Partial-Wave Dispersion Relations for the Process $\pi+\pi \rightarrow N+\bar{N}$*

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The problems of pion-nucleon and nucleon-nucleon scattering and nucleon electromagnetic structure involve the matrix element for two pions producing a nucleon-antinucleon pair. By use of the Mandelstam representation we are able to write dispersion relations for the partial-wave scattering amplitudes of this process. In the low-energy range these dispersion relations can be transformed into integral equations whose kernels are simply related to pion-nucleon and pion-pion scattering amplitudes.

I. INTRODUCTION

A METHOD for calculating the behavior of systems of strongly interacting particles has been developed recently by Chew and Mandelstam,[1] and has been applied by them to the problem of pion-pion scattering. Their procedure is based on Mandelstam's generalization of dispersion relations,[2] which prescribes a method of analytic continuation of scattering amplitudes into the complex plane as a function of both the energy and momentum-transfer variables. This simultaneous extension of both variables into the complex plane permits one to write dispersion relations for partial-wave amplitudes. Applying the unitarity condition and using the "effective-range" approximation—i.e., determining the behavior of an analytic function by considering only near-by singularities—one can transform these partial-wave dispersion relations into a system of integral equations.

We have applied this method to the calculation of the matrix element for the production of a nucleon-antinucleon pair by two pions. This matrix element enters into many of the problems of strong interactions, such as pion-nucleon scattering and photoproduction, the nucleon-nucleon interaction, and the nucleon electromagnetic structure. In pion-nucleon scattering the structure of the Mandelstam representation forces one to consider simultaneously the three processes shown in Fig. 1. In the nucleon-nucleon interaction problem, knowledge of the process $\pi+\pi \leftrightarrow N+\bar{N}$ will permit calculation of the two-pion exchange contribution. In the nucleon electromagnetic structure this process in the state of total angular momentum one, together with the pion form factor, dominates the isotopic vector properties.[3] This application will be discussed in the following paper.

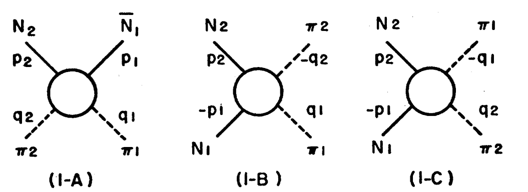

FIG. 1. The three channels of the two-nucleon, two-pion problem.

In Sec. II the kinematics and isotopic spin analysis are treated. The partial-wave decomposition is carried out in Sec. III. We follow there the work of Jacob and Wick,[4] in terms of helicity states rather than orbital angular momenta. In Sec. IV the Mandelstam representation and its properties are described, and in Sec. V they are used to study the structure of the singularities of the partial-wave amplitudes. In Sec. VI the dispersion relations are transformed into integral equations and a method of approximate solution in the low-energy (unphysical) region is given.

II. KINEMATICS

Let the four-vector momenta of the pions be q_1 and q_2, and those of the antinucleon and nucleon be p_1 and p_2, respectively [Fig. 1(a)]. Define the variables[5]

$$t=-(q_1+q_2)^2=4(q^2+\mu^2)=4(p^2+m^2), \quad (2.1a)$$

$$s=-(p_1-q_1)^2=-p^2-q^2+2pq\cos\theta, \quad (2.1b)$$

$$\bar{s}=-(p_1-q_2)^2=-p^2-q^2-2pq\cos\theta, \quad (2.1c)$$

where q and p are the magnitudes of the pion and nucleon momenta, and $\cos\theta=\mathbf{p}_2\cdot\mathbf{q}_2/pq$, all in the barycentric system. Momentum conservation leads to the relation

$$s+\bar{s}+t=2m^2+2\mu^2. \quad (2.2)$$

The Lorentz invariants defined by Eqs. (2.1a, b, c) are just the squares of the energies in the barycentric system of the corresponding process in Fig. 1(a), (b), (c). The structure of the Mandelstam representation[2] forces us to consider these three processes simultaneously.

