Single Particle Energies in the Theory of Nuclear Matter*

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The excitation spectrum of real and virtual excitations is discussed from the point of view of the perturbation series expansion. The effect of the rearrangement energy terms in the single-particle virtual excitation spectrum is evaluated and seen to be very small. It is also noted that the virtual particle energies are real due to the effects of off-energy-shell propagation.

Starting from the reaction matrix approximation for the ground state energy, the energies of real particle and hole excitations are defined which satisfy the separation energy theorem. It is shown that the energies so defined depend on an infinite sequence of terms giving the rearrangement energy for multiple excitations. Appropriate termination and evaluation of this series are discussed.

I. INTRODUCTION

IT has been shown by Brueckner[1] and by Hugenholtz and van Hove[2] that an approximate theory of a many-body system of strongly interacting particles will usually lead to a violation of the equality between the mean binding energy and the separation energy of the last particle. The equality follows quite generally from the linear relation between energy and total particle number for a saturating system. These authors have analyzed the origin of this difficulty from the particular viewpoint of the many-body theory in the approximation of Brueckner and also from the general viewpoint of perturbation theory.

In this paper we shall first discuss from a perturbation viewpoint[3] the excitation spectrum of real or energy conserving excitations and of virtual excitations, the latter spectrum determining the energies required to evaluate the reaction-matrix. Such virtual excitations, since they typically exist only for the very short times characteristic of two-body collisions, are changed only slightly by rearrangement effects. A consequence of this is that the convergence of the cluster expansion for the ground-state energy is nearly unaffected by rearrangement terms in the single-particle virtual excitation spectrum.

We next consider the definition of real particle and hole energies, starting from the reaction matrix approximation for the ground state energy. We shall show within this framework how single-particle energies can be defined which automatically satisfy the separation energy theorem and also how they may be evaluated.

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[1] K. A. Brueckner, Phys. Rev. **110**, 597 (1958).

[2] N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).

[3] D. J. Thouless [Phys. Rev. **112**, 906 (1958)] has also given a detailed discussion of particle energies from the viewpoint of perturbation theory.

II. SINGLE-PARTICLE ENERGIES IN PERTURBATION THEORY

In this section we consider the single-particle energies from the viewpoint of perturbation theory. This discussion will not be intended to lead to a quantitative theory, which can be done only starting from a more rigorous formulation, but rather to indicate some characteristic features of the spectra. In particular we shall be interested in showing the changes in the energies which occur when the off-energy-shell effects typical of virtual excitations are included.

The single-particle energy in the first two orders of perturbation theory is given in Fig. 1, not including the effects of exchange explicitly. The matrix elements corresponding to a and b in Fig. 1 are included in the usual definition of the K-matrix.[4] The single-particle energy computed in this approximation will be complex, since an excited particle can transfer energy to the unexcited medium and bring about energy-conserving transitions. These transitions lead to a finite lifetime and hence a complex energy.

The process represented in Fig. 1(c) arises from the identity of the particles. It can be viewed as the correction for the alteration of effects of the exclusion principle on a ground-state pair. In the presence of a particle in a formerly empty state, a ground-state pair can no longer make transitions into this state. This is shown in Fig. 1(d) which gives an equivalent diagram, the matrix element corresponding to this diagram having the same structure as that corresponding to

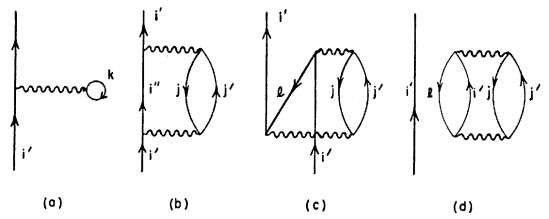

FIG. 1. Single-particle energies in the first [(a)] and second [(b)−(d)] orders of perturbation theory.

[4] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

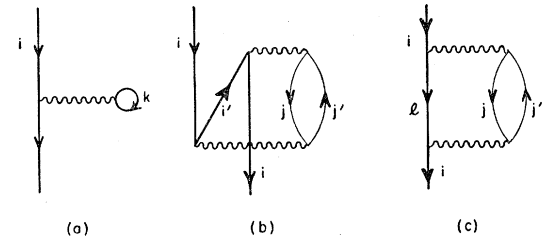

FIG. 2. Hole energies in the first [(a)] and second [(b) and (c)] orders of perturbation theory.

