = 0. \tag{32}$$

In (32) the quantity $\epsilon_{(\mu\nu)}$ is the symmetric part of $\epsilon_{\mu\nu}$ and is therefore the strain tensor of the crystal. The second term accounts for internal damping and the third term accounts for the elastic forces. $B_\nu{}^\alpha$ and $Y^{\alpha\beta}$ are normalized to unit mass density. Again p^α is a unit vector tangent to the world lines. Since r^μ can be arbitrarily specified, we may write

$$\frac{\delta^2 \epsilon_{\mu\nu}}{\delta s^2} + B_\nu{}^\alpha \frac{\delta \epsilon_{(\alpha\mu)}}{\delta s} + Y^{\alpha\beta} \frac{\delta^2 \epsilon_{(\mu\nu)}}{\delta x^\alpha \delta x^\beta} + R_{\nu\alpha\mu\beta} p^\alpha p^\beta + R_{\nu\alpha\gamma\beta} p^\alpha p^\beta \epsilon_\mu{}^\gamma = 0. \tag{33}$$

In (33) the fourth term is clearly symmetric in the indices ν and μ. The last term in (33) may ordinarily be dropped because it is many orders smaller than the fourth one. For the strain tensor we may therefore write

$$\frac{\delta^2 \epsilon_{(\mu\nu)}}{\delta s^2} + B_\nu{}^\alpha \frac{\delta \epsilon_{(\alpha\mu)}}{\delta s} + Y^{\alpha\beta} \frac{\delta^2 \epsilon_{(\mu\nu)}}{\delta x^\alpha \delta x^\beta} \approx -R_{\nu\alpha\mu\beta} p^\alpha p^\beta. \tag{33A}$$

We now consider a special case of (33A), namely excitation of longitudinal waves in an isotropic medium. An approximate form suitable for the present discussion, for waves in the direction x^1 of an orthogonal coordinate system (with the time direction tangent to the world lines), is

$$y \frac{\partial^2 \epsilon}{\partial (x^1)^2} - \rho \frac{\partial^2 \epsilon}{\partial t^2} - b \frac{\partial \epsilon}{\partial t} = c^2 \rho R^1{}_{010}. \tag{34}$$

If (34), ρ is the density, y is an appropriate modulus, and b is a damping constant. We assume that $R^1{}_{010}$ has its origin in incident sinusoidal gravitational waves so that

$$-c^2 R^1{}_{010} = f \exp[i(\omega t - \beta_j x^j)]. \tag{35}$$

In (35) the index j runs from 1 to 3. Let v_s be the sound velocity $(y/\rho)^{\frac{1}{2}}$, λ_s be the wavelength of sound, $k_s = 2\pi/\lambda_s$, $\alpha = b/\rho v_s$, and $\gamma = \alpha + ik_s$. Then to a good approximation the solution of (34) may be written

$$\epsilon = [A\gamma \cosh\gamma x^1 - f\omega^{-2} \exp(-i\beta_j x^j)] e^{i\omega t}. \tag{36}$$

Making use of the boundary condition that ϵ vanishes at the ends leads to

$$A = -\frac{f\lambda_s \cos\beta_1 l}{2\pi\omega^2(\alpha l \sin k_s l + \cos k_s l)}. \tag{37}$$

In (37), l is half the length of the crystal. The first term of (36) gives the contribution of the acoustic waves and the second term gives the strains which would be set up if there were no internal forces at all† (37) must be modified if the crystal is piezoelectric.

(37) has maxima when $k_s l$ is an odd multiple of $\pi/2$; however, it is clear from the denominator that the largest maximum is the first one, for which the total length is half an acoustic wavelength. The system composed of the two masses and spring (Fig. 1) must be described by an equation such as (34), when the spacing of the masses approaches half an acoustic wavelength. The largest value we can expect from (28) will occur when r is half an acoustic wavelength in the spring. This is an important limitation because the velocity of acoustic waves is about five orders smaller than the velocity of light, so the cross sections implied by (28) are limited to values ten orders smaller than would be the case if the elastic forces of the spring were propagated with the velocity of light. Such a limitation could be overcome in a number of ways. One might employ restoring forces transmitted by electric and magnetic fields, with the velocity of light. The piezoelectric effect may be employed, in which case the polarization charges in the crystal faces may give rise to some stress components which do not change sign every half acoustic wavelength.