The S matrix for the process $\pi+\pi\rightarrow N+\bar{N}$ can be written

$$S_{fi}=-(2\pi)^4 i\delta^4(p_1+p_2-q_1-q_2)\frac{m}{(4E_1E_2\omega_1\omega_2)^{\frac{1}{2}}}\tau_{fi}, \quad (2.3)$$

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[1] G. F. Chew and S. Mandelstam, University of California Radiation Laboratory Report UCRL-8728, April 15, 1959 (unpublished).
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958) and Phys. Rev. **115**, 1741 and 1752 (1959).
[3] W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).
[4] M. Jacob and G. C. Wick, Ann. Phys. **7**, 404 (1959).
[5] **Notation:** We use the metric such that $p\cdot q=\mathbf{p}\cdot\mathbf{q}-p_0q_0$. The Dirac equation reads $(i\gamma\cdot p+m)u=0$, $(i\gamma\cdot p-m)v=0$. The spinors are normalized to $\bar{u}u=1$, $\bar{v}v=-1$. The nucleon mass is m, the pion mass is μ. The coupling constant g_r is defined so that $g_r^2/4\pi\approx 14$.

where E_1 and E_2 are the antinucleon and nucleon energies, ω_1 and ω_2 are the meson energies, and

$$\tau_{fi}=\bar{u}(p_2)Tv(p_1). \tag{2.4}$$

The decomposition of T into spin-independent functions has been carried out by Chew, Goldberger, Low, and Nambu for pion-nucleon scattering.[6] Making the substitution $q_2\rightarrow -q_2$, $p_1\rightarrow -p_1$, we find for the process $\pi+\pi\rightarrow N+\bar{N}$

$$T=-A+\tfrac{1}{2}i\gamma\cdot(q_1-q_2)B, \tag{2.5}$$

where A and B are functions of s, $\bar{s}$, and t, and matrices in isotopic spin space. As in pion-nucleon scattering, the most general form consistent with charge independence[6] is

$$A_{\beta\alpha}=\delta_{\beta\alpha}A^{(+)}+\tfrac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)}, \tag{2.6}$$

and similarly for $B_{\beta\alpha}$. The Pauli principle requires

$$A^{(\pm)}(s,\bar{s},t)=\pm A^{(\pm)}(\bar{s},s,t), \tag{2.7a}$$

$$B^{(\pm)}(s,\bar{s},t)=\mp B^{(\pm)}(\bar{s},s,t). \tag{2.7b}$$

Note that, according to Eq. (2.1), $s\leftrightarrow\bar{s}$ means $\cos\theta\leftrightarrow -\cos\theta$. It is also evident from the symmetry properties of the $(\pm)$ amplitudes that they are proportional to the two possible eigenamplitudes of total isotopic spin. As shown in the Appendix,

$$A^{(+)}=(1/\sqrt{6})A^0,\quad A^{(-)}=\tfrac{1}{2}A^1. \tag{2.8}$$

III. ANGULAR-MOMENTUM DECOMPOSITION

In the barycentric system the differential cross section for $\pi+\pi\rightarrow N+\bar{N}$ is

$$d\sigma/d\Omega=\sum(p/q)|(m/2E)(\tau^{(\pm)}/4\pi)|^2, \tag{3.1}$$

where $\sum$ represents a sum over final spin states. The quantity τ, defined by Eq. (2.4), can be written as a matrix element between Pauli spinors, χ, in the form

$$\tau^{(\pm)}=\chi_N^\dagger(h_1^{(\pm)}\boldsymbol{\sigma}\cdot\mathbf{p}+h_2^{(\pm)}\boldsymbol{\sigma}\cdot\mathbf{q})\chi_{\bar{N}}, \tag{3.2}$$

where

$$h_1^{(\pm)}=-(1/m)\{A^{(\pm)}+[B^{(\pm)}/(m+E)]\mathbf{p}\cdot\mathbf{q}\}, \tag{3.3}$$

$$h_2^{(\pm)}=(E/m)B^{(\pm)}. \tag{3.4}$$

We have denoted by E the nucleon total energy in the barycentric system; i.e., $t=4E^2$.

We could now write partial-wave amplitudes corresponding to each value of the orbital angular momentum l of the nucleon-antinucleon system; however, as we shall show later, the amplitudes introduced by Jacob and Wick[4] have simpler analytic properties. Therefore we shall carry out the partial-wave decomposition by their method, defining

$$d\sigma/d\Omega=\sum(p/q)|\mathfrak{F}(\lambda\bar{\lambda})|^2, \tag{3.5}$$

where $\mathfrak{F}(\lambda\bar{\lambda})$ is the amplitude for production of a nucleon with helicity λ and an antinucleon with helicity $\bar{\lambda}$. We have suppressed the superscripts $(\pm)$ in Eq. (3.5). Equations (31) and (44) of reference 4 then give

