Radiation Damping of an Electron in a Uniform Magnetic Field*

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The analytic solutions are given for the pair of differential equations obtained from classical theory which express the time rate of change of the angle between the momentum vector and the magnetic field vector, and the time rate of change of the energy of the electron.

I. INTRODUCTION

ENERGETIC electrons (~1 Mev), which coexist with low-temperature ions and neutral molecules to form a tenuous plasma (Lorentzian gas), can be kept under observation for extremely long times[1] (many seconds) when they are trapped in a constant magnetic field geometry such as is utilized in controlled fusion research, e.g., the mirror machine. If radiative effects are negligible, then a minimum vacuum-containment time of ions, under certain conditions, can be inferred from the observed electron containment times since for a given magnetic field configuration charged particles moving in the same direction at a given point in space and having the same Larmor radius will follow the same trajectory. This can be shown in the following manner. The force acting on a charged particle is given by

$$d\mathbf{p}/dt=(e/c)\mathbf{v}\times\mathbf{B}. \tag{1}$$

Neglecting radiation this equation can be rewritten as

$$\rho_0\frac{d}{ds}\left(\frac{\mathbf{v}}{v}\right)=(\mathbf{v}/v)\times(\mathbf{B}/B), \tag{2}$$

where $\rho_0=pc/eB=\rho/|(\mathbf{v}/v)\times(\mathbf{B}/B)|$ if $\rho=p_\perp c/eB$ is the Larmor radius, S is the distance measured along the trajectory, and $\mathbf{v}/v$ and $\mathbf{B}/B$ are the unit vectors. The conclusion that ions have long vacuum-containment times is important for some of the schemes proposed for creating a plasma by high-energy injection[2] since the plasma buildup times are equally long and during the early stages of the buildup a single particle model is applicable. However, for comparable momenta an electron traverses a trajectory much faster than an ion, and consequently radiates more. The purpose of this paper is to determine the importance of radiation damping on the motion of an electron with regard to its energy and containment time.

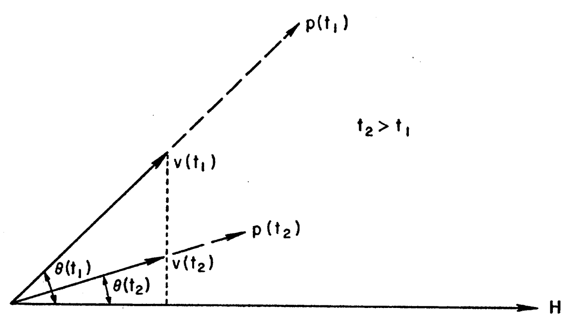

FIG. 1. Qualitative picture of the change in the velocity and momentum vectors with time.

The vector potential for the magnetic field of a mirror machine may to a close approximation be written analytically; however, the equations of motion for a charged particle in such a field have not been solved analytically. Theoreticians have approached the problem in several ways.

A direct method of attack is to solve the equations of motion, for a given field configuration and given initial conditions, numerically over a length of the particle path where accumulative errors are negligible.[3] This can be done for the order of 10^2 reflections in a mirror machine, whereas observed containment times correspond to 10^8 reflections.

It can be shown rigorously from a Störmer type analysis, where radiation is neglected, that a certain class of particles, all of which encircle the axis, is absolutely confined within the mirrors.

Approximate solutions may be obtained when the physical situation warrants making the assumption that the relative change in the magnetic field is small over a Larmor diameter. This approximation yields the adiabatic invariants. One of the adiabatic invariants is the magnetic moment of the particle, and this property of the particle motion leads to the loss cone concept,[4] i.e., if the velocity vector of a particle at a given point makes an angle, θ, with the magnetic field, B, which is less than some critical angle, θ_c, the particle is not contained within the magnetic mirrors; whereas if $\theta>\theta_c$ the particles are contained.

A loss mechanism for trapped adiabatic particles is the scattering of their velocity vector into the loss cone. In a sufficiently good vacuum the effects of radiation

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[2] Gibson, Lamb, and Lauer, Phys. Rev. 114, 937 (1959).

