Equivalent Hamiltonians in Scattering Theory*

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This paper is a contribution to the discussion of the question: to what extent does the scattering matrix determine the Hamiltonian? The Hamiltonians considered are nonrelativistic, but in extension of previous studies, "nonlocal" potentials and many-body potentials are allowed. A large class of unitary transformations is found which produce Hamiltonians leading to the same S matrix. In the last section, it is shown that this equivalence is only a special consequence of the general axiomatic formulation of scattering in field theory.

I. INTRODUCTION

THE extent to which the scattering matrix determines the Hamiltonians has been the subject of many studies.[1–4] These are limited to the case in which the interaction is a potential, i.e., commutes with the position operator. However, there are no good reasons, either from prime principles[5] or from phenomenological analysis[6] to exclude "nonlocal" interactions which may be, for instance, integral operators in coordinate representation. There is some interest, then, in considering in a more general manner the extent to which the Hamiltonian is left indeterminate by a given scattering matrix. The subject of this paper is both wider and narrower than those of preceding studies; we exclude bound states, but we consider scattering by any number of nonrelativistic scalar particles, the interaction operator being composed of many-body operators, not necessarily diagonal in x representation. A class of Hamiltonians, equivalent in the sense that they lead to the same S operator, can be defined by a class of unitary operators which transform a given Hamiltonian. This class is so wide that one is tempted to say that "any reasonable" unitary transformation leaves a given Hamiltonian within its equivalence class.

This result may be practically useful because it provides a large class of unitary transformations by which a given Hamiltonian can be put into a more tractable form without the necessity of "reinterpreting" the transformed wave function.

It is natural to wonder about the physical reason for the equivalence of apparently very dissimilar Hamiltonians. If the problem is considered in the formalism of second quantization, the results become understandable by adopting the view that the "basic" field operators have no observable meaning.

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[1] V. Bargmann, Revs. Modern Phys. **27**, 30 (1949).
[2] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **36A**, 11 (1949).
[3] R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952).
[4] V. A. Marchenko, Doklady Akad. Nauk S.S.S.R. **104**, 695 (1955).
[5] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. **27**, 281 (1941).
[6] M. Moshinsky, Phys. Rev. **109**, 933 (1958).

In Sec. III, a slightly revised form of Haag's postulates for the physical interpretation[7] of field theory is formulated. From this point of view, the equivalent Hamiltonians appear as expressions of the same energy operator in terms of different physical-particle creation operators.

II. EQUIVALENT HAMILTONIANS

We consider a nonrelativistic theory of scalar bosons, interacting through two- or many-body potentials. There are no bound states. The theory is assumed to be invariant under space rotation, translation in time and space, and reflections in time and space. There exists a nondegenerate vacuum state which is invariant under all symmetry operations, and a single subspace of one-particle states.

The usual creation and destruction operators $a(k)$, $a^\dagger(k)$ are used, and the Hamiltonian H commutes with the number operator

$$N=\int a^\dagger(k)a(k)d^3k. \tag{2.1}$$

Space vectors are denoted by italics except where a confusion with four-vectors is possible.

The Hamiltonian is of the form

$$H=H_0+V=\int a^\dagger(k)(k^2/2m)a(k)d^3k+V, \tag{2.2}$$

where the operator V annuls the vacuum and all one-particle states. We assume the existence of wave operators

$$\exp(iHt)\exp(-iH_0t)\rightarrow\Omega_\mp\ (t\rightarrow\pm\infty), \tag{2.3}$$

where the arrow stands for strong convergence.[8] Since we have excluded bound states, the wave operators are unitary.[9]

[7] R. Haag, Phys. Rev. **112**, 669 (1958).
[8] The existence of the strong limit was proved for some Hamiltonians by J. M. Cook, J. Math. and Phys. **36**, 82 (1957), and M. N. Hack, Nuovo cimento **9**, 731 (1958).
[9] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1142 (1954).