$$\mathfrak{F}_{++}=\mathfrak{F}_{--}=(1/q)\sum_J(J+\tfrac{1}{2})T_+{}^JP_J(\cos\theta), \tag{3.6}$$

$$\mathfrak{F}_{+-}=-\mathfrak{F}_{-+}=\frac{1}{q}\sum_J\frac{J+\frac{1}{2}}{[J(J+1)]^{\frac{1}{2}}}T_-{}^J\sin\theta P_J'(\cos\theta), \tag{3.7}$$

where we have used the abbreviation $\pm$ for $\pm\frac{1}{2}$. The scattering amplitudes $T_\pm{}^J$ for the state of total angular momentum J are related to the corresponding S-matrix elements as follows:

$$T^J=-i(q/p)^{\frac{1}{2}}S^J. \tag{3.8}$$

Our next step is to relate the $T_\pm{}^J$ to the invariant functions $A^{(\pm)}$ and $B^{(\pm)}$. This can easily be done by choosing the z axis along $\mathbf{p}$ and evaluating Eq. (3.2) for the helicity states. We find

$$\mathfrak{F}_{++}=(ie^{i\alpha}m/8\pi E)(h_1p+h_2q\cos\theta), \tag{3.9}$$

$$\mathfrak{F}_{+-}=(ie^{i\alpha}m/8\pi E)h_2q\sin\theta, \tag{3.10}$$

where the arbitrary phase α, arising from the relation between Eqs. (3.1) and (3.5), will be adjusted later. Introducing Eqs. (3.9) and (3.10) into (3.6) and (3.7), and using the orthogonality properties [Eq. (23) of reference 4], we obtain

$$T_+{}^J(t)=\frac{ie^{i\alpha}mq}{8\pi Ep}\int_{-1}^{1}dx\,P_J(x)\times[p^2h_1(x)+pqxh_2(x)], \tag{3.11}$$

$$T_-{}^J(t)=\frac{ie^{i\alpha}mq^2}{8\pi E}\left(\frac{J+1}{J}\right)^{\frac{1}{2}}\int_{-1}^{1}dx\,h_2(x)\times[xP_J(x)-P_{J+1}(x)], \tag{3.12}$$

where $x=\cos\theta$. Let us now define new amplitudes

$$f_+{}^J=(p/q)[E/(pq)^J]T_+{}^J, \tag{3.13}$$

$$f_-{}^J=(p/q)[1/(pq)^J]T_-{}^J, \tag{3.14}$$

which will be shown in Sec. V to have simple analytic properties. Using Eqs. (3.3) and (3.4), we finally find

$$f_+{}^J(t)=\frac{1}{8\pi}\left(-\frac{p^2}{(pq)^J}A_J+\frac{m}{(2J+1)(pq)^{J-1}}\times[(J+1)B_{J+1}+JB_{J-1}]\right), \tag{3.15}$$

$$f_-{}^J(t)=\frac{1}{8\pi}\frac{[J(J+1)]^{\frac{1}{2}}}{2J+1}\frac{1}{(pq)^{J-1}}(B_{J-1}-B_{J+1}), \tag{3.16}$$

[6] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

where we have defined

$$(A_J{}^{(\pm)};B_J{}^{(\pm)})=\int_{-1}^{1}dx\,P_J(x)(A^{(\pm)};B^{(\pm)}).\quad(3.17)$$

The arbitrary phase α has been adjusted in Eqs. (3.15) and (3.16) so that $f_{\pm}{}^J$ are real when A and B are real. Notice that the Pauli principle, as expressed by Eq. (2.7), implies that for J even,

$$A_J{}^{(-)}=B_J{}^{(+)}=0,$$

whereas for J odd

$$A_J{}^{(+)}=B_J{}^{(-)}=0.$$

IV. THE MANDELSTAM REPRESENTATION

We assume that the invariant functions $A^{(\pm)}$ and $B^{(\pm)}$ satisfy the spectral representation proposed by Mandelstam[2]:

$$\begin{aligned}B^{(\pm)}(s,\bar{s},t)=&\frac{g_r{}^2}{m^2-s}\mp\frac{g_r{}^2}{m^2-\bar{s}}\\&+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}ds'\int_{(m+\mu)^2}^{\infty}d\bar{s}'\frac{b_{12}{}^{(\pm)}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}\\&+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}ds'\int_{4\mu^2}^{\infty}dt'\frac{b_{13}{}^{(\pm)}(s',t')}{(s'-s)(t'-t)}\\&+\frac{1}{\pi^2}\int_{(m+\mu)^2}^{\infty}d\bar{s}'\int_{4\mu^2}^{\infty}dt'\frac{b_{23}{}^{(\pm)}(\bar{s}',t')}{(\bar{s}'-\bar{s})(t'-t)}.\end{aligned}\quad(4.1)$$