[3] Garren, Riddell, Smith, Bing, Henrich, Northrop, and Roberts, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Geneva, Switzerland* (United Nations, Geneva, 1958), A/Conf. 15/P/383.

[4] E. Fermi, Phys. Rev. 75, 1169 (1949).

due to the helical motion of the electrons about the field may also contribute to the loss of the electrons which have a kinetic temperature of interest for thermonuclear work. Indirectly radiation affects the containment time of the electrons since the scattering cross section increases as the particle energy decreases. It may affect the containment time directly if the reaction force due to the radiation affects the orientation of the velocity vector relative to the field. In a dense plasma of course, aside from the complicating effects associated with cooperative effects (instabilities, etc.), the absorption of the radiation must be considered, since it may reduce the effects mentioned above. A relativistic effect is the emission of the radiation in the higher harmonics of the basic gyromagnetic frequency, and it has been shown[5] that this radiation may escape from the plasma.

To ascertain the importance of the radiation damping without absorption in the various containment geometries the motion of a single electron in a uniform magnetic field is investigated.

II. EQUATIONS

From the expression for radiation damping in terms of the external field (which is found in several tests[6]) the following equations may be derived[7]:

$$d\theta/d\tau = -\sin\theta\cos\theta/\epsilon, \tag{3}$$

and

$$d\epsilon/d\tau = -(\epsilon^2-1)\sin^2\theta, \tag{4}$$

where θ is the angle between the magnetic field and the momentum vector, ϵ is the total energy of the electron in units of m_0c^2 (the rest energy of the electron $=0.51$ Mev), and τ is the time in units of $\frac{3}{2}(m_0c/r_0^2H^2)=5.1\times10^2$ sec$/[H^2/(\text{kilogauss})^2]$, where H is the magnetic field strength, and r_0 is the classical electron radius.

The analytic solutions to Eqs. (3) and (4) are

$$\theta(\tau)=\cot^{-1}\left[\left(\frac{1}{B\epsilon(0)}\sinh B\tau+\cosh B\tau\right)\cot\theta(0)\right], \tag{5}$$

and

$$\epsilon(\tau)=\epsilon(0)\frac{1}{B\epsilon(0)}\frac{\sinh B\tau+B\epsilon(0)\cosh B\tau}{\cosh B\tau+B\epsilon(0)\sinh B\tau}, \tag{6}$$

where

$$B=(1/\epsilon(0))\sin\theta(0)\{[\epsilon(0)]^2+[\cot\theta(0)]^2\}^{\frac{1}{2}}$$

and $\epsilon(0)$, $\theta(0)$ are the initial values. As $\tau\rightarrow\infty$; $\theta\rightarrow0$ and $\epsilon\rightarrow1/B$ for $\theta\neq\pi/2$.