The scattering operator is defined by[10]

$$\exp(iH_0t)\exp(-2iHt)\times\exp(iH_0t)\to S=\Omega_-^\dagger\Omega_+\ (t\to\infty). \quad (2.4)$$

Clearly, the vacuum ψ_0 and the one-particle states $\psi^{(1)}$ are eigenfunctions of the S operator with eigenvalue 1:

$$S\psi_0=\psi_0;\quad S\psi^{(1)}=\psi^{(1)}. \quad (2.5)$$

We consider those Hamiltonians H' which may be obtained from H by a unitary transformation

$$H'=UHU^\dagger,\quad UU^\dagger=U^\dagger U=1, \quad (2.6)$$

and we ask for the class of equivalent Hamiltonians H', i.e., those which lead to the same S operator

$$\exp(iH_0t)\exp(-2iH't)\times\exp(iH_0t)\to S'=S\ (t\to\infty), \quad (2.7)$$

and have the same action on the subspace which consists of the vacuum and one-particle states.

If Eq. (2.6) is used in Eq. (2.7), one obtains

$$\exp(iH_0t)U\exp(-iH_0t)\exp(iH_0t)\exp(-2iHt)\times\exp(iH_0t)\exp(-iH_0t)U^\dagger\exp(iH_0t)\to S'. \quad (2.8)$$

Since the product of the 4th to 6th factors converges strongly to S by Eq. (2.4), it is consistent to assume that the separate strong limits

$$\begin{aligned}&\exp(iH_0t)U\exp(-iH_0t)\to U_+,\ (t\to\infty)\\&\exp(-iH_0t)U^\dagger\exp(iH_0t)\to U_-^\dagger,\ (t\to\infty)\end{aligned} \quad (2.9)$$

exist.

We are going to assign further properties to the operator U and show that this particular class leads to equivalent Hamiltonians. The requirement of equivalence on the irreducible subspace is expressed by

$$UHU^\dagger a_k^\dagger\psi_0=Ha_k^\dagger\psi_0. \quad (2.10)$$

It is consistent with this equation if we require that U should act as a unit operator on the subspaces of vacuum and one-particle states:

$$\begin{aligned}Ua_k^\dagger\psi_0&=a_k^\dagger\psi_0,\\U\psi_0&=U^\dagger\psi_0=\psi_0.\end{aligned} \quad (2.11)$$

It follows that U has the form

$$U=1+D, \quad (2.12)$$

where D is a destruction operator which annuls both the vacuum and one-particle states.

The operator U may be formally expanded into a series of normal products of creation and destruction operators:

$$U=1+\sum_{N=0,M=2}\int G_{NM}(p_1\cdots p_N,p_1'\cdots p_M')\times\prod_{i=1}^N a^\dagger(p_i)dp_i\prod_{i=1}^M a(p_i')dp_i'. \quad (2.13)$$

We require that U commute with the number operator, so that H' has this property. It follows that $N=M$. We require further the translational invariance of U in order to guarantee that of H'. It follows that the functions G_{NM} must have a factor of the form $\delta(\sum p_i-\sum p_i')$. Hence, the form of U is

$$U=1+\sum_{N=2}\int F_N(p_1\cdots p_N,p_1'\cdots p_N')\times\exp[ik\cdot(\sum p_i-\sum p_i')]\times\prod_1^N a^\dagger(p_i)dp_i\prod_1^N a(p_i')dp_i'. \quad (2.14)$$

We assume that such an expansion exists with square-integrable functions F_N.

We consider the sequence of operators

$$\exp(iH_0t)U\exp(-iH_0t)=1+\sum_{N=2}\int\exp it[\sum^N E(p_i)-\sum E(p_i')]\times\exp ik\cdot(\sum^N p_i-\sum^N p_i')F_N\prod_1^N a^\dagger(p_i)\times dp_i\prod_1^N a(p_i')dp_i'dk. \quad (2.15)$$

In the limit $t\to\pm\infty$, the sum on the right-hand side vanishes strongly in virtue of the Riemann-Lebesgue lemma.[11] We have, by Eq. (2.9)

$$U_\pm=1. \quad (2.16)$$

Comparison with Eq. (2.8) shows that under these conditions, $S'=S$, i.e., the Hamiltonians H' and H are indeed equivalent.