Although the variables s, $\bar{s}$, and t are related by Eq. (2.2), we shall often write them explicitly in order to show the full symmetry of the representation. The functions $A^{(\pm)}$ satisfy a similar representation, excluding the first two terms. We shall not consider the possibility of subtraction terms in Eq. (4.1), since we shall use the representation only to determine the location of singularities in the partial-wave amplitudes. The spectral functions $b_{ij}{}^{(\pm)}$ and $a_{ij}{}^{(\pm)}$ are not independent; it follows from Eqs. (2.7) that

$$a_{12}{}^{(\pm)}(x,y)=\pm a_{12}{}^{(\pm)}(y,x),\quad(4.2)$$

$$b_{12}{}^{(\pm)}(x,y)=\mp b_{12}{}^{(\pm)}(y,x),\quad(4.3)$$

$$a_{13}{}^{(\pm)}(x,y)=\pm a_{23}{}^{(\pm)}(x,y),\quad(4.4)$$

$$b_{13}{}^{(\pm)}(x,y)=\mp b_{23}{}^{(\pm)}(x,y).\quad(4.5)$$

As shown by Mandelstam, one can easily derive from Eq. (4.1) one-dimensional dispersion relations with either s, $\bar{s}$, or t held fixed. In order to derive dispersion relations for partial-wave amplitudes for $\pi+\pi\rightarrow N+\bar{N}$, we need the representation which makes explicit the dependence on the momentum transfer (s) for fixed energy (t):

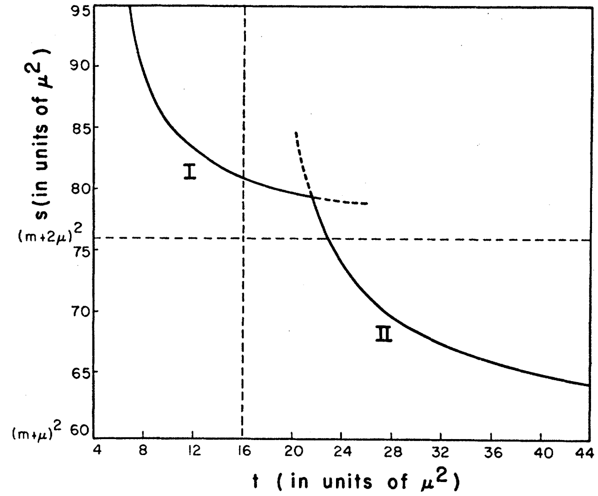

FIG. 2. Boundary curve of the spectral functions $b_{13}(s,t)$, $a_{13}(s,t)$.

$$\begin{aligned}B^{(\pm)}(s,\bar{s},t)=&\frac{g_r{}^2}{m^2-s}\mp\frac{g_r{}^2}{m^2-\bar{s}}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{b_1{}^{(\pm)}(s',t)}{s'-s}\\&+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\bar{s}'\frac{b_2{}^{(\pm)}(\bar{s}',t)}{\bar{s}'-\bar{s}}.\end{aligned}\quad(4.6)$$

Then Eq. (4.1) shows

$$\begin{aligned}b_1{}^{(\pm)}(s',t)=&\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{b_{13}{}^{(\pm)}(s',t')}{t'-t}\\&+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}d\bar{s}'\frac{b_{12}{}^{(\pm)}(s',\bar{s}')}{\bar{s}'+s'+t-2m^2-2\mu^2},\end{aligned}\quad(4.7)$$

and

$$\begin{aligned}b_2{}^{(\pm)}(\bar{s}',t)=&\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{b_{23}{}^{(\pm)}(\bar{s}',t')}{t'-t}\\&+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\frac{b_{12}{}^{(\pm)}(s',\bar{s}')}{s'+\bar{s}'+t-2m^2-2\mu^2}.\end{aligned}\quad(4.8)$$

Using Eq. (4.3), one finds

$$b_2{}^{(\pm)}(s',t)=\mp b_1{}^{(\pm)}(s',t).\quad(4.9)$$

Relations similar to Eqs. (4.6–4.9) hold for $A^{(\pm)}$, but without the pole terms and with the $\mp$ inverted.