In a piezoelectrical crystal a strain results in an electric polarization P_μ given by

$$P_\mu = \epsilon_{\alpha\beta} \mathcal{E}_\mu{}^{\alpha\beta}.$$

Here $\mathcal{E}_\mu{}^{\alpha\beta}$ is the piezoelectric stress tensor. The electric polarization gives rise to an electric field over the crystal. Its integrated value may give a terminal voltage large enough to be observed with a low-noise

† *Note added in proof.*—Integration of (36) gives relative displacements. If the result is applied to effects of gravitational waves interacting with the earth, the contribution of the first term of (36) is found to be very small. Apparatus on the earth's surface acts therefore as if it were in free fall insofar as the waves are concerned. This is a consequence of the fact that the velocity of sound is much smaller than the velocity of light.

radio receiver. Measurement of this voltage measures components $R_{\alpha 0\beta 0}$ of the Riemann tensor if a crystal with suitable constants is employed.

The system of stresses in the crystal is modified in a significant way if it is piezoelectric. Additional terms involving the piezoelectric constants need to be added to Eq. (33). We consider a very simple example. Suppose a single longitudinal mode is excited, with sound velocity in the x^1 direction. Let the thickness in the x^2 direction be small and assume that the crystal faces normal to the x^2 direction are plated with a conductor. The piezoelectric relations[10] are

$$\begin{aligned} -T &= \epsilon Y_0 + DH/4\pi, \\ E &= D/K + H\epsilon. \end{aligned} \tag{38}$$

In (38), T is the stress, K is the dielectric constant, ϵ is the strain, Y_0 is the elastic modulus, E is the electric field intensity, and D is the electric displacement. Both D and E are assumed to have components in the x^2 direction only. H is the piezoelectric constant relating open-circuit voltage to strain. A study of (38) and the equations of motion of mass elements of the crystal indicates that a wave equation similar to (34) results with

$$y = Y_0[1 - H^2K/4\pi].$$

Since the crystal surface normal to the x^2 direction is plated with a conductor, $\partial E/\partial x^1 = 0$. At the free ends of the crystal, $T=0$. If the crystal is coupled to an external impedance Z, we may write

$$-\int E dx^1 = Z\frac{\partial}{\partial t}\int D dx^1 dx^3.$$

These boundary conditions and the wave equation (34) then lead to the result

$$\epsilon = [A_1\gamma \cosh\gamma x^1 - f\omega^{-2}\exp(-i\beta_j x^j)]e^{i\omega t}, \tag{36A}$$

where γ, β, and f are as defined earlier; this has the same form as (36), but now the constant A_1 is given in terms of the length l_1 in the x^1 direction, and lengths l_2 and l_3 in directions x^2 and x^3 and the "clamped" capacitance $C/4\pi$ as

$$A_1 = \frac{(f/\beta_1\omega^2)[2\pi\beta_1 l_2(1+i\omega CZ)(Y_0 - H^2K/4\pi)\cos(\beta_1 l_1/2) + iH^2K^2 l_3\omega Z\sin(\beta_1 l_1/2)]}{2\pi l_2(Y_0 - H^2K/4\pi)[\gamma\cosh(\gamma l_1/2)](1+i\omega CZ) + i\omega ZH^2K^2 l_3\sinh(\gamma l_1/2)}, \tag{39}$$

and the voltage which appears at the crystal terminals when coupled to an impedance Z is

$$\begin{aligned} V_Z = {} & [2i\omega ZHKl_3/(1+i\omega CZ)] \\ & \times[A_1\sinh(\gamma l_1/2) - (f/\beta_1\omega^2)\sin(\beta_1 l_1/2)]. \end{aligned} \tag{40}$$

