$a \geqslant c > 2a/3$ we obtain

$$x_3^{(\gamma)} = 8t^2\left(1+\frac{2t}{a-2t}+\frac{(a-c)(c-2t)}{2t(a-2t)}\right), \quad (4.12)$$

provided we have $\frac{1}{2}c \leqslant 2t \leqslant 2c-a$, and

$$x_3^{(\gamma)} = 8t^2\{1+[2t/(a-2t)]+(a-2t)/8t\} \quad (4.13)$$

for $2c-a \leqslant 2t < a$. For cases where we have $0 \leqslant c \leqslant 2a/3$, we find Eq. (4.13) provided $\frac{1}{3}a \leqslant 2t < a$. In the special case of the electromagnetic form factors for the nucleon, we have for the isotopic vector part $a = M + m_\pi$, $c = 2m_\pi$, and $2t = M$, which gives

$$x_3^{(\beta)} = 2m_\pi^2 2M/(2M-m_\pi),$$
$$x_3^{(\gamma)} = (M/2m_\pi)(2M+m_\pi). \quad (4.14)$$

The isotopic scalar part requires $c = 3m_\pi$ and leads to

$$x_3^{(\beta)} = 3m_\pi^2 M/(M-m_\pi),^{12}$$
$$x_3^{(\gamma)} = (M/2m_\pi)(2M+m_\pi). \quad (4.15)$$

In problems related to the question of consistency of quantum electrodynamics, it is sometimes useful to know some analytic properties of the electron-photon vertex function. From the direct representation, we can say only the following: if one is willing to introduce a small, auxiliary photon mass $\lambda > 0$ such that we have $x = m_e^2$, $a = b = m_e + \lambda$, $c = 3\lambda$, then the singularities in the z_3 plane are restricted to a finite region and the static cut $x_3 \geqslant (3\lambda)^2$, $y_3 = 0$. The real boundary points of the region with complex singularities are given by Eqs. (4.15) with M replaced by m_e, and m_π by λ. Note that for $\lambda \to 0$ the mass variable $x_1 = x_2 = m_e^2$ coincides with the static cut $x \geqslant a^2 = \lim_{\lambda\to 0}(m_e+\lambda)^2$, $y = 0$, and the singular region covers the whole z_3 plane.

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Kinematics of General Scattering Processes and the Mandelstam Representation

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The kinematics of an arbitrary process involving two incoming and two outgoing particles is studied in terms of the invariants used in Mandelstam's representation, treating the three processes described by the same Green's function simultaneously. It is shown that the physical regions for these processes are bounded by a cubic curve in the plane of the two independent invariants. The unitarity conditions are discussed in the approximation of neglecting intermediate states of more than two particles. The formula for the spectral functions of the double dispersion relation is obtained explicitly in terms of the invariants chosen.

1. INTRODUCTION

MANDELSTAM[1] has recently proposed a representation of the scattering amplitude for meson-nucleon scattering, which is obtained from a plausible assumption about its behavior as an analytic function of two variables, the energy and momentum transfer. He has also been able to show,[2] for a more general process, that the representation is satisfied by the lower orders of the perturbation series, and that this series can actually be constructed from the representation and the unitarity relations,[3] in a two-particle approximation. In this paper we shall discuss certain aspects, mainly kinematical, of the extension of this representation to a general process. We consider together the three processes

$$\text{I}: \quad 1+2 \to 3+4,$$
$$\text{II}: \quad 1+\bar{3} \to \bar{2}+4,$$
$$\text{III}: \quad 1+\bar{4} \to \bar{2}+3.$$

The complications of spin and isotopic spin will be ignored, and all the particles will be assumed to be stable.

In Sec. 2 we shall find the physical regions for the three scattering processes in terms of the three invariants r, s, t, whose sum is equal to the sum of squared masses of the four particles. These invariants may be regarded as homogeneous coordinates in a plane, and the physical regions are then bounded by a cubic curve in this plane. The curve has three branches corresponding to the physical regions for the three scattering processes, and also a closed branch within the rst-triangle. The interior of this closed curve would correspond to the physical region for the decay process

$$\text{IV}: \quad 1 \to \bar{2}+3+4$$

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[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[2] S. Mandelstam, Phys. Rev. **115**, 1741 (1959).