The spectral functions b_{ij}, a_{ij} are nonzero in regions whose boundaries have been calculated by Mandelstam.[2] For completeness we reproduce his results here. The spectral functions b_{13} and a_{13} are bounded by the following two curves (see Fig. 2):

I. $$(t-4\mu^2)[s-(m+2\mu)^2][s-(m-2\mu)^2]-16\mu^4(s+3m^2-3\mu^2)=0,\quad(4.10a)$$

II. $$(t-16\mu^2)[s-(m+\mu)^2][s-(m-\mu)^2]-64\mu^4s=0.\quad(4.10b)$$

The bounding curves for b_{23} and a_{23} can be obtained from these equations by changing s to $\bar{s}$. The spectral functions b_{12} and a_{12} are bounded by[7a]

$$[\bar{s}-(m+2\mu)^2][\bar{s}-(m-2\mu)^2][s-(m+\mu)^2]\times[s-(m-\mu)^2]-16s\bar{s}m^2\mu^2+16\mu^2(m^2-\mu^2)^2(s+\bar{s}-m^2-2\mu^2)=0, \quad (4.11)$$

and by a second curve obtained by interchanging s and $\bar{s}$. From Eqs. (4.10) and (4.11) it is evident that the regions in which the spectral functions are nonzero are asymptotically bounded by the lower limits of integration in Eq. (4.1).

V. ANALYTIC PROPERTIES OF THE PARTIAL-WAVE AMPLITUDES

Let us now use the analytic properties of the invariant functions $A^{(\pm)}(s,\bar{s},t)$ and $B^{(\pm)}(s,\bar{s},t)$, as given in Sec. IV, to make an analytic continuation into the complex t plane of the partial-wave amplitudes $f_\pm{}^J(t)$ defined by Eqs. (3.15) and (3.16). In order to do this let us consider, for example, the term $A_J/(pq)^J$. Using Eqs. (3.17), (4.6), and (4.9), we find

$$\frac{A_J(t)}{(pq)^J}=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} ds'\, a_1(s',t)I_J(s',t), \qquad (5.1)$$

where

$$I_J(s',t)=\frac{1}{(pq)^J}\int_{-1}^{1}dx\, P_J(x)\left(\frac{1}{s'+p^2+q^2-2pqx}+\frac{(-1)^J}{s'+p^2+q^2+2pqx}\right). \qquad (5.2)$$

By inspection of these two equations and Eq. (4.7), it is a straightforward task to determine the nature and location of the singularities of $A_J/(pq)^J$. The vanishing of the denominator of the first term in Eq. (4.7) produces a series of branch cuts on the positive real t axis associated with the thresholds of the possible intermediate states between two pions and a nucleon-antinucleon pair. The lowest occurs at $t=(2\mu)^2$, the next at $t=(4\mu)^2$, and so on. The threshold of the physical region comes at $t=(2m)^2$. The apparent singularity from the vanishing of the second denominator in Eq. (4.7) was introduced artificially through the separation into partial fractions of one of the terms in Eq. (4.1). This singularity can easily be seen to vanish after the integration in Eq. (5.1) is performed.

The other two sets of branch cuts, arising from the vanishing of the denominators in Eq. (5.2), are coincident and lie on the negative real axis. It can easily be shown that the branch cuts extend from $t=-\infty$ to

$$t=[4m^2\mu^2-(s'-m^2-\mu^2)^2]/s'. \qquad (5.3)$$

[7a] *Note added in proof.*—We thank Dr. James Ball for suggesting this formula, which is a correction of Eq. (4.3a) of reference 2.

Since s' is the energy variable for pion-nucleon scattering, Eq. (5.3) means that there will be a branch point in t arising from each threshold for the states which can be produced by a pion and a nucleon. Therefore the first branch point, lying at $t=0$, corresponds to the lower limit of integration in Eq. (5.1). The second, corresponding to the threshold for pion production, occurs at $t\approx-10\mu^2$.

It should further be noticed that $I_J(s',t)$ contains no singularities other than those arising from vanishing denominators in the Mandelstam representation. Since it can easily be shown that the integral in Eq. (5.2) vanishes at $p=0$ or $q=0$ as $(pq)^J$, no pole is introduced by dividing by this factor. Finally, since only even powers of pq are present in $I_J(s',t)$, no branch points arising from kinematical factors occur.