In order to make H' a Hamiltonian with the required invariance properties, we further require that U commutes with the symmetry operators of rotation and of space and time reflection. This gives rise to constraints on the functions F_N. In particular, a transformation which leaves the field operator unchanged and adds a function of the field to the canonical momentum does not produce an equivalent Hamiltonian. For, if $\Phi(\mathbf{x})$ is the field operator, which we assume to be time-reversal invariant as usual, and if f is a real-valued functional, then the unitary operator $U=\exp[if\{\Phi(\mathbf{x})\}]$ does not commute with the time-reversal operator T, but anticommutes with it. Therefore, the operator $H'=UHU^\dagger$ does not commute with T and is not an acceptable Hamiltonian. If a scattering operator S' were defined

[10] If $A(t)$ and $B(t)$ are sequences of unitary operators, and if $A(t)\to A$, $B(t)\to B$, then $A(t)B(t)\to AB$. The proof is found in B. v. Sz. Nagy, *Spektraldarstellungen Linearer Transformationen des Hilbertschen Raumes* (Springer-Verlag, Berlin, 1942), p. 11.

from it through Eq. (2.4), it would not, in general, have the time-reversal property $S'^{\dagger}=TS'T^{-1}$ required of scattering operators.

III. UNITARY TRANSFORMATIONS IN SCATTERING PROBLEMS

In calculations of scattering amplitudes, it is often convenient to consider unitary transforms of a given Hamiltonian, so that the new form $H'=UHU^{\dagger}$ is easier to manipulate. In particular, the transformation is often used to make the "perturbation" smaller or, at least, less important for the particular physical situation.

It is understood that after such a unitary transformation the wave function must be "reinterpreted." By this one means either that a wave function (say, an eigenfunction of H') must be transformed by $\psi=U^{\dagger}\psi'$ to have a clear physical meaning or that one connects the calculated quantities ψ' to observables by intuitive arguments. The first procedure is often self-defeating, because the operators U are usually given as formal exponential functions of other operators, so that an actual evaluation amounts to an expansion into an infinite series, while the second procedure is doubtful. In particular, the calculation of the S matrix from H' by the use of Eq. (2.4) or equivalent methods, is justified only in special cases. As an obvious example of fallacy, one may consider the transformation

$$H'=\Omega H\Omega^{\dagger}=H_0, \tag{3.1}$$

which carries H into H_0. Equation (2.4) would then give $S'=1$ for the scattering operator, which is absurd.

We can use the result of Sec. II to determine those particular unitary operators which allow the use of the standard Eq. (2.4) to calculate the S matrix from the new Hamiltonian, without the necessity of reinterpreting.

The conditions on the operator U are so unspectacular that one may be inclined to think that "any reasonable" unitary transformation will lead to the same S operator. He may notice, however, that U cannot be a Møller wave operator as in Eq. (3.1), since this operator is known to have a δ_+ function in its second term [Eq. (2.14)], so that F would not be square-integrable. Our result is not in contradiction to the statement that a scattering operator without bound states completely determines the potential. The Hamiltonians produced from a potential interaction by our transformation U will produce "nonlocal" interactions described by an integral kernel in coordinate representation.

As an illustration, we exhibit a simple operator U. Consider the effect of U on the two-body subspace. The operator U is represented by a functional operator u which acts on the two-body Fock functions $\psi(x,y)$:

$$(u\psi)(x,y)=\psi(x,y)+\int(x,y|G|x'y')\psi(x'y')d^3x'd^3y'. \tag{3.2}$$

Let u act as a unit operator on the center-of-mass coordinate $x+y=R$. A special choice is

$$(u\psi)(R,r)=\psi(R,r)-2\left[\int f^2(r)d^3x\right]^{-1}\times f(r)\int f(r')\psi(R,r')d^3x'. \tag{3.3}$$

If $r=|x-y|$ and f is any real square-integrable function, this operator is unitary and real, i.e., orthogonal, and therefore commutes with the time-reversal operator. It also commutes with the operators of space-translation and rotation.