[3] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).

if this were possible. The form of the double dispersion relation for such processes has been given by Mandelstam.[2]

In Sec. 3 we shall discuss the unitarity condition for a typical process in the two-particle approximation. By making use of the determinant of scalar products of the independent momenta, the unitarity condition can be written in the form of an integral over the invariants. From this we are able to derive the relation giving the spectral functions in terms of the absorptive parts, in a similar form. The boundaries of the regions where the spectral functions are nonzero will be given by the vanishing of the determinant.

2. KINEMATICS

For convenience, we shall assume that the masses of the four particles involved in the processes I, II, III satisfy the inequalities

$$m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0. \tag{1}$$

If

$$m_1 > m_2 + m_3 + m_4, \tag{2}$$

then the decay process IV is also energetically possible (although in that case we assume that it has vanishing probability). We shall choose the momenta of the incoming and outgoing particles to be p_i and $-p_i$, respectively, so that the conservation equation is always

$$p_1 + p_2 + p_3 + p_4 = 0. \tag{3}$$

The metric is chosen so that $p_i^2 = m_i^2$.

In addition to the masses, there are two independent scalar products. It is, however, convenient to use the three invariants

$$\begin{aligned} r &= (p_1+p_2)^2 = (p_3+p_4)^2, \\ s &= (p_1+p_3)^2 = (p_2+p_4)^2, \\ t &= (p_1+p_4)^2 = (p_2+p_3)^2, \end{aligned} \tag{4}$$

which satisfy

$$r + s + t = K \equiv m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{5}$$

In the center-of-mass system for process I, the momenta are $p_1 = (E_1, \mathbf{q}_1)$, $p_2 = (E_2, -\mathbf{q}_1)$, $p_3 = (-E_3, -\mathbf{q}_3)$, $p_4 = (-E_4, \mathbf{q}_3)$. The invariant r is then the square of the total energy,

$$r = W^2, \quad W = E_1 + E_2 = E_3 + E_4.$$

The magnitudes of the spatial momenta are given by

$$\begin{aligned} 4rq_1^2 &= [r - (m_1+m_2)^2][r - (m_1-m_2)^2], \\ 4rq_3^2 &= [r - (m_3+m_4)^2][r - (m_3-m_4)^2], \end{aligned}$$

and the invariants s and t may be related to the scattering angle by

$$\begin{aligned} 2s &= K - r + 4q_1q_3z - (m_1^2 - m_2^2)(m_3^2 - m_4^2)/r, \\ 2t &= K - r - 4q_1q_3z + (m_1^2 - m_2^2)(m_3^2 - m_4^2)/r, \end{aligned} \tag{6}$$

where $z = \cos(\mathbf{q}_1, \mathbf{q}_3)$.

The conditions for a physical scattering process may now be expressed in terms of r, s, t. The necessary condition

$$\pm p_i \cdot p_j > m_i m_j$$

yields

$$r > (m_1+m_2)^2 \quad \text{or} \quad r < (m_1-m_2)^2, \tag{7}$$

and similar inequalities for other pairs of masses. The requirement that the scattering angle be real can be stated in the form

$$\begin{vmatrix} p_1^2 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ p_2 \cdot p_1 & p_2^2 & p_2 \cdot p_3 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3^2 \end{vmatrix} > 0.$$

This may be written as a homogeneous inequality in r, s, t,

$$rst > (r+s+t)^2(ar+bs+ct) \tag{8}$$

where the dimensionless constants a, b, c are given by

$$\begin{aligned} K^3a &= (m_1^2m_2^2 - m_3^2m_4^2)(m_1^2 + m_2^2 - m_3^2 - m_4^2), \\ K^3b &= (m_1^2m_3^2 - m_2^2m_4^2)(m_1^2 + m_3^2 - m_2^2 - m_4^2), \\ K^3c &= (m_1^2m_4^2 - m_2^2m_3^2)(m_1^2 + m_4^2 - m_2^2 - m_3^2). \end{aligned}$$