Similar considerations hold for the terms proportional to B_J in $f_\pm{}^J$. However, the pole terms in Eq. (4.6) produce an additional branch point at $t=4\mu^2(1-\mu^2/4m^2)$. Thus we can conclude that the functions $f_\pm{}^J(t)$ are analytic in the complex t plane except for branch cuts on the real axis extending from $-\infty$ to $4\mu^2(1-\mu^2/4m^2)$ and from $4\mu^2$ to ∞. We remark here that the amplitudes $f_\pm{}^J$ corresponding to definite helicities are clearly more convenient than the usual amplitudes corresponding to definite orbital angular momenta, since the latter contain additional singularities of a purely kinematical origin (such as factors of E).

In order now to be able to write dispersion relations for the partial-wave amplitudes we must consider their asymptotic behavior. The unitarity condition tells us that $T_\pm{}^J(t)$ are bounded as $t\to\infty$. Therefore we see from Eqs. (3.13) and (3.14) that as $t\to\infty$, $f_-{}^J(t)$ goes to zero at least as fast as t^{-J}, and $f_+{}^J(t)$, as $t^{-J+\frac{1}{2}}$. Guided by these considerations we write for $J\neq0$

$$f_\pm{}^J(t)=\frac{1}{\pi}\int_{-\infty}^{a}\frac{\mathrm{Im}f_\pm{}^J(t')dt'}{t'-t-i\epsilon}+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{\mathrm{Im}f_\pm{}^J(t')dt'}{t'-t-i\epsilon}, \qquad (5.4)$$

where $a=4\mu^2(1-\mu^2/4m^2)$.

For $J=0$, $f_-{}^0(t)=0$, from Eq. (3.16). This is obvious physically from the fact that $f_-{}^J$ refers to states in which the projection of **J** along **p** is unity. Moreover, the quantity $f_+{}^0/p^2$ remains finite at $p=0$ and has the necessary asymptotic behavior as $t\to\infty$.

Our next task is to evaluate $\mathrm{Im}f_\pm{}^J(t)$ on the left-hand branch cut $(-\infty<t\leq a)$. In this region $a_1(s',t)$ and $b_1(s',t)$ are real [Eq. (4.7)], and we find, from Eq. (5.1),

$$\mathrm{Im}\frac{A_J(t)}{(pq)^J}=-\int_{(m+\mu)^2}^{L(t)} ds'\, a_1(s',t)\frac{P_J(z)}{(p_-q_-)^{J+1}}, \qquad (5.5)$$

where

$$L(t)=m^2+\mu^2+2p_-q_--t/2,\quad p_-=(m^2-t/4)^{\frac{1}{2}},\quad q_-=(\mu^2-t/4)^{\frac{1}{2}}, \qquad (5.6)$$

and

$$z=\frac{s'-q_-^2-p_-^2}{2q_-p_-}.$$

A relation similar to Eq. (5.5) but including the contribution of the pole terms holds for $B_J(t)$, leading to

$$\mathrm{Im}f_+{}^J(t)=-\frac{1}{8\pi(p_-q_-)^J}\bigg\{\pi g_r{}^2mz_0P_J(z_0)+\theta(-t)\int_{(m+\mu)^2}^{L(t)}ds'\,[(p_-/q_-)a_1(s',t)+mzb_1(s',t)]P_J(z)\bigg\},\quad(5.7)$$

$$\mathrm{Im}f_-{}^J(t)=\frac{[J(J+1)]^{\frac{1}{2}}}{8\pi(2J+1)(p_-q_-)^J}\times\bigg\{\pi g_r{}^2[P_{J+1}(z_0)-P_{J-1}(z_0)]+\theta(-t)\int_{(m+\mu)^2}^{L(t)}ds'b_1(s',t)[P_{J+1}(z)-P_{J-1}(z)]\bigg\},\quad(5.8)$$

where

$$z_0=(m^2-q_-^2-p_-^2)/2q_-p_-.$$

As we have stated previously, the left-hand branch cut is associated with pion-nucleon scattering. From Eqs. (4.6) and (4.9) it follows, for $t<0$,