Fig. 1(c). This correction term has been called a rearrangement term,[5] since it arises from the effect of the change in the state of one particle on the energies of the other bound particles. The rearrangement corrections vanish at low density in comparison with the K-matrix terms in the energy, since, in the former terms, an additional intermediate state is involved in which transitions into levels of the Fermi gas are required.

The energy of a hole in the first two orders of perturbation theory is given in Fig. 2. Again the matrix elements corresponding to Fig. 2(a) and Fig. 2(b) are included in the definition of the K-matrix. In the hole energy, in contrast to the particle energy, the K-matrix terms are real since no intermediate state conserving energy is possible. The matrix element corresponding to Fig. 2(c) is similar to that corresponding to Fig. 1(c) since again it occurs only because of the identity of the particles. The presence of a hole allows transitions to occur from other hole states and so alters the energy of other bound particles. This type of term is consequently also called a "rearrangement" term and vanishes at low density compared to the other K-matrix terms. The hole energy is complex due to the occurrence of this term since the hole-hole transitions which it includes can conserve energy and so lead to a finite hole life time.

We next examine the change in the matrix elements giving the single-particle energies when they appear as insertions into the ground-state energy, which is represented by the second order term of Fig. 3(a). The contributions from the insertions represented by Fig. 3(b) and Fig. 3(c) are similar to Fig. 1(a) and Fig. 1(b), except that the system is off-the-energy-shell by the excitation energy

$$\Delta E=E_i+E_k-E_{i'}-E_{k'}. \tag{1}$$

Thus the K-matrix term represented by Fig. 3(c) must be computed "off-the-energy-shell." The off-energy-shell corrections arising from this change in the K-matrix have been discussed in detail and evaluated by Brueckner and Gammel.[4] There is a considerable shift in the energy resulting from this effect and also, more important, the K-matrix so computed is real since the off-energy-shell energy shift makes intermediate-state energy-conserving transitions impossible.

The rearrangement energy corrections to the particle energy are given in Fig. 3(d). We also give three other topologically equivalent diagrams in Fig. 3(e), Fig. 3(f), and Fig. 3(g). These give corrections to the energy of the same form as Fig. 3(d) and, to be consistent, must be included. They can be generated from the diagram of Fig. 3(d) simply by reordering the time sequence of the vertices. The separate appearance of these diagrams is a characteristic complication of the introduction of hole theory and is readily avoided in, for example, the ordinary Rayleigh-Schrödinger formulation of the perturbation theory.[6]

Before collecting all terms of rearrangement energy form together, we first note that Fig. 3(d) and Fig. 3(g) as well as Fig. 3(f) and Fig. 3(e) arise from each other simply by reflection of the diagram about its vertical axis. Both sets of diagrams should therefore not be separately counted when summing freely over all particle coordinates. The rearrangement terms of Fig. 3(d) and Fig. 3(e) can easily be written out explicitly. They combine to give the simple result

$$v_{ik,i'k'}\frac{1}{E_i+E_k-E_{i'}-E_{k'}}\left[v_{jl,i'j'}\frac{1}{E_j+E_l-E_{i'}-E_{j'}}v_{i'j',jl}\right]\times\frac{1}{E_i+E_k-E_{i'}-E_{k'}}v_{i'k',ik}. \tag{2}$$

The quantity in the square bracket is precisely the matrix element corresponding to Fig. 1(c), with no correction in the energy denominator for the excitation energy of the rest of the diagram. This decoupling of the energy denominators is a consequence of the combination of the various contributions coming from different time sequences. We thus obtain the important result that the matrix element giving the particle rearrangement energy is unaltered by off-energy-shell effects when inserted into a ground-state diagram.

Consider next the hole energy. The insertions into the ground state energy are given in Fig. 4. The

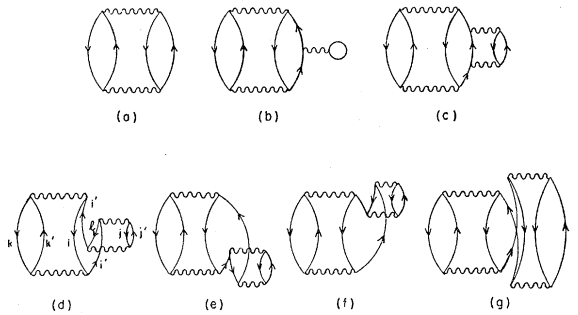

FIG. 3. The ground-state energy diagram (a) with first (b) and second (c)–(g) order corrections to the virtual particle energy.