The variables r, s, and t may now be regarded as homogeneous coordinates in a plane, in which the line at infinity is $r+s+t=0$. The region (8) is bounded by a cubic curve in this plane, whose asymptotes are $r=0$, $s=0$ and $t=0$. Moreover, the curve intersects its asymptotes on the line

$$ar + bs + ct = 0. \tag{9}$$

The shape of the curve is shown in Fig. 1, in which the regions marked I, II, and III are the physical regions for the corresponding processes. We note that, by the assumed inequalities for the masses, the constants a, b and c satisfy

$$a \geq b \geq c,$$

and that a and b are necessarily positive, although c may have either sign. If c is negative, the line (9) passes within the rst-triangle, and therefore the region

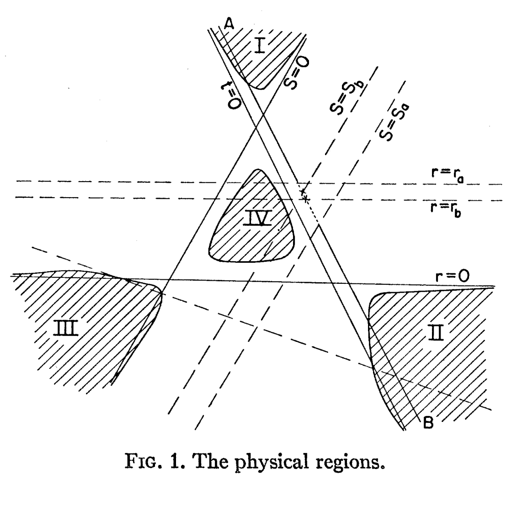

FIG. 1. The physical regions.

III in Fig. 1 includes part of this triangle. In other words, all three invariants can be positive for this process.

The bounding lines of the region defined by (7) can be shown to be tangential to the curve, so that the entire boundary of each of the physical regions is a part of the curve.[4] If the condition (2) is satisfied, then the excluded strips

$$\begin{aligned}(m_1-m_2)^2<r<(m_1+m_2)^2,\\ (m_3-m_4)^2<r<(m_3+m_4)^2\end{aligned} \qquad (10)$$

do not overlap, so that the region IV is allowed kinematically, as we should expect. If (2) is not satisfied, however, the strips (10) overlap, and the region IV is excluded.

3. MANDELSTAM REPRESENTATION AND UNITARITY CONDITIONS

We shall assume that for the processes in question the ordinary dispersion relations are valid. Thus for a fixed value of t, say, one can write a dispersion relation which will be an integral along a line such as AB in Fig. 1. The poles will occur on lines $r=r_b$ and $s=s_b$, and the continuous integrals will begin on lines $r=r_a$ and $s=s_a$, as indicated. Here r_b and r_a are the squared masses of the single-particle[5] and lowest two-particle intermediate states in the process I, respectively. Using the same assumptions as in his treatment of meson-nucleon scattering,[1] Mandelstam[2] has given a double-variable representation for these processes, involving three spectral functions, here denoted[6] by A_{rs}, A_{rt}, and A_{st}. The function A_{rs}, for example, will be nonzero in a region lying within the triangle defined by $r>r_a$, $s>s_a$. To find the exact boundary, C_{rs}, of this region, as shown in Fig. 2, we must use the unitarity relation. We shall assume that intermediate states with three or more particles may be neglected in this relation, and further that only one pair of particles contributes to the two-particle intermediate states. If there is more than one such pair, we have only to sum the contributions from each. In this approximation, the unitarity condition for the process I is

$$A_{1r}(rs_1)=\tfrac{1}{2}(2\pi)^{-2}\int d^4p_5d^4p_6\delta(p_5^2-m_5^2)\theta(p_{50})\delta(p_6^2-m_6^2)\times\theta(p_{60})\delta(p_5+p_6-p_1-p_2)A_3^*(rs_3)A_2(rs_2), \qquad (11)$$

where A_1, A_2, and A_3 now refer to the processes

$$\begin{aligned}\mathrm{I}_1:&\ 1+2\rightarrow 3+4,\\ \mathrm{I}_2:&\ 1+2\rightarrow 5+6,\\ \mathrm{I}_3:&\ 3+4\rightarrow 5+6,\end{aligned}$$