The electrical network theorems now permit straightforward calculation of the power which can be delivered by the detector to a radio receiver. For a crystal with constants similar to polarized barium titanate on which sinusoidal gravitational waves are incident the power which can be transferred is, roughly,

$$P_A \approx 10^{-19}\omega^{-1}VQ_t t_{0r} \text{ ergs/second.} \tag{41}$$

In (41), ω is again the angular frequency and t_{0r} is the incident gravitational power flow in ergs per square centimeter per second. V is the volume of the crystal. Q_t is the Q of the crystal and associated electric circuit. A cubic meter of crystal at $\omega \sim 10^3$ gives a cross section for absorption $\sim 10^{-10}$ cm^2. While this is a small quantity it appears sufficiently large to start some experiments. For a continuous spectrum of gravitational radiation with gravitational power flow having a power spectrum function $t_{0r}(\omega)$, the power absorbed is about

$$P_A \approx 10^{-19}Vt_{0r}(\omega_0) \text{ ergs/second.} \tag{42}$$

(41) and (42) provide a basis for discussion of sensitivity. In microwave spectroscopy it has been found that all spurious effects other than random fluctuations can be recognized. A similar assumption will be made here. The random fluctuations are partly thermal in origin, partly the result of spontaneous emission processes. For synchronous detection of sinusoidal waves the power output of the detector must exceed the noise[11] power P_{N1} given by

$$P_{N1} = N\hbar\omega/[8\tau_A(e^{\hbar\omega/kT} - 1)],$$

where k is Boltzmann's constant, T is the gravitational antenna temperature, and N is the noise factor of the receiver which is expected to be less than 25 and more than 1. τ_A is the averaging time. A different expression is required if radiation with a continuous spectrum is being studied. In this case the power delivered by the detector must exceed[11]

$$P_{N2} = [\pi^3\omega/(64\tau_A Q)]^{\frac{1}{2}} N\hbar\omega/[e^{\hbar\omega/kT} - 1].$$

Experiments are being planned to search for interstellar gravitational radiation[12] using methods de-

[10] See, for example, W. P. Mason, *Electromechanical Transducers and Wave Filters* (D. Van Nostrand Co., Princeton, N. J., 1948), Chap. VI, Second Edition.

[11] R. H. Dicke, Rev. Sci. Instr. **17**, 268 (1946).

[12] J. A. Wheeler has noted [Onzième Conseil de l'Institut International de Physique Solvay, *La Structure et l'Evolution de l'Universe* (Editions Stoops, Brussels, 1958), p. 112] that the density of gravitational radiation could be as high as 10^{-29} to 10^{-28} grams/cm^3 ($\sim 10^3$ ergs/cm^2 second) and still be consistent with present information about the rate of expansion of the universe. He and M. Schwarzschild (private communication) have subsequently noted that if this radiation were set free by the same process which caused the inhomogeneous collection of matter into galaxies, it would be characterized at that time, and therefore also now, by the same scale of lengths, of the order of 10^{24} cm today (10^6 years vibration period).

$$(\partial g_{\text{typical}}/\partial x)^2 \sim \rho Gc^{-2} \sim 0.2\times 10^{-56} \text{ cm}^{-2},$$
$$\delta g_{\text{typical}} \sim 0.5\times 10^{-28} \text{ cm}^{-1} (10^{24} \text{ cm}) \sim 10^{-4}.$$

This would appear to be not too small, but too slow to measure, by these methods.

scribed here.‡ For the first method the earth itself is the block of material constituting the antenna. The normal modes of the earth (about 1 cycle per hour) are excited by incident gravitational waves. This procedure is limited by the relatively low Q of the earth and the high noise temperature of its core. The apparatus of Fig. 2 is employed in the second method, in which the strains induced in the crystal are employed. Search at frequencies $\sim 10^3$ cycles per second is planned. The earth rotates the apparatus. If radiation is incident from some given direction it may be observed from the diurnal change in amplifier noise output. The arrangement of Fig. 3 should not require rotation. If radiation is incident it will cause correlated outputs. All sources of internal fluctuations will be uncorrelated. Low-noise amplifiers such as masers[13] may be employed.