IV. CONNECTION WITH THE POSTULATES OF GENERAL SCATTERING THEORY

The equivalences established in the previous sections seem somewhat unmotivated and surprising. It will be shown now that they are just special consequences of general scattering theory.

In wave mechanics, the physical meaning of the operators is given at the outset, and scattering theory is a straightforward application of established principles. In field theory, on the other hand, the physical significance of the field operators is obscure; even in the one-particle subspaces, the position operators "get lost" and must be reintroduced by special postulates.[12] It is necessary to adopt additional principles for the physical interpretation of the theory.

In the modern literature, several such principles have been proposed.[13–15] In these papers certain "bare" or "basic" field operators are of fundamental importance, although they are not, at finite times, directly related to observables. We adopt a slightly modified version of Haag's formulation[15] in which the basic fields play no role.[16]

We assume, as usual, that the Hilbert space is the basis of a reducible representation (up to a factor) of a group of symmetry operators. This group may be the Lorentz or the Galilei group. There are irreducible subspaces, which correspond to one-particle states.

Instead of introducing field operators with simple transformation properties, but with obscure physical significances, we define creation operators $\mathbf{c}_k$ which, acting on the vacuum, produce one-particle states:

$$\mathbf{c}(k)\psi_0=\psi_k^{(1)}, \tag{4.1}$$

where $\psi_k^{(1)}$ is a simultaneous eigenstate of the space-

[12] T. D. Newton and E. P. Wigner, Revs. Modern Phys. **21**, 400 (1949).

[13] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo cimento **1**, 205 (1955).

[14] A. S. Wightman, *Problèmes Mathématiques de la Théorie Quantique des Champs*, Colloques Internationaux (Centre National de la Recherche Scientifique, Paris, 1959), p. 1.

[15] R. Haag, Phys. Rev. **112**, 669 (1958).

[16] This is a more precise version of the principles proposed in H. Ekstein, Nuovo cimento **4**, 1017 (1956).

and time-translation operators.[17] For scalar particles, the momentum k fully specifies the one-particle function; for particles with spin, another variable must be added.

We require that the operators $\mathbf{c}$ have the same transformation properties under translation and rotation in general as they must have by Eq. (4.1) when acting on the vacuum. In particular, if $\mathbf{P}$ is the generator of space translations, the translation $\mathbf{r}$ has the effect

$$\exp(i\mathbf{P}\cdot\mathbf{r})\mathbf{c}(\mathbf{k})\exp(-i\mathbf{P}\cdot\mathbf{r})=\exp(-i\mathbf{k}\cdot\mathbf{r})\mathbf{c}(\mathbf{k}). \quad (4.2)$$

The operators $\mathbf{c}(\mathbf{k})$ should create "physical particles" in the reducible part of Hilbert space. Intuitively, we mean that if a particle is created by means of $\mathbf{c}$ in addition to other existing particles, and if the added particle is sufficiently distant from the pre-existing ones, the action of the symmetry operators (in particular, the Hamiltonian) on such a state should be "additive," i.e., the Hamiltonian should act on the pre-existing set as well as on the added particle as if they were the only existent ones.

Consider the operator

$$\mathbf{c}_{fr}=\int f(k)e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{c}(k)d^3k, \quad (4.3)$$

which, for large distances r, creates a particle distant from other pre-existing particles, and a general state $\mathbf{C}\psi_0$. We require

$$H\mathbf{c}_{fr}\mathbf{C}\psi_0-\mathbf{c}_{fr}H\mathbf{C}\psi_0-\mathbf{C}H\mathbf{c}_{fr}\psi_0\rightarrow 0, \quad (4.4)$$

as a strong convergence for large distances $|\mathbf{r}|$. The same condition should hold if several particles are created at large distances:

$$(H\prod_n \mathbf{c}_{f(n)r}\mathbf{C}-\prod_n \mathbf{c}_{f(n)r}H\mathbf{C}-\mathbf{C}H\prod_n \mathbf{c}_{f(n)r})\times\psi_0\rightarrow 0. \quad (4.5)$$

Equation (4.5) will be referred to as postulate 1.