respectively. The invariants are defined[7] as in Sec. 2. We now wish to convert (11) into an integral over invariants. To do this, we perform the p_6-integration using the δ-function, and convert the p_5-integration into one over p_5^2, $(p_5-a)^2$, $(p_5-b)^2$, $(p_5-c)^2$, where a, b and c are any three fixed timelike vectors. The Jacobian for this transformation is

$$J=\tfrac{1}{16}[-\Delta(a,b,c,p_5)]^{-\frac{1}{2}},$$

where

$$\Delta(a,b,c,p)=\begin{vmatrix}a^2 & a\cdot b & a\cdot c & a\cdot p\\ b\cdot a & b^2 & b\cdot c & b\cdot p\\ c\cdot a & c\cdot b & c^2 & c\cdot p\\ p\cdot a & p\cdot b & p\cdot c & p^2\end{vmatrix}. \qquad (12)$$

It is convenient to choose $a=p_1+p_2$, $b=p_1$, $c=-p_3$, so that

$$(p_5-a)^2=p_6^2,\quad (p_5-b)^2=s_2,\quad (p_5-c)^2=s_3.$$

Then, using the definitions of invariants, we find

$$\Delta(a,b,c,p)=\tfrac{1}{16}\begin{vmatrix}2r & r+m_1^2-m_2^2 & r+m_3^2-m_4^2 & r+m_5^2-m_6^2\\ r+m_1^2-m_2^2 & 2m_1^2 & m_1^2+m_3^2-s_1 & m_1^2+m_5^2-s_2\\ r+m_3^2-m_4^2 & m_1^2+m_3^2-s_1 & 2m_3^2 & m_3^2+m_5^2-s_3\\ r+m_5^2-m_6^2 & m_1^2+m_5^2-s_2 & m_3^2+m_5^2-s_3 & 2m_5^2\end{vmatrix}=\Delta(r;s_1s_2s_3),\ \text{say.} \qquad (13)$$

This transformation is not one-to-one, since the scalar products are unaltered by changing the sign of the component of p_5 perpendicular to a, b and c. This introduces an extra factor of 2. Finally, we obtain

$$A_{1r}(rs_1)=(1/64\pi^2)\int ds_2ds_3\,[-\Delta(r;s_1s_2s_3)]^{-\frac{1}{2}}\times A_3^*(rs_3)A_2(rs_2), \qquad (14)$$

where the integration is over the region where Δ is negative.

The function Δ can of course also be expressed in terms of s_1, t_2, t_3 by interchanging m_5 and m_6 in (13), and similarly it can be expressed in terms of t_1, s_2, t_3 or t_1, t_2, s_3.

Now, in order to find an expression for the spectral functions, we have to substitute in (14) the ordinary dispersion relations for A_2 and A_3 in which r is held fixed. If we choose the value of r to be such that

$$r>(m_1+m_2)^2,\quad r>(m_3+m_4)^2,\quad r>(m_5+m_6)^2, \qquad (15)$$

and take s_1 to be in the physical region for the process I_1, then it is easy to see that the condition $\Delta<0$ implies

[4] In the case of elastic scattering, the curve degenerates into a straight line and a hyperbola.

[5] Of course there may be more than one such particles and hence more than one pole, or there may be none at all.

[6] Mandelstam denotes the corresponding functions by A_{12}, A_{13}, and A_{23}. That notation would, however, be likely to cause confusion with the A_1, A_2, and A_3 introduced below.

[7] Note that if we define the signs of the momenta in I_1 and I_2 according to the convention (3), then two of the momenta in I_3 have the "wrong" sign. Thus we must define, for example, $s_3=(p_3-p_5)^2$.