The discussion given here predicts that gravitational flux with a power spectrum $t_{0r}(\omega) \sim 10^{-4}$ ergs/cm² second cycle should be detectable.

ROTATIONS INDUCED BY GRAVITATIONAL RADIATION

Dirac[14] has suggested that astronomical anomalies might be correlated with effects of gravitational radiation. To discuss this and to consider detection by observing rotations we return to expression (5). Let a group of masses be situated near the space origin of a coordinate system and let the infinitesimal vector r^μ be the position vector of one of the masses. Let $v_{\mu\alpha\beta\kappa}$ be the Levi Civita tensor density. Multiply (5) by $v_{\mu\alpha\beta\kappa}r^\beta$ to obtain

$$v_{\mu\alpha\beta\kappa}r^\beta \frac{\delta}{\delta s}\frac{dx^\alpha}{ds} = \frac{d}{ds} v_{\mu\alpha\beta\kappa}r^\beta \frac{dx^\alpha}{ds} + v_{\mu\alpha\beta\kappa}r^\beta \Gamma^\alpha{}_{\gamma\delta} p^\gamma p^\delta = v_{\mu\alpha\beta\kappa}r^\beta F^\alpha/mc^2. \quad (43)$$

Here again p^δ is a unit vector tangent to the world lines and in the second term of (43) we have used the identity $v_{\mu\alpha\beta\kappa}p^\alpha p^\beta = 0$. Let the world line of the origin be a path for which the Christoffel symbols vanish.

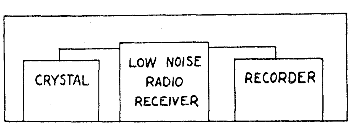

FIG. 2. Schematic diagram of piezoelectric crystal for detection of gravitational waves.

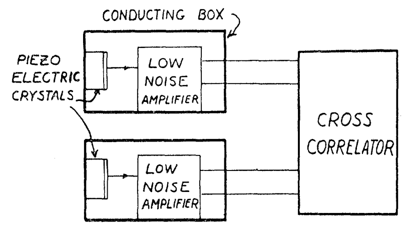

FIG. 3. Schematic diagram of cross correlation detection of gravitational waves.

Then it follows that (43) can be written

$$v_{\mu\alpha\beta\kappa}r^\beta \frac{\delta}{\delta s}\frac{dx^\alpha}{ds} = \frac{d}{ds} v_{\mu\alpha\beta\kappa}r^\beta \frac{dx^\alpha}{ds} + v_{\mu\alpha\beta\kappa}r^\beta \frac{\partial \Gamma^\alpha{}_{\gamma\delta}}{\partial x^\omega} p^\gamma p^\delta r^\omega. \quad (44)$$

In these coordinates $R^\alpha{}_{\gamma\omega\delta} = \partial\Gamma^\alpha{}_{\gamma\delta}/\partial x^\omega$, so (44) becomes

$$v_{\mu\alpha\beta\kappa}r^\beta \frac{\delta}{\delta s}\frac{dx^\alpha}{ds} = \frac{d}{ds} v_{\mu\alpha\beta\kappa}r^\beta \frac{dx^\alpha}{ds} - v_{\mu\alpha\beta\kappa}R^\beta{}_{\gamma\omega\delta} p^\gamma p^\delta r^\alpha r^\omega. \quad (45)$$