$$a_1{}^{(\pm)}(s',t)=\mathrm{Im}A^{(\pm)}(s',t),\quad(5.9)$$

and the same for b_1. Although in Eqs. (5.7) and (5.8) the energy variable s' is in the physical range for pion-nucleon scattering, the upper limit $L(t)$ is such that $\cos\phi\leq-1$, where ϕ is the pion-nucleon scattering angle in the barycentric system. Therefore we must make an analytic continuation from the physical region. A well-known method of continuation is to expand $\mathrm{Im}A(s,t)$ in Legendre polynomials[6]:

$$[a_1{}^{(\pm)}(s,t);\,b_1{}^{(\pm)}(s,t)]=4\pi\bigg(\frac{(W+m;\,1)}{E_s+m}\sum_l[\mathrm{Im}f_{l+}{}^{(\pm)}P_{l+1}'(\cos\phi)-\mathrm{Im}f_{l-}{}^{(\pm)}P_{l-1}'(\cos\phi)]-\frac{(W-m;\,-1)}{E_s-m}\sum_l P_l'(\cos\phi)\times[\mathrm{Im}f_{l-}{}^{(\pm)}-\mathrm{Im}f_{l+}{}^{(\pm)}]\bigg),\quad(5.10)$$

where

$$W^2=s,\quad E_s=(W^2+m^2-\mu^2)/2W,$$

and

$$k^2=E_s{}^2-m^2,\quad \cos\phi=1+t/2k^2.$$

The region of convergence of this Legendre polynomial expansion can be determined from Eq. (4.7). Since a function $f(\cos\phi)$ that is analytic inside an ellipse with foci at -1 and $+1$ can be expanded in Legendre polynomials, we must find the position of the nearest singularity in $\cos\phi$. This singularity can be seen to come from the vanishing of the denominator of the first term in Eq. (4.7) in the region where $b_{13}(s,t)\neq0$. Using Eq. (4.10) for the boundary curve of this region, we find that the expansion converges on the left-hand branch cut as long as $t\gtrsim-26\mu^2$. For comparison we state the result rigorously proved by Lehmann,[7b] that the expansion in Eq. (5.10) converges at least for values of t greater than $-32\mu^2(2m\mu+\mu^2)/3(2m\mu-\mu^2)\approx-12\mu^2$.

Beyond the region of convergence of the polynomial expansion more subtle methods of analytic continuation will be necessary. However, on the basis of the effective range approximation, which we shall discuss in the next section, the contribution of $\mathrm{Im}f_\pm{}^J(t)$ to the scattering amplitude for $t\lesssim-26\mu^2$ might be considered unimportant.

VI. THE INTEGRAL EQUATIONS

In reference 3 we have outlined an approximate method of solution of Eq. (5.4) for the $J=1$ state, based on the effective range approximation and therefore appropriate to the low-energy unphysical region. We shall now generalize this method to states of arbitrary J.

In order to do this we conjecture that in the region $(2\mu)^2\lesssim t\lesssim(4\mu)^2$ the phase of the amplitudes $f_\pm{}^J$ defined by Eqs. (3.15) and (3.16) is equal to the pion-pion scattering phase shift δ_J in the corresponding angular momentum and isotopic spin state. This conjecture can be verified for the $J=1$ state if one accepts the validity of the dispersion-relation treatment of the nucleon electromagnetic structure,[8,9] where this phase condition is necessary to maintain the reality of the spectral functions.[10] In the general case the reason for imposing the phase condition on the amplitudes $f_\pm{}^J$, rather than $T_\pm{}^J$ for example, is that these amplitudes have the property of being real in the region $(2\mu)^2\leq t\leq(4\mu)^2$ for $\delta_J=0$.

Use of the phase condition permits the construction of a solution for $f_\pm{}^J$. Consider the quantity $f_\pm{}^J(t)e^{-u_J(t)}$, where

$$u_J(t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\delta_J(t')}{t'-t-i\epsilon}.\quad(6.1)$$

If this integral does not converge, define instead

$$u_J'(t)=\frac{t}{\pi}\int_{4\mu^2}^{\infty}dt'\frac{\delta_J(t')}{t'(t'-t-i\epsilon)}.\quad(6.2)$$

[7b] H. Lehmann, Nuovo cimento **10**, 579 (1958).

[8] G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958).

[9] P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958).

[10] The precise relation between the spectral functions and the amplitudes $f_\pm{}^1(t)$ will be given in the following paper.