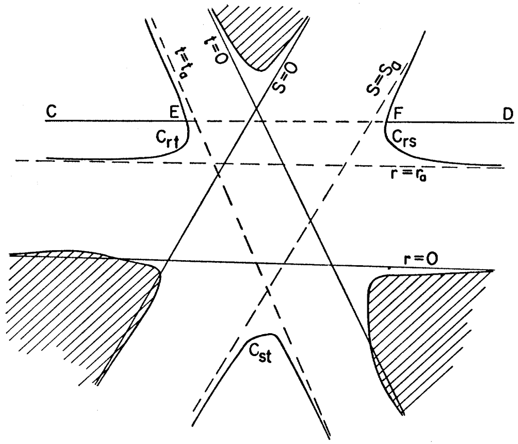

FIG. 2. The regions in which the spectral functions are nonzero.

that s_2 and s_3 are in the physical regions for the processes I_2 and I_3, respectively. Thus the denominators of the dispersion integrals for A_2 and A_3 will never vanish in the region of integration in (14), and we may ignore their small imaginary parts. For values of r which do not satisfy (15) we must have recourse to analytic continuation in the masses.[8]

It is now possible to perform the s_2 and s_3 integrations in (14) explicitly. This can be done most simply by introducing the center-of-mass variables

$$z_1=\cos(\mathbf{q}_1,\mathbf{q}_3),\quad z_2=\cos(\mathbf{q}_1,\mathbf{q}_5),\quad z_3=\cos(\mathbf{q}_3,\mathbf{q}_5),$$

which are linearly related to s_1, s_2, s_3 by the analogs of (6), as is done by Mandelstam.[2] We find by combining the rows and columns of the determinant (13) in a suitable way that

$$\Delta(r;\,s_1s_2s_3)=rq_1^2q_3^2q_5^2k(z_1z_2z_3),$$

where

$$k(z_1z_2z_3)=z_1^2+z_2^2+z_3^2-1-2z_1z_2z_3.$$

Thus the s_2 and s_3 integrations reduce to an integral already evaluated by Mandelstam.[9] The values of the spectral functions A_{1rs} and A_{1rt} may now be found from (14) by evaluating the discontinuity across the real s_1-axis. The function A_{1r} is easily seen to be an analytic function of s_1 for fixed real values of r, except for these cuts, indicated by the lines CE and FD in Fig. 2. The expression for A_{1rs}, obtained by evaluating the discontinuity along FD, may be concisely expressed in terms of the original invariants. It is[10]

$$A_{1rs}(rs_1)=(1/32\pi^2)\left\{\int ds_2ds_3\,[\Delta(r;\,s_1s_2s_3)]^{-\frac{1}{2}}\right.$$
$$\times A_{3s}^*(rs_3)A_{2s}(rs_2)+\int dt_2dt_3\,[\Delta(r;\,s_1t_2t_3)]^{-\frac{1}{2}}$$
$$\left.\times A_{3t}^*(rt_3)A_{2t}(rt_2)\right\}.\quad(16)$$

Here the region of integration in both terms is part of the region where $\Delta>0$, and is bounded by one branch of the curve $\Delta=0$. In the first integral, s_2 and s_3 are always positive, and in the second, t_2 and t_3 are. There are of course two other branches of the curve, corresponding to positive t_1 rather than s_1, which bound the regions where A_{1rt} is nonzero.

It should be remarked that Eq. (16) is remarkably similar to the relation (14) for A_{1r} itself, except for the fact that (14) is an integral over the physical region, whereas (16) is entirely over part of the unphysical region.

The boundary C_{rs} of the region where A_{1rs} is nonzero will clearly be given by the appropriate branch of the curve $\Delta=0$, in which the arguments s_2 and s_3 are given their minimum values, provided that these are attainable simultaneously. In the general case, we must consider all those four-cornered diagrams which are such that none of the four internal masses (of one or more particles) can be decreased.

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[8] This situation is discussed in detail in reference 2.

[9] See reference 1, Eq. (3.5).

[10] The fourth-order contribution to A_{rs} may be alternately evaluated by finding the discontinuity in A_s on crossing the real r-axis (see reference 3). The consistency of the two methods of calculation is assured by the invariance of Δ under the simultaneous interchange $m_2^2\leftrightarrow m_3^2$, $m_5^2\leftrightarrow s_2$, $m_6^2\leftrightarrow s_3$, $r\leftrightarrow s_1$. In fact Δ has a great deal more symmetry than this. It is invariant under a transitive permutation group on its ten arguments r, s_1, s_2, s_3, m_1^2, $\cdots m_6^2$, isomorphic to the symmetric group of degree 5.