Time-translation has an effect similar to that of space-translation in that it increases the distances between particles indefinitely, i.e., decreases the probability of finding two particles near each other. The equation

$$\exp(-iHt)\int \mathbf{c}(k)f(k)d^3k\psi_0 =\int \exp[-iE(k)t]f(k)\mathbf{c}(k)d^3k\psi_0, \quad (4.6)$$

suggests that the analog of Eq. (4.3) is

$$\mathbf{c}_{ft}=\int \exp(-iE_kt)f(k)\mathbf{c}(k)d^3k, \quad (4.7)$$

and that of Eq. (4.5) is

$$(H\mathbf{c}_{ft}\prod_n \mathbf{c}_{g(n)}-\prod_n \mathbf{c}_{g(n)}H\mathbf{c}_{ft}-\mathbf{c}_{ft}H\prod_n \mathbf{c}_{g(n)})\psi_0\rightarrow 0 \quad (t\rightarrow\pm\infty). \quad (4.8)$$

Equation (4.8) will be referred to as postulate 2.

Equation (4.8) may be applied to the product of two operators $\mathbf{c}_{f(1)t(1)}\mathbf{c}_{f(2)t(2)}$, with subsequent limits, for the times t_1 and t_2:

$$(H\mathbf{c}_{f(1)t(1)}\mathbf{c}_{f(2)t(2)}-\mathbf{c}_{f(2)t(2)}H\mathbf{c}_{f(1)t(1)} -\mathbf{c}_{f(1)t(1)}H\mathbf{c}_{f(2)t(2)})\psi_0\rightarrow 0\ (t(1)\rightarrow\infty,\ t(2)\rightarrow\infty). \quad (4.9)$$

By Eqs. (4.1) and (4.6), the two last terms may be written

$$\int[E(k_1)+E(k_2)]f(k_1)f(k_2)\mathbf{c}(k_1)\mathbf{c}(k_2) \times\exp\{-i[E(k_1)t_1+E(k_2)t_2]\}d^3k_1d^3k_2\psi_0.$$

More generally, for a product of creation operators

$$\Big(H\prod_n \mathbf{c}_{f(n)t(n)}-\int[\sum E(k_n)]\prod_n f_n(k_n)\prod_n \mathbf{c}(k_n) \times\exp[-i\sum E(k_n)t_n]\prod_n dk_n\Big)\psi_0\rightarrow 0\ (t_{(n)}\rightarrow\infty). \quad (4.10)$$

We have defined a particle theory by the requirement that operators satisfying Eqs. (4.5) and (4.8) exist. This may be considered as a formal expression of an empirical fact. This postulate must be supplemented by two assumptions of rather technical nature: (2a) In Eq. (4.10), the time limits may be taken simultaneously, i.e., $t(n)$ may be replaced by the single parameter t. (2b) The length of the vector on the left-hand side of Eq. (4.10) decreases asymptotically faster than t^{-1}. It will be shown now that the existence and general properties of the scattering states can be derived from these postulates.

Consider the state vector

$$\exp(iHt)\prod_{n=1}^{N}\mathbf{c}_{f(n)t}\psi_0=Q(t). \quad (4.11)$$

If this quantity has a strong limit for $t\rightarrow\pm\infty$, then the existence of asymptotic scattering states, and from there the theory of the S matrix, can be established.[15,16]

The Cauchy test for convergence is

$$\|Q(T_1)-Q(T_2)\|<\epsilon(T),$$

where $T_1>T$ and $T_2>T$. The convergence is proved if

$$\left\|\int_{T_1}^{T_2}\frac{dQ}{dt}dt\right\|<\epsilon,$$

[17] It is understood that such state vectors are improper and that a "smearing out" is necessary to make them meaningful.

or if

$$\int_{T1}^{\infty}\left\|\frac{dQ}{dt}\right\|dt<\epsilon,$$

that is, if $\|dQ/dt\|$ is integrable to infinity.