[5] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).

[6] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).

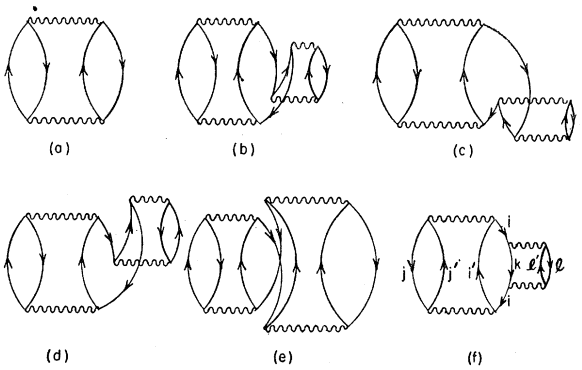

FIG. 4. The ground-state energy diagram with first (a) and second (b)–(f) order corrections to the virtual hole energy.

K-matrix hole energy diagrams are given in Fig. 4(a–e). Again we see that Fig. 4(d) and Fig. 4(e) give the same matrix elements as Fig. 4(b) and Fig. 4(c). The matrix elements of Fig. 4(b) and Fig. 4(c) combine to give a K-matrix insertion for the hole energy which is not altered by the excitation energy of the rest of the diagram. Thus the K-matrix contribution to the hole energies is not affected by off-energy-shell propagation.

The hole rearrangement term is given in Fig. 4(f). In this case, in contrast to the particle rearrangement energy, there are no other matrix elements of similar structure but different time sequence. Consequently the matrix element is

$$v_{ij,i'j'}\frac{1}{E_i+E_j-E_{i'}-E_{j'}}\times\left[v_{kl,ii'}\frac{1}{E_k+E_l+E_j-E_{i'}-E_{j'}-E_{l'}}v_{il',kl}\right]\times\frac{1}{E_i+E_j-E_{i'}-E_{j'}}v_{i'j',ij}, \quad (3)$$

where the quantity in the square brackets differs from the rearrangement energy of a hole [Fig. 2(c)] since the denominator includes the excitation energy of the rest of the diagram. The presence of this energy removes the possibility of energy-conserving transitions and the hole-rearrangement energies become real.

To summarize these results, we observe that in the first two orders of perturbation theory, the contributions to the single particle energies have the following properties:

A. Real excitation: The particle K-matrix energy is complex; the particle rearrangement energy is real; the hole K-matrix energy is real; the hole rearrangement energy is complex.

B. Virtual excitations as ground state insertions: The particle K-matrix goes off-energy-shell and becomes real; the particle rearrangement energy is unaffected; the hole K-matrix energy is unaffected; the hole rearrangement energy goes off-energy-shell and becomes real.

This asymmetrical behavior of particles and holes in the K-matrix approximation has already been taken into account by Brueckner and Gammel[4] in their study of nuclear matter. The asymmetry is also trivially apparent in the Rayleigh-Schrödinger form of the linked cluster expansion.[6]

We now note some consequences of these corrections to the real excitation energies. The off-energy-shell corrections to the K-matrix particle energies are repulsive and give an appreciable upward shift to the single particle energy for states above the Fermi surface. On the other hand, the rearrangement corrections are repulsive and are made less so for the hole energies by the off-energy-shell effects (to which we return in more detail later). Consequently the virtual hole and particle energies are shifted in opposite directions relative to the energies of real excitations.

Another consequence of the off-energy-shell corrections is to alter somewhat the singularity near the Fermi surface which may for some states be present for attractive forces. If, as has been done in the studies of nuclear matter, the excitation energy of Eq. (1) is replaced by an average value, then a large energy gap appears in the virtual excitation spectrum at the Fermi surface. This prevents the occurrence of "dangerous denominators" in the K-matrix equation which in this approximation can then be solved without difficulty. These comments are, of course, not intended to imply that the characteristic singularities near the Fermi surface associated with attractive forces are necessarily absent, but only that in a practical calculation, the difficulty well may not appear.