If we now use (43) and (45) and sum over all masses, we obtain

$$\sum_{\text{masses}} \frac{d}{ds} v_{\mu\alpha\beta\kappa}r^\beta \frac{dx^\alpha}{ds} = \sum_{\text{masses}} v_{\mu\alpha\beta\kappa}R^\beta{}_{\gamma\omega\delta} p^\gamma p^\delta r^\alpha r^\omega - \sum_{\text{masses}} v_{\mu\alpha\beta\kappa} r^\alpha F^\beta/mc^2. \quad (46)$$

(46) is a generalization of the relation between torque and change of angular momentum. If there are no nongravitational forces acting and if we take the time direction tangent to the world lines (46) becomes

$$\sum_{\text{masses}} \frac{d}{ds} v_{\mu\alpha\beta0}r^\beta \frac{dx^\alpha}{ds} = \sum_{\text{masses}} v_{\mu\alpha\beta0}R^\beta{}_{0\omega0} r^\alpha r^\omega. \quad (47)$$

We have applied (47) to the calculation of the irregular fluctuations in the period of rotation of the earth caused by incident gravitational radiation with a continuous spectrum. Under these conditions a straightforward calculation leads to the result

$$\langle I^2\rangle_{\text{Av}}/I_a{}^2 \approx 25\pi G t_{0r}\omega^{-2}c^{-3}. \quad (48)$$

Here $\langle I^2\rangle_{\text{Av}}$ is the mean square fluctuation in the earth's angular momentum; I_a is the angular momentum of rotation; t_{0r} is the total gravitational wave flux in ergs per square centimeter per second, assuming its Fourier transform to be concentrated near zero frequency; ω is the angular frequency of rotation. If we arbitrarily assume that all the earth's rotational anomalies are due to incident gravitational waves, t_{0r} is calculated to be 5×10^8 ergs per square centimeter per second. It is clear from this that the earth's rotation is not a useful detector unless the size of the anomaly can be reduced. The other astronomical anomalies lead to larger figures.

‡ *Note added in proof.*—Experimental work along these lines has begun recently. It is being carried out by Dr. David M. Zipoy and Mr. Robert L. Forward, in collaboration with the author. The piezoelectric effect gives enhanced sensitivity when a mass which is many acoustic wavelengths on a side is used. At low frequencies this is not important because a mass which is one half acoustic wavelength long is already quite large and may not be obtainable as a piezoelectric crystal. Excitation of resonant acoustic vibrations in a block of metal, in accordance with expressions (36) and (37) is being considered along with the arrangements of Figs. 2 and 3. Detection of the motion of the crystal ends is by the change in capacitance to a nearby electrode.

[13] A review of these developments has been published [J. Weber, Revs. Modern Phys. **31**, 681 (1959)].

[14] P. Dirac (private communication).

GENERATION OF GRAVITATIONAL WAVES

It would be very desirable to be able to generate gravitational waves with sufficient energy to be detected in the laboratory. A number of important experiments could be done.

For a spinning rod, Einstein,[1] and later Eddington,[2] gave the formula for the radiated power P_R as

$$P_R = 1.73\times10^{-59} I_m{}^2 \omega^6 \text{ ergs/sec.} \quad (49)$$

Here I_m is the moment of inertia and ω is the angular frequency. ω can be increased until the rod ultimately breaks. If we write the maximum value of ω in terms of the tensile strength and express the result in terms of the elastic modulus and strain, we obtain for the length l the formula

$$l = \lambda_s (2\delta)^{\frac{1}{2}}/\pi. \quad (50)$$

In (50), δ is the maximum allowed strain for the material and λ_s is the wavelength of sound in the rod at the angular frequency of rupture. The implication of (50) is that the wavelength of the gravitational waves which can be radiated by a rod is at least 1 000 000 times the length of the rod. Also the moment of inertia is limited to values less than about

$$10^{-3}\rho\lambda_s{}^5\delta^{5/2}/(12\pi^5). \quad (51)$$

In (51), ρ is again the density, and we are considering a fairly slender rod, for which the length is an order larger than the lateral dimensions. Employing (51) in (49) leads to

$$P_R < 4\times10^{-63}\rho^2 v_s{}^{10}\delta^5\omega^{-4} \text{ ergs/sec.} \quad (52)$$

(52) shows that contrary to the appearance of (49), low-frequency operation with large rods gives more radiation than high-frequency spinning of small rods. About 10^{-30} erg per second can be radiated by such a one-meter rod.

