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Fine Structure in the Decline of the Ferromagnetic Resonance Absorption with Increasing Power Level

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A theory of ferromagnetic resonance at high signal powers is developed. The stationary response at high power levels is investigated for the case in which the unstable spin waves have the same frequency as the applied signal. It is found that a fine structure should be superimposed on the general decline of the resonance absorption with increasing power level. This fine structure arises from the discrete nature of the spin-wave spectrum. It should be observable even if the frequency separation of adjacent spin-wave modes is much smaller than the inverse of their relaxation times. The fine structure appears as a series of kinks superimposed on the general decline of the resonance absorption with increasing power level. The separation of subsequent kinks increases with decreasing sample volume and increasing exchange field. An interpretation of experimental data along the lines suggested in this paper should yield information about the strength of the exchange coupling.

I. INTRODUCTION

IT has been known since the work of Damon[1] and of Bloembergen and Wang[2] that at sufficiently high power levels nonlinear effects can be observed in ferromagnetic resonance experiments. These effects usually set in at a fairly well-defined threshold. At power levels below this threshold the response of the material to an rf magnetic field is linear; i.e., the susceptibility is independent of the power level. Beyond the threshold the susceptibility at resonance decreases with increasing power level. The general features of the observed phenomena can be understood quite well in terms of a theory developed by Suhl.[3] According to this theory the nonlinear effects are due to the fact that certain spin waves become excited as soon as the amplitude of the uniform mode (which is driven by the applied microwave field) exceeds a certain critical value. Two possible mechanisms may be distinguished. For one of them, the unstable spin waves have half the frequency of the applied signal; for the other, they have the same frequency as the applied signal. It has been shown by Suhl that the first process (which involves spin waves whose frequency equals half the signal frequency) is not allowed at resonance if the signal frequency exceeds a certain critical value which for spherical samples is $\frac{2}{3}\gamma 4\pi M$.

Green[4] has recently observed a fine structure in the decline of the susceptibility with increasing power level which is superimposed on the general downward trend. Similar effects have been observed by Seiden[5] and by Martin.[6] The experimental conditions in each case were such that the unstable spin waves must have the same frequency as the signal. In the present paper a theory will be developed which accounts quantitatively for the observed effects and allows an interpretation of the experimental data in terms of fundamental physical properties of the material. The theory is essentially an extension of Suhl's work, but a number of refinements not considered previously are necessary to explain the observed effect.

The basic physical reason for the fine structure is the fact that the spin-wave spectrum is discrete rather than continuous. If periodic boundary conditions are imposed, the excitations have the form of plane waves and the components of the wave number vector are integer multiples of $2\pi/L$, where L is the length of the periodicity cube. The spin waves may thus be represented by an array of points in k space such as shown in Fig. 1. It was shown by Suhl that at resonance and at sufficiently high frequencies (larger than approximately 3000 Mc/sec for spherical samples of YIG), the unstable spin waves propagate along the direction of the magnetic field (z axis of Fig. 1). The unstable spin waves must also satisfy the condition that their frequency ω_k is at least approximately equal to the signal frequency ω.

The curved lines a and b in Fig. 1 represent surfaces of constant frequency in k space. Spin waves close to the surface characterizing the signal frequency ω tend to become unstable before spin waves located away from this surface. In the situation described in Fig. 1 the spin waves with the lowest threshold are those located in the layer $n=0$. Spin waves in subsequent layers $n\geq 1$ have a slightly higher threshold.

It is not at all obvious that the discreteness of the spin-wave spectrum can produce a fine structure of the observed kind. Under the usual experimental conditions the frequency separation of adjacent z-directed spin waves is of the order of 1 Mc/sec. The frequency separation between a z-directed spin wave and adjacent spin waves whose wave vectors have the same z component is typically of the order of 1 kc/sec. The line width associated with the z-directed spin waves can be inferred from measurements of the critical power level at which the nonlinear behavior sets in. This line width is typically of the order of several Mc/sec. It may,

[1] R. W. Damon, Revs. Modern Phys. **25**, 239 (1953).
[2] N. Bloembergen and S. Wang, Phys. Rev. **93**, 72 (1954).
[3] H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).
[4] J. J. Green, Bull. Am. Phys. Soc. **4**, 177 (1959), and private communication.
[5] P. Seiden (private communication).
[6] R. L. Martin (private communication).

therefore, be argued that the spin-wave spectrum should behave as if it were continuous even though the wave number vector assumes only discrete values. It will be shown below that this argument is not correct. The time-varying coupling between spin-wave pairs which arises from the excitation of the uniform mode is essentially equivalent to a reduction of the damping constant associated with these modes. In the region of interest adjacent layers of spin waves oriented perpendicular to the z axis are well separated.

II. GENERAL OUTLINE OF THE THEORY

The theory described below goes beyond Suhl's[3] theory in two respects.