We finally turn to the question of the magnitude of the effects of the rearrangement terms on the ground-state energy. To do so we restrict ourselves to the second order rearrangement terms only, although higher order terms also contribute appreciably. A typical term of third order, for example, is shown in Fig. 5(a) together with its appearance in Fig. 5(b) as an insertion into the second order ground-state diagram. This term can be viewed as a correction for the depletion of the Fermi gas as a result of interactions, the particle with momentum i' in Fig. 5(a) interacting not with the unper-

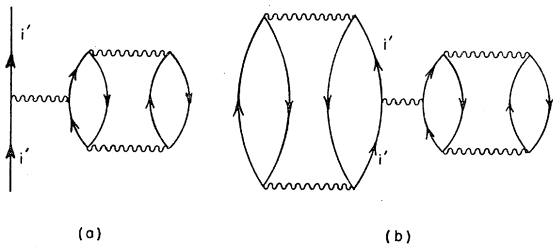

FIG. 5. Third order rearrangement energy diagram (a) and its insertion into a ground-state diagram (b).

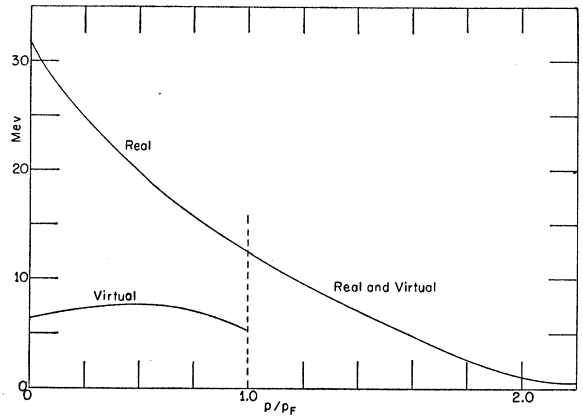

FIG. 6. Rearrangement energy as a function of momentum, for real and virtual particle and hole excitations.

turbed Fermi gas but instead with particles and holes excited by the interaction. An exactly equivalent description, which will turn out to be more useful in the next section, is that the effective mass of the bound particles is altered by the presence of an excitation. The effective mass change then in turn causes a change in the bound-particle binding.

Restricting ourselves to the second order rearrangement effects, we have evaluated their magnitude and energy dependence using a Yukawa interaction of meson Compton wavelength range. We have also for simplicity normalized the rearrangement energy for the excitation of a real hole at the Fermi surface to the computed value of 12.5 Mev as determined by Brueckner and Gammel.[4] To determine the rearrangement energy of the virtual holes, we have set the mean excitation energy of Eq. (1) equal to the full excitation energy of the Fermi gas, i.e.,

$$(E_i+E_k-E_{i'}-E_{k'})_{\mathrm{Av}}=-2[E(p_F)-E(0)]\equiv-\Delta. \quad (4)$$

This is probably an underestimate since, due to the great strength of the interactions, the typical excitation is considerably greater than the energy spread of the Fermi gas.

In this approximation, the rearrangement energy appearing in Eq. (3) is

$$E_R(i)=\sum_{kll'} v_{kl,il'}{}^2\frac{1}{E_k+E_l-E_i-E_{l'}-\Delta}. \quad (5)$$

We use the effective-mass approximation for the energies, i.e.,

$$E_i=p_i^2/2M^*+\text{constant}.$$

The evaluation of Eq. (5) is carried out in Appendix A. The result is given in Fig. 6, for on-energy-shell and off-energy-shell propagation. In the former case, we give only the real part of the energy, defined by taking the principal part of the singular integral involved. For the hole states, the off-energy-shell rearrangement energy is reduced by a factor of about 2.8 at the Fermi surface and 5.3 in the lowest state. The virtual rearrangement energy also becomes nearly constant at about 7 Mev, changing by only about 10% from the Fermi surface to the lowest state.