A new method for generation of gravitational waves is suggested by the field equations of general relativity. The source of the gravitational field is the stress-energy tensor. Time-dependent stresses can be produced electrically in a piezoelectric crystal, and these give rise to radiation. The weak-field solutions are entirely adequate for this kind of discussion since the metric is expected to differ from the Lorentz metric by perhaps one part in 10^{30}. The weak-field solutions are

$$\psi_\mu{}^\nu = h_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu h = (4G/c^4)\int r^{-1}(T_\mu{}^\nu)_{\text{retarded}}\, d^3x. \quad (53)$$

In (53), $T_\mu{}^\nu$ is the stress-energy tensor, r is the distance from the source to the observer, and $h_\mu{}^\nu$ is defined by $g_\mu{}^\nu = \delta_\mu{}^\nu + h_\mu{}^\nu$, where $\delta_{\mu\nu}$ is the Lorentz metric. The coordinate conditions required for (53) are $\partial\psi_\mu{}^\nu/\partial x^\nu = 0$.

In order to apply (53) to the problem of radiation by a crystal, we first assume that acoustic resonance is employed and that one-dimensional compressional waves are set up. The components of $T_\mu{}^\nu$ are then given by

$$T_0{}^0 \approx -\rho c^2[1-(V_p/v_s)\cos(\omega t)\cos(k_s x^3)] \times[1-U(x^3-l-A_0\cos\omega t) - U(-x^3-l-A_0\cos\omega t)], \quad (54)$$

$$T_0{}^3 = \rho V_p c \sin\omega t \sin k_s x^3[1-U(x^3-l-A_0\cos\omega t) - U(-x^3-l-A_0\cos\omega t)], \quad (55)$$

$$T_3{}^3 = -\rho V_p v_s \cos\omega t \cos k_s x^3[1-U(x^3-l-A_0\cos\omega t) - U(-x^3-l-A_0\cos\omega t)]. \quad (56)$$

In these expressions it is assumed that the waves travel in the direction x^3; V_p is the particle velocity; v_s is the sound velocity; U is a step function defined by $U(x)=0$ for $x<0$, $U(x)=1$ for $x>0$. A_0 is the vibration amplitude of the free end. Making use of these expressions in (46) and employing the Einstein form of the stress-energy pseudotensor enables the total radiated power to be calculated. The maximum value of A_0 is determined by the maximum strain which is allowed before rupture takes place. The maximum strain corresponds to a value of A_0 about four orders smaller than the acoustic wavelength λ_s.

For quartz the result for acoustic resonance is

$$P < [(16/15)G\rho^2 S^2 v_s (v_s/c)^5 \times 10^{-6}]_\omega + [(\pi/15)G\rho^2 S^2 n_\lambda{}^2 v_s (v_s/c)^5\times10^{-12}]_{2\omega} \text{ ergs/sec.} \quad (57)$$

In (57), S is the cross-sectional area, the term with subscript ω gives the radiated power at the fundamental frequency, and the second term gives the power radiated at twice the fundamental frequency. The resonator must be a multiple n_λ of a half acoustic wavelength long. The first term of (57) is seen to be independent of n_λ. This is because for $n>1$ the crystal is essentially an assemblage of electric quadrupoles with a given quadrupole driven out of phase with respect to its nearest neighbors. The resulting fundamental-frequency radiation is approximately that of a single quadrupole. Each half-wave section has an equivalent moment $\sim MA_0\lambda_s/2\pi$, where M is the mass of a single (acoustic) half-wave resonator; A_0 and λ_s are defined by (54), (55), and (56). If a large number of separate resonators are located within a region of dimensions less than a gravitational-wave half wavelength, the radiated power will be proportional to the square of the total number of crystals. (57) depends on the frequency through the requirement that each resonator be a multiple of a half wavelength long. In order to radiate 10^{-15} erg per second at the fundamental frequency, 10^6 crystals would be needed, each one-half acoustic wavelength thick and with a cross-sectional area of 50 cm^2. A complex phasing arrangement would be needed in order to drive the array properly.