1. Line broadening processes are taken into account in a more detailed fashion. It was pointed out before[7] that this is rather important if the line width is predominantly caused by inhomogeneity broadening such as discussed by Clogston *et al.*,[8] Geschwind and Clogston,[9] and the present author.[10] In the present context this refinement is probably not very important because the experiments that showed the fine structure were performed on highly-perfect single crystals of yttrium-iron garnet and gadolinium-iron garnet. In these cases the inhomogeneity contribution to the line width is believed to be rather small. The refinement is taken into account for the sake of generality and because it simplifies the quantitative development of the theory.

2. The nonlinear interaction between the excited spin waves is taken into consideration. The mathematical development of the theory is greatly complicated by this refinement, and for the sake of simplicity only the essential parts of this interaction are retained.

It will be shown that the essential part of the nonlinear interaction can be taken into account by replacing the conventional spin-wave frequencies by effective frequencies which are dependent on the excitation of the uniform mode and on that of all other spin waves. The physical basis for this effect is that the excitation reduces the z component of the magnetic moment and hence the demagnetizing field. Thus, the effective spin-wave frequencies increase with increasing power level.

The fine structure can now be understood qualitatively by reference to Fig. 1. Let the surface of constant frequency a characterize the signal frequency at a given power level. At a higher power level all spin-wave frequencies are shifted upwards. Thus the line representing the (constant) signal frequency in Fig. 1 is shifted downwards to position b. With increasing power level a series of situations is realized in which the signal frequency coincides with the effective frequency of a z-directed spin wave. It is apparent that this will give rise to a fine structure in the decline of χ'' with power level. It will now be necessary to develop the theory in more quantitative terms in order to compare the predicted effects with the observed phenomena.

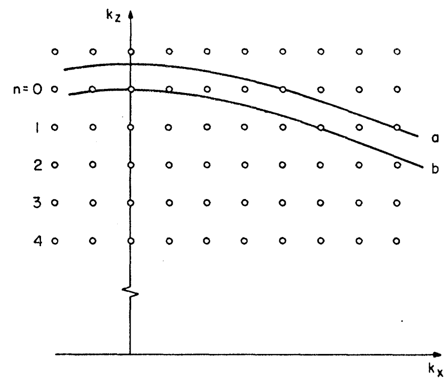

FIG. 1. Schematic representation of spin-wave modes in k space. Each point corresponds to a possible mode of oscillation consistent with the boundary conditions. The curves a and b represent surfaces of constant frequency. The index n characterizes layers of spin waves whose wave vectors have the same z component.

III. QUANTITATIVE DEVELOPMENT OF THE THEORY

The formulation of the equations of motion in the present paper deviates slightly from the conventional. In Suhl's work the equations of motion are expressed in terms of the Fourier components of the transverse components of the magnetization vector. In the present paper we shall first introduce new variables which are related to the transverse components of the magnetization vector and in fact reduce to these variables in the limit of small excitation. The advantage of this procedure is that the equations of motion appear in their canonical form. They can be derived very easily from the Hamiltonian which is proportional to the total energy of the system. The contributions to the energy considered explicitly comprise the Zeeman energy, the exchange energy, and the dipolar interaction energy. Other contributions such as crystalline anisotropy and strain-induced anisotropy can also be taken into account.

Let $\boldsymbol{\alpha}$ be a unit vector pointing in the direction of the magnetic moment

$$\mathbf{M}=M\boldsymbol{\alpha}. \tag{1}$$

Conventionally the equations of motion are expressed in terms of the Fourier components of $\alpha_x(\mathbf{r})$ and $\alpha_y(\mathbf{r})$. It is known that in the limit of small amplitudes α_x and α_y behave essentially as conjugate coordinates and momenta. It is, therefore, convenient to introduce new variables p and q which behave in this way even for finite amplitudes. The new variables are related to the

[7] E. Schlömann, Bull. Am. Phys. Soc. Ser. II, 4, 53 (1959).

[8] Clogston, Suhl, Walker, and Anderson, J. Phys. Chem. Solids 1, 129 (1956).

[9] S. Geschwind and A. M. Clogston, Phys. Rev. **108**, 49 (1957).

[10] E. Schlömann, J. Phys. Chem. Solids **6**, 242 (1958).

old variables by

$$\alpha_x = p[1-\tfrac{1}{4}(p^2+q^2)]^{\frac{1}{2}},$$
$$\alpha_y = q[1-\tfrac{1}{4}(p^2+q^2)]^{\frac{1}{2}}, \tag{2}$$
$$\alpha_z = 1-\tfrac{1}{2}(p^2+q^2).$$

We then introduce a complex variable

$$s = p+iq, \tag{3}$$

and the Fourier components s_k of this variable

$$s(\mathbf{r}) = \sum_k s_k e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

s is the classical equivalent of