The correction to the energy spectrum arising from the rearrangement effects is given in Fig. 7, taking for the uncorrected spectrum that obtained by Brueckner and Gammel.[4] Their results are given for propagation which above the Fermi momentum is off-the-energy shell by the amount given in Eq. (3), i.e., the full excitation energy of the Fermi gas. The appearance of a marked gap at the Fermi surface in the virtual spectrum is apparent. We also note that the rearrangement effects are rather smaller than the other off-energy-shell effects. The effect on the ground-state energy of these shifts in the K-matrix energies has been shown by Brueckner and Gammel[4] to be somewhat less than 1 Mev, so that the smaller rearrangement effects must certainly give a shift in the ground-state energy of no more than a fraction of a Mev. These corrections are of at least fourth order in the linked cluster expansion for the energy, and their magnitude thus is consistent with the estimates of convergence of the linked cluster series, which in third order gives a few tenths of a Mev correction.

To conclude this section, we summarize our results which are that (a) the virtual particle and hole energies are real due to the effects of off-energy-shell propagation, and (b) that rearrangement effects in the virtual excitation spectrum can be neglected within the accuracy of the K-matrix approximation, which determines the ground-state energy with an error of less than a Mev.

III. NONPERTURBATION THEORY OF HOLE AND PARTICLE ENERGIES

As we have seen within the framework of perturbation theory in the preceding section and as also has been shown in other work on the nuclear matter problem,[6,7] a good approximation to the ground-state energy can be obtained in the K-matrix approximation. The total energy in this approximation is

$$E_{\mathrm{total}}=\sum_i n_i\frac{k_i^2}{2M}+\tfrac{1}{2}\sum_{ij} n_i n_j(K_{ij,ij}-K_{ij,ji}), \quad (6)$$

with the K-matrix defined by the equation

$$K_{kl,ij}=v_{kl,ij}+\sum_{mn} v_{kl,mn}\frac{(1-n_m)(1-n_n)}{E_i+E_j-E_m-E_n}K_{mn,ij}. \quad (7)$$

In Eq. (6) and Eq. (7), n_i is the expectation value of the number operator for the state i, and is equal to one for an occupied state and 0 for an empty state. The effect of the appearance of the number operators in

[7] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

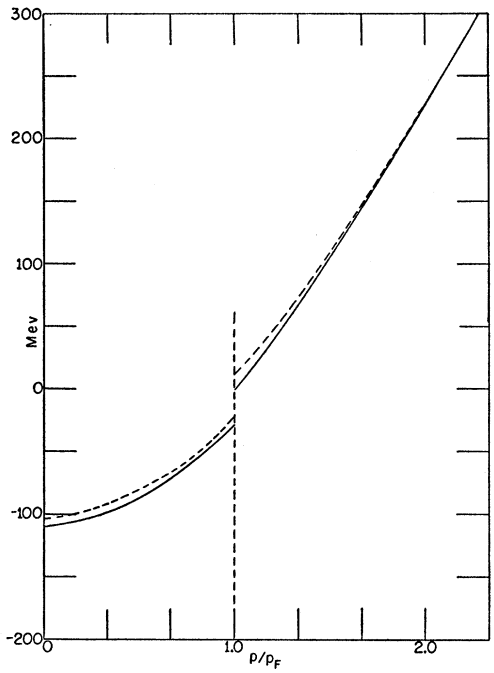

FIG. 7. Rearrangement corrections to virtual excitation spectrum. The solid curve gives the self-consistent spectrum calculated by Brueckner and Gammel in the K-matrix approximation. The dashed curve shows the rearrangement correction.

Eq. (6) and Eq. (7) is to restrict the sums to states in the Fermi gas or above, respectively.

The single-particle energies as customarily defined in the existing applications of the theory to nuclear matter and liquid He^3 are (ignoring the complications of off-energy-shell propagation)

$$E_i=k_i^2/2M+\sum_j n_j(K_{ij,ij}-K_{ij,ji}). \quad (8)$$

This definition, as we have shown in the perturbation approximation of the preceding section, is a sufficiently accurate definition of the virtual particles and hole energies which are required in Eq. (7) to evaluate the K-matrix. The definition leads, however, to violation of the expected equality

$$E(k_F)=(1/N)E_{\text{total}}. \quad (9)$$

We now seek a more exact definition, based on Eq. (6), for the energies of real excitations. As we have shown in Sec. II, these will in general be somewhat shifted from the energies as defined in Eq. (8).