It appears better to suppress the acoustic resonance vibrations and create, by the piezoelectric effect or by electrostriction, mechanical stress components which do not reverse sign every acoustic half-wavelength. In order to see that this is possible it is only necessary to study the solution given in the literature[15] for the component T_{11} of the stress in a longitudinally vibrating crystal with x^3 as the thickness direction, with a conducting plating on the faces normal to x^3. The thickness is assumed small, for simplicity, and an externally applied electric field parallel to x^3 drives the crystal. The component T_{11} is then

$$T_{11}=(d_{131}E_3 \sin\omega t)\{\sin[\omega(L-x)/v_s] +\sin[\omega x/v_s]-\sin[\omega L/v_s]\}/\sin(\omega L/v_s). \quad (58)$$

In (58), d_{ijk} is the tensor relating stress to electric field, E_3 is the electric field, and L is the total length. At resonance $\omega L/v_s=\pi$ and losses would have to be taken into account in (58) by a hyperbolic function in the denominator. However, off resonance, for example at $\omega L/v_s=\pi/2$, (58) becomes

$$T_{11}=d_{131}E_3[\cos(\omega x/v)+\sin(\omega x/v)-1]\sin\omega t. \quad (59)$$

(59) is seen to have a component $-d_{131}E_3 \sin\omega t$ which does not reverse sign every half acoustic wavelength. A single large crystal, driven in this manner, will then give volume-integrated stress components which are large. The radiated power would be expected to be

$$P_R\approx GP_{\max}{}^2\lambda^4\pi^2/(120c^3). \quad (60)$$

In (60), $P_{\max}$ is the effective tensile strength and again λ is the gravitational wave wavelength.

Waves one meter long could be radiated by a crystal with dimensions about fifty centimeters on a side. If it is driven just below the breaking point, each crystal would radiate $\sim 10^{-13}$ erg per second, assuming $P_{\max}$ to be its static published value. Single-crystal detectors of the type considered earlier may detect a power of about 10^{-3} erg per second at these wavelengths. A large gap therefore still exists between what can be generated and what can be detected in a small laboratory. Complex detection and generation arrays can narrow this gap. Large amounts of electrical power would have to be dissipated in crystals driven to the fracture point—perhaps 10^8 watts in a crystal fifty centimeters on a side. This might well be substantially reduced if low-temperature operation can be achieved. Also one might hope that low-temperature high-frequency operation might raise the effective tensile strengths. All of these issues need careful experimental investigation. If the numbers employed earlier cannot be improved upon, it would require a crystal roughly one hundred meters on a side, and a large detection system, to generate and detect the gravitational radiation. We are not proposing that this be done. We are suggesting some investigations of crystals at low temperatures, for the purpose of exploring the possibility of improvements.

[15] W. P. Mason, *Piezoelectric Crystals and Their Applications to Ultrasonics* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1950), p. 64.

CONCLUSION

The detectors which have been proposed are sufficiently good to search for interstellar gravitational radiation. Further advances are necessary in order to generate and detect gravitational waves in the laboratory. If we compare a crystal which is excited as described above with a spinning rod of the same linear dimensions, we find that the radiation from the crystal is about seventeen orders greater and the frequency radiated by the crystal about one million times greater. If both the rod and the crystal radiate at the same frequency the crystal radiation is about thirty-nine orders greater than that of the rod. We acknowledge, with thanks, the helpful criticism of F. A. E. Pirani, P. G. Bergmann, and J. A. Wheeler. We have had very helpful discussions with R. H. Dicke.