By Eq. (4.11),

$$\|dQ/dt\|=\left\|\exp(iHt)\int i[H-\sum E(k_n)] \times\prod f_n(k_n)\mathbf{c}(k_n)\exp[-i\sum E(k_n)t]d^3k_n\psi_0\right\|. \quad (4.12)$$

The asymptotic vanishing of this limit follows from Eq. (4.10) if the uniform limit $t(n)=t$ is taken according to postulate (2a). The integrability follows from postulate (2b). From this point on, the theory of the S operator may be established as usual.[15] In particular, Haag has shown that all operators $\mathbf{c}$ which satisfy our requirements lead to the same S matrix.

The results of Secs. II and III can now be recognized to be special applications of the general theory. We consider a special class of theories in which the operators $\mathbf{c}(k)$ may be chosen so that the canonical commutation relations hold:

$$[\mathbf{c}(k),\mathbf{c}^{\dagger}(k')]=\delta(k-k'), \quad (4.13)$$

and that the adjoints of $\mathbf{c}_k$ are destruction operators:

$$\mathbf{c}(k)^{\dagger}\psi_0=0. \quad (4.14)$$

These are the theories without vacuum polarization. For this class of theories, Eq. (4.11) may be written under the form

$$Q(t)=\exp(iHt)\exp(-iKt)\prod_1^N \mathbf{c}_{f(n)}\psi_0, \quad (4.15)$$

where

$$K=\int\mathbf{c}(k)E(k)\mathbf{c}^{\dagger}(k)d^3k, \quad (4.16)$$

since

$$\exp(-iKt)\mathbf{c}(k)\exp(+iKt)=\mathbf{c}(k)\exp(-iE_kt), \quad (4.17)$$

and

$$K\psi_0=0, \quad \exp(iKt)\psi_0=\psi_0. \quad (4.18)$$

A set of equivalent operators may be generated by the substitution

$$\mathbf{c}'(k)=U\mathbf{c}(k)U^{\dagger}. \quad (4.19)$$

For the new operators $\mathbf{c}'$, the appropriate form of $Q(t)$, according to Eq. (4.11), is

$$Q(t)=\exp(iHt)\exp(-iK't)\prod\mathbf{c}_{f(n)}'\psi_0, \quad (4.20)$$

where K' is the function K [Eq. (4.16)] of the new operators, i.e.,

$$K'=UKU^{\dagger}. \quad (4.21)$$

We have

$$Q'(t)=\exp(iHt)\exp(-iKt)\exp(iKt)U \times\exp(-iKt)U^{\dagger}U\prod\mathbf{c}_{f(n)}U^{\dagger}\psi_0. \quad (4.22)$$

If $U^{\dagger}\psi_0=\psi_0$, and

$$\exp(iKt)U\exp(-iKt)\rightarrow 1, \quad (4.23)$$

the limits of Q and Q', and hence the S matrix, are identical.

We have shown that the equivalent Hamiltonians considered in Sec. II may be considered as being identical operators, expressed as functions of different but equivalent physical-particle operators.

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APPENDIX

The "asymptotic vanishing of destruction operators" in theories without vacuum polarization is not new. For convenience, and without any claim to originality, we give a somewhat detailed proof.

The square-integrable function F_N in Eq. (2.15) may be uniformly approximated by sums of products of square-integrable functions of one-particle variables k_i, k_i'. Hence, we only have to prove

$$a_{ft}\psi\equiv\int f(k)e^{-iE(k)t}a(k)d^3k\psi\rightarrow 0\ (t\rightarrow\pm\infty),$$

with square-integrable f. If we restrict ourselves to a particle number L smaller than L_m, then a_{ft} is bounded, and we may, for the purpose of the proof, consider only an everywhere dense set of states ψ'. The assertion that if $a_{ft}\psi'\rightarrow 0$ then $a_{ft}\psi\rightarrow 0$, for *all* states ψ, is proved as follows. By the definition of "everywhere dense," for every ψ there exists a ψ' such that $\|\psi'-\psi\|<\epsilon/2c$ where ϵ and c are positive numbers. If a_{ft} is bounded there exists a number c such that $\|a_{ft}\phi\|\leq c\|\phi\|$. Then,

$$\|a_{ft}\psi\|=\|a_{ft}(\psi-\psi')+a_{ft}\psi'\|\leq\|a_{ft}\|\|\psi-\psi'\| +\|a_{ft}\psi'\|\leq\epsilon/2+\|a_{ft}\psi'\|.$$