Instead of Eq. (8), we now adopt the following physically reasonable definition of the real single-particle energies. We regard the energy E_{total} as a function of the occupations numbers n_i, and define the energy of a particle in state α as

$$E_\alpha^{(R)}=\partial E_{\text{total}}/\partial n_\alpha. \quad (10)$$

This clearly is the energy required to remove a particle from the system, leaving a hole in the state α, and so is equivalent (within the K-matrix approximation for the total energy) at the Fermi surface to the definition of energy given by Hugenholtz and Van Hove. We shall now show that, using the definition of Eq. (9), we find new terms in the single-particle energy not included in the previous definition of Eq. (8). These terms correspond in the leading orders to the corrections discussed in Sec. II. More generally, additional terms appear as an infinite series of corrections which must be terminated to give a closed expression.

To proceed, we carry out the differentiation indicated in Eq. (9). The result is

$$E_\alpha^{(R)}=E_\alpha+\tfrac{1}{2}\sum_{ij} n_i n_j \frac{\partial}{\partial n_\alpha}(K_{ij,ij}-K_{ij,ji}), \quad (11)$$

with E_α as defined in Eq. (8). The rearrangement effects appear in the last term of Eq. (11). To evaluate this term, we use the definition of the K-matrix in Eq. (7), and find

$$\frac{\partial K_{ij,ij}}{\partial n_\alpha}=\sum_{mn} v_{ij,mn}\frac{\partial}{\partial n_\alpha}\left\{\frac{(1-n_m)(1-n_n)}{E_i+E_j-E_m-E_n}K_{mn,ij}\right\}. \quad (12)$$

This equation is of a form often encountered in scattering theory, and is readily solved with the result

$$\frac{\partial K_{ij,ij}}{\partial n_\alpha}=\sum_{mn} K_{ij,mn}'\left[\frac{\partial}{\partial n_\alpha}\frac{(1-n_m)(1-n_n)}{E_i+E_j-E_m-E_n}\right]K_{mn,ij}, \quad (13)$$

where

$$K_{ij,mn}'=v_{ij,mn}+\sum_{kl} v_{ij,kl}\frac{(1-n_k)(1-n_l)}{E_i+E_j-E_k-E_l}K_{kl,mn}'. \quad (14)$$

That Eq. (13) is the solution of Eq. (12) can be seen most easily by substituting Eqs. (13) and (14) into Eq. (12).

Carrying out the indicated differentiation in Eq. (12) gives

$$\begin{aligned}\frac{\partial K_{ij,ij}}{\partial n_\alpha}=&-2\sum_m K_{ij,m\alpha}'\frac{(1-n_m)}{E_i+E_j-E_m-E_\alpha}K_{m\alpha,ij}\\&-\sum_{mn} K_{ij,mn}'\frac{(1-n_m)(1-n_n)}{(E_i+E_j-E_m-E_n)^2}\\&\times\left[\frac{\partial}{\partial n_\alpha}(E_i+E_j-E_m-E_n)\right]K_{mn,ij}. \quad (15)\end{aligned}$$

The first term in Eq. (15) arises from the change in the exclusion effect as the state α is emptied, a new intermediate state transition becoming available. This term, which is second order in the reaction matrix, corresponds in the perturbation approximation to the second order rearrangement term already considered in Sec. II. The second term in Eq. (15) comes from the change in the self-consistent spectrum or, equivalently, in the effective mass, as a particle is removed from state α. To evaluate the derivative we use Eq. (8) for E_α. The

result is

$$\frac{\partial E_i}{\partial n_\alpha}=K_{i\alpha,i\alpha}-K_{i\alpha,\alpha i}+\sum_j n_j\frac{\partial}{\partial n_\alpha}(K_{ij,ij}-K_{ij,ji}). \quad (16)$$

More generally, if the rearrangement effects had been included in the virtual hole and particle energies, Eq. (16) would also include the second derivatives of the K-matrix with respect to the number operators. The equation for the second derivative then would lead to an equation involving the third derivative. The first derivative is related to the rearrangement energy due to single-particle excitation and can be pictured as the diagrams in Figs. 1–5. The higher derivatives are similarly related to the rearrangement energy for multiple particle removal. The coupled nature of this succession of equations makes it impossible to utilize exactly the definition of particle energy in Eq. (10) and consequently the relationship between $E(k_F)$ and $E_{Av}=(1/N)E_{\text{total}}$ cannot be exactly satisfied if we start from the K-matrix approximation.