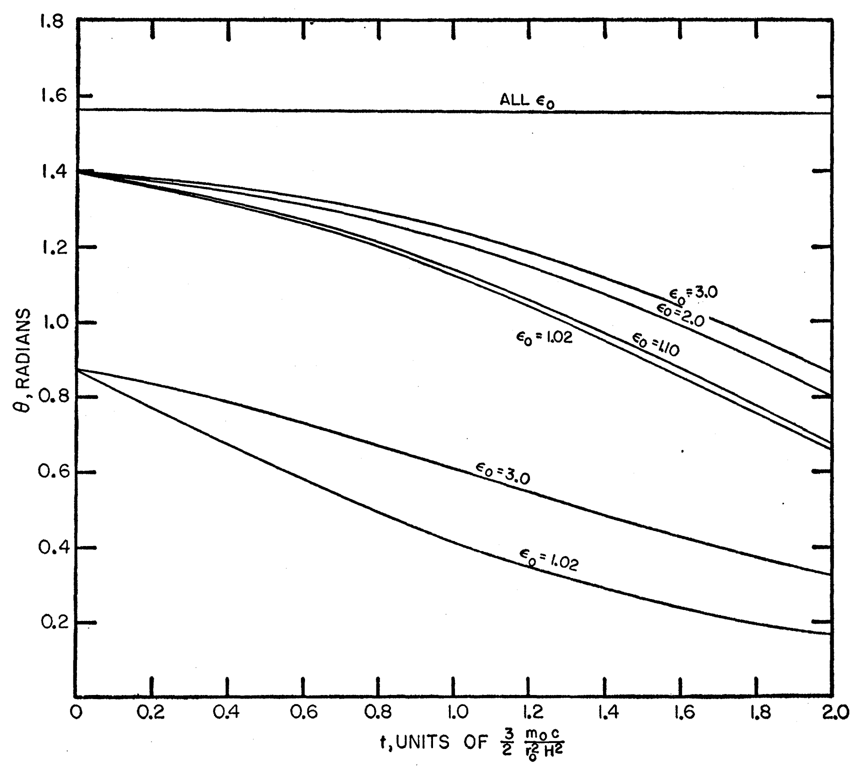

FIG. 2. θ vs time for different initial energies and velocity orientations.

[5] B. A. Trubnikov and V. S. Kudryavtsev, reference 3, A/Conf. 15/P/2213.
[6] For example, W. Heitler, *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954); or L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, 1951).
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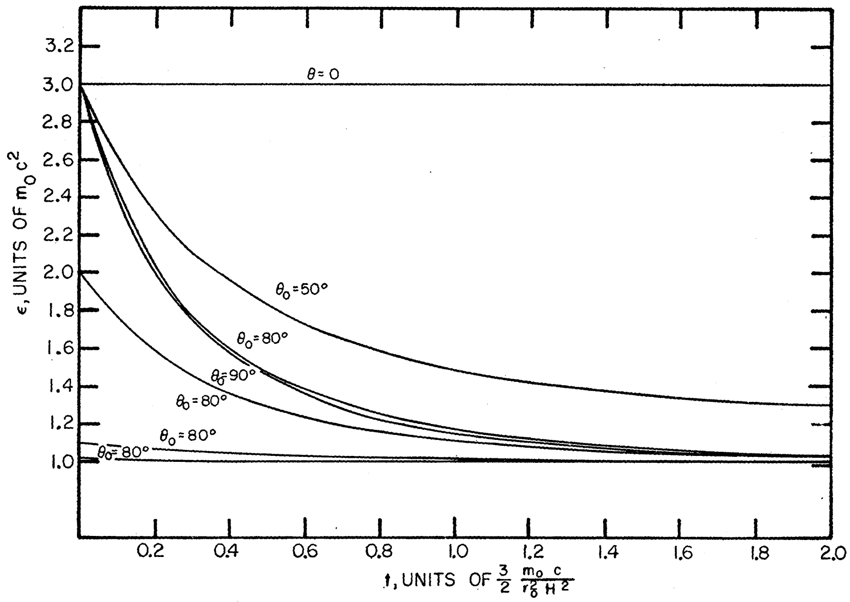

FIG. 3. Energy vs time for different initial energies and velocity orientations.

III. DISCUSSION

At nonrelativistic speeds the reaction force on the electron is normal to the field, however, the damping force more nearly approaches the direction opposite to the velocity vector as the electron becomes more relativistic. For all speeds the velocity component in the direction of the field is constant. At ultrarelativistic velocities the angle between the velocity or momentum vector and the direction of the field does not change appreciably with time; however, the magnitude of the momentum vector decreases because of the mass change resulting from the radiation of energy, whereas, the magnitude of the velocity vector is essentially constant ($\sim c$). At other than ultrarelativistic velocities the momentum and velocity vectors decrease in magnitude and move in the direction of the magnetic field in the manner illustrated in Fig. 1.

Because of radiation the magnetic moment of the particle motion is reduced, and no particles though they interact only with the magnetic field can be absolutely bound in a mirror machine except for those special cases of $v_{\parallel}=0$ in the central plane and the planes of the mirrors. The motions of electrons started with different initial conditions are shown in Figs. 2 and 3. For these cases the velocity vector would enter the loss cone in a time of the order of $\tau=1$. During this time the kinetic energy of these particles may decrease more than an order of magnitude because of the radiation. For a field of 22.5 kilogauss $\tau=1$ corresponds to 1 sec.

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