Now we can write a dispersion relation for $f_{\pm}^{J}(t)e^{-u_J(t)}$, provided the function approaches zero sufficiently rapidly. Since this function is real in the region $(2\mu)^2 \leq t \leq (4\mu)^2$, the dispersion relation is

$$f_{\pm}^{J}(t)e^{-u_J(t)}=\frac{1}{\pi}\int_{-\infty}^{a}dt'\frac{e^{-u_J(t')}\,\mathrm{Im}f_{\pm}^{J}(t')}{t'-t-i\epsilon}+\frac{1}{\pi}\int_{16\mu^2}^{\infty}dt'\frac{\mathrm{Im}[f_{\pm}^{J}(t')e^{-u_J(t')}]}{t'-t-i\epsilon}. \quad (6.3)$$

As a first effective-range approximation to $f_{\pm}^{J}(t)$ for small t in the region $t>4\mu^2$ we can neglect the second integral on the right-hand side of Eq. (6.3) by virtue of the size of its denominator. Moreover, we expect the phase condition to be approximately satisfied for values of t considerably above the next threshold at $16\mu^2$. This has been found by Capps to be true for photoproduction.[11] That being the case, the numerator in the second term of Eq. (6.3) will be small in a region extending considerably beyond the lower limit of integration, thus giving further justification to our approximation.

Equation (6.3) is in general a very complex set of coupled integral equations since the imaginary part on the left-hand cut is related through Eqs. (5.7) and (5.8) to pion-nucleon scattering, which in turn involves the process $\pi+\pi \rightarrow N+\bar{N}$. However, as a first approach to the problem one can use experimental information about pion-nucleon scattering, so that $\mathrm{Im}f_{\pm}^{J}(t)$ becomes a known function within the region of convergence of the Legendre polynomial expansion. We can again use the effective-range approach, hoping that the contribution beyond this region is small and therefore cutting off the first integral in Eq. (6.3) at $t=-26\mu^2$. Thus, if the pion-pion phase shift δ_J is known, Eq. (6.3) gives an explicit solution.[12]

We have applied this method of solution to the $J=1$ state, which enters in the nucleon electromagnetic structure problem. Results will be given in the following paper.

[11] R. H. Capps, Phys. Rev. Letters **2**, 475 (1959).

[12] It is shown in the following paper that linear combinations of f_+^J and f_-^J exist (namely those which enter into the nucleon structure problem) that have improved asymptotic behavior.

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APPENDIX: ISOTOPIC SPIN PROJECTION OPERATORS

The isotopic spin decomposition of the invariant functions A and B can be written, assuming charge independence, as

$$\langle jk|A|\beta\alpha\rangle=\sum_{I=0,1}A^I\langle jk|\mathcal{G}^I|\beta\alpha\rangle, \quad (A.1)$$

where $j=n,\,p;\; k=\bar{n},\,\bar{p}$; and $\alpha,\beta=1, 2, 3$ are the isotopic spin indices of the pions; and the projection operator $\mathcal{G}^I$ is defined as

$$\mathcal{G}^I=\sum_{I_z}|I,I_z(n)\rangle\langle(\pi)I,I_z|. \quad (A.2)$$

The symbols (n) and (π) denote normalized nucleon-antinucleon and two-pion states, respectively. Then we write

$$\langle jk|\mathcal{G}^I|\beta\alpha\rangle=\chi_j^{\dagger}\mathcal{G}_{\beta\alpha}{}^I\chi_k. \quad (A.3)$$

From Eq. (A.2) one can easily show

$$\mathrm{Tr}\mathcal{G}_{\alpha'\beta'}{}^{I\dagger}\mathcal{G}_{\alpha\beta}{}^I=P_{\alpha'\beta',\alpha\beta}{}^I, \quad (A.4)$$

where the trace is taken in the nucleon isotopic spinor space, and where

$$P_{\alpha'\beta',\alpha\beta}{}^I=\sum_{I_z}\langle\alpha'\beta'|I,I_z(\pi)\rangle\langle(\pi)I,I_z|\alpha\beta\rangle. \quad (A.5)$$

The P^I, which are the normalized isotopic-spin projection operators for the pion-pion scattering, are given implicitly by Eqs. (II.8) and (II.4) of reference 1.

One can easily verify that the operators

$$\mathcal{G}_{\beta\alpha}{}^0=(1/\sqrt{6})\delta_{\beta\alpha}, \quad \mathcal{G}_{\beta\alpha}{}^1=\tfrac{1}{4}[\tau_\beta,\tau_\alpha],$$

satisfy Eq. (A.4). This method has the advantage of avoiding the use of explicit representations for the isotopic spin eigenstates.