To reduce the problem to manageable size, we make the approximation of neglecting the rearrangement energy of more than one particle. The justification for this approximation lies in the rapid convergence of the K-matrix method. Previous work[4] has shown that the one-particle rearrangement energy shifts the single-particle energy at the Fermi surface by about 20%. Similarly, the two-particle rearrangement energy may be expected to shift the single-particle energy by the order of a few percent. The neglect of the two-particle (and higher) rearrangement energies is also equivalent to the assumption that these energies do not depend on the momentum of the particles involved, since in this case the rearrangement terms cancel when energy differences are computed.

We now rewrite Eq. (16), dropping all but the leading terms, as

$$\partial E_i/\partial n_\alpha \cong K_{i\alpha,i\alpha}-K_{i\alpha,\alpha i}, \quad (17)$$

and find for Eq. (14)

$$\frac{\partial K_{ij,ij}}{\partial n_\alpha}=-2\sum_m{}' K_{ij,m\alpha}\frac{(1-n_m)}{E_i+E_j-E_m-E_\alpha}K_{m\alpha,ij}$$
$$-\sum_{mn}{}' K_{ij,mn}\frac{(1-n_m)(1-n_n)}{(E_i+E_j-E_m-E_n)^2}(K_{i\alpha,i\alpha}+K_{j\alpha,j\alpha}$$
$$-K_{m\alpha,m\alpha}-K_{n\alpha,n\alpha}+\text{exchange terms})K_{mn,ij}. \quad (18)$$

A similar equation holds for the derivative of the exchange term $K_{ij,ji}$. In the form of Eq. (18), the correspondence of the last term to the third order term described in Sec. II and Fig. 5 is now obvious.

In the evaluation of the first term in Eq. (18), a problem previously encountered arises in the possible singularity of the energy denominator. If a particle is removed from the state α, a transition may occur in which the intermediate state energy is the same as the initial energy. In Eq. (18) this would mean that the energies E_i+E_j and E_m+E_α may be equal, which was not possible in previously evaluating the single-particle energy. The occurrence of this singularity is, of course, related to the finite lifetime of the real hole and particle states, so that the singularity should be defined by the addition of a small positive pure imaginary term to the energy denominator.

The solution to Eq. (11) and Eq. (18) includes the second and third order terms given in Figs. 1–5 together with the higher order terms which would arise from a perturbation expansion of the K-matrices and energy denominators in these equations. The energies so obtained will at the Fermi surface satisfy Eq. (9) very accurately and so, within the basic framework of the K-matrix approximation, provide the basis of an improved determination of the properties of the systems.

To apply these methods in the actual solution of the basic K-matrix equations is of course nontrivial. Even if the perturbation solution only is needed, the second and third order terms in the rearrangement energy can be estimated to give comparable contribution and can be evaluated only if detailed knowledge of the diagonal and nondiagonal elements of the K-matrix is available. Fortunately, the differentiation of the K-matrix in Eq. (12) can be carried out readily in the course of the numerical solution of the K-matrix equation, without using the explicit result of Eq. (18). This can be done by making a definite shift in the population of the Fermi gas near the desired momentum and so determining $\partial K_{ij,ij}/\partial n_\alpha$ of Eq. (12) from the finite shift in the K-matrix. Such differencing is easily carried out within the framework of the general numerical method. The derivative so evaluated can then be used to correct the single-particle energies according to Eq. (11), and the process repeated to improve the accuracy of the result. This study is now being carried out: the results will be reported in a separate communication.

APPENDIX A. EVALUATION OF THE REARRANGEMENT ENERGY [EQ. (5)]

The expression we wish to evaluate is

$$E_R(p_i,\Delta)=\sum_{kll'} v_{kl,il'}{}^2\frac{1}{E_k+E_l-E_i-E_{l'}-\Delta}. \quad (A.1)$$

$v_{kl,il'}$ is the Fourier transform of the two-body interaction. It depends solely on the momentum transfer between the vacant state l and the excited state l'. Thus a natural coordinate to take is

$$\mathbf{x}\equiv(\mathbf{p}_l-\mathbf{p}_{l'}). \quad (A.2)$$

In terms of x, for a Yukawa potential of the form $V(r)=-V_0e^{-r/\alpha}/r$, we find (for units of $\hbar=1$) $v(x)=-f(x)/\Omega$, with $f(x)=-4\pi V_0/(x^2+\alpha^2)$.