The first inequality is the "triangle inequality." The assertion is thereby proved. For the set ψ', we choose, in Fock representation, finite sums of products of one-particle functions $\Phi,\cdots\Phi_l$ for the l-particle subspace. These form an everywhere dense set. By the definition of the destruction operators

$$a_{ft}\Phi,\cdots\Phi_e=(1/l!)[(f_t^*,\Phi_1)\Phi_2\cdots\Phi_l +\Phi_1(f_t^*,\Phi_2)\cdots\Phi_l+\cdots+\Phi_1\cdots(f_t^*,\Phi_l)],$$

where

$$f_t=f(k)\exp[-iE(k)t].$$

It is now sufficient to show that

$$(f_t^*,\Phi)=\int \exp[-iE(k)t]f(k)\Phi(k)d^3k \to 0.$$

A change of variables from k to E and two angle variables Ω gives

$$(f_t^*,\Phi)=\int \exp(-iEt)f(E,\Omega)\Phi(E,\Omega)JdEd\Omega,$$

where J is the Jacobian. By Schwarz's inequality, the product of two square-integrable functions f and Φ is absolutely integrable with the weight function J. The Riemann-Lebesgue lemma then asserts the asymptotic vanishing of the norm $\|a_{ft}\psi\|$.

The "asymptotic vanishing of destruction operators" was used previously by Coester and Kummel.[18]

[18] F. Coester and H. Kummel, Nuclear Phys. **9**, 225 (1959), Appendix II.

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Canonical Variables for General Relativity

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The general theory of relativity is cast into normal Hamiltonian form in terms of two pairs of independent conjugate field variables. These variables are explicitly exhibited and obey ordinary Poisson bracket relations. This form is reached by imposing a simple set of coordinate conditions. It is shown that those functionals of the metric used as invariant coordinates do not appear explicitly in the Hamiltonian and momentum densities, so that the standard differential conservation laws hold. The bearing of these results on the quantization problem is discussed.

I. INTRODUCTION

In the program of quantization of general relativity according to the Schwinger action principle it has previously been shown that the classical theory can, in principle, be reduced to canonical form in terms of two pairs of independent, unconstrained canonical variables.[1] This canonical form has been given explicitly for the linearized theory in I, where the analysis was in complete analogy to the quantization of electromagnetic theory. In II, a general study of the dynamics of the full classical theory led to the exhibition of four unconstrained variables whose specification fully determines the state of the system, but which were not canonical. The precise definition of these variables depended upon the specification of the four remaining field variables (the gauge functions) as invariant coordinates. In general, these four field variables will appear explicitly in the Hamiltonian density which arises when the dynamical variables are rearranged into canonical form. Thus, the Hamiltonian density will in general depend upon the coordinates explicitly and hence the system will appear to be nonconservative for a closed system. The remaining problem, as was stressed in II, was thus to achieve canonical form while simultaneously choosing as coordinates a set of gauge variables which will not appear in the resulting Hamiltonian.

This paper exhibits explicitly a simple set of canonical variables and coordinate conditions which solve the above problem.

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[1] Two previous papers in this series, R. Arnowitt and S. Deser, Phys. Rev. **113**, 745 (1959); and R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **116**, 1322 (1959), will be referred to as I and II, respectively. Notation and units are as in II with the exception that g^{ij} here denotes the three dimensional matrix inverse to g_{ij}. Natural units are employed: $\kappa=16\pi\gamma c^{-4}=1$, $c=1$, where γ is the Newtonian gravitational constant. Latin indices run from 1 to 3, and Greek from 0 to 3 and $x^0=t$. Ordinary differentiation is denoted by a comma in a subscript or by the symbol ∂_μ.

II. FORMULATION OF THE PROBLEM

We begin with the action integral obtained in II when the algebraic constraints have been eliminated. Written in terms of variables appropriate to a 3+1 dimensional breakup it becomes

$$I=\int d^4x\, \mathfrak{L}, \tag{2.1}$$