Substituting Eq. (A.2) in Eq. (A.1) and summing over the state k yields

$$E_R(p_i,\Delta)=\sum_{xl}\frac{1}{\Omega^2}\frac{f^2(x)M^*}{-\mathbf{p}_i\cdot\mathbf{x}+\mathbf{p}_l\cdot\mathbf{x}-M^*\Delta}. \quad (A.3)$$

Replacing the summations of Eq. (A.3) by integrations, we obtain

$$E_R(p_i,\Delta)$$

$$=\frac{1}{(2\pi)^6}\int_{\substack{|\mathbf{p}_i-\mathbf{x}|\leqq p_F\\ |\mathbf{p}_l-\mathbf{x}|\geqq p_F}}\int d\mathbf{x}d\mathbf{p}_l\frac{f^2(x)M^*}{\mathbf{x}\cdot\mathbf{p}_l-\mathbf{x}\cdot\mathbf{p}_i-M^*\Delta}. \quad (A.4)$$

The restrictions on the integrations arise from the conditions that $p_k\leqq p_F$ and $p_{l'}\geqq p_F$. To evaluate the integral over $p_{l'}$, we make use of the rapid convergence of the integral over x. If x is much smaller than p_F, then

$$p_l^2-2p_lx\mu\geqq p_F^2, \quad (A.5)$$

where $\mu=\cos(\mathbf{p}_l,\mathbf{x})$. If x is small, then Eq. (A.5) together with the condition that $p_l\leqq p_F$ shows that

$$p_l=p_F-O(x).$$

From Eq. (A.5), to order x, we then find

$$p_l\geqq p_F+\mu x, \quad (A.6)$$

and for this inequality to hold, μ must always be negative. Thus we can rewrite Eq. (A.4) approximately as

$$E_R(p_i,\Delta)=\frac{M^*}{(2\pi)^6}\int_{|\mathbf{p}_i-\mathbf{x}|\leqq p_F}f^2(x)d\mathbf{x}$$

$$\times\int_{-1}^{0}\frac{2\pi}{p_Fx\mu-\mathbf{x}\cdot\mathbf{p}_i-M^*\Delta}\int_{p_F+\mu x}^{p_F}p_l^2dp_l. \quad (A.7)$$

The angular integral is now easily evaluated, with the result (we now drop constants since we intend to normalize to the value at the Fermi surface)

$$E_R(p_i,\Delta)=\text{constant}\int_{|\mathbf{p}_i-\mathbf{x}|\leqq p_F}d\mathbf{x}\,f^2(x)$$

$$\times\left[1-\frac{\mathbf{x}\cdot\mathbf{p}_i+M^*\Delta}{xp_F}\ln\frac{xp_F+\mathbf{x}\cdot\mathbf{p}_i+M^*\Delta}{\mathbf{x}\cdot\mathbf{p}_i+M^*\Delta}\right]. \quad (A.8)$$

At the Fermi surface, if $\Delta=0$, then

$$E_R(p_F,0)=\text{constant}2\pi\int_0^{2p_F}x^2f^2(x)dx$$

$$\times\int_0^1d\mu'\left[1-\mu'\ln\frac{1+\mu'}{\mu'}\right], \quad (A.9)$$

where $\mu'=\cos(\mathbf{p}_i,\mathbf{x})$. This integral can be evaluated analytically and gives

$$E_R(p_F,0)=\text{constant}\frac{\pi}{2\alpha}\left[\tan^{-1}\left(\frac{2p_F}{\alpha}\right)\right.$$

$$\left.-\frac{2\alpha p_F}{\alpha^2+4p_F^2}\right]. \quad (A.10)$$

If we take the normal nuclear density and the meson Compton wavelength for the Yukawa range, then $p_F/\alpha\cong 2$. Equation (A.10) then fixes the value of the multiplicative constant.

The evaluation of $E_R(p_i,\Delta)$ for other values of p_i and Δ is now straightforward and will not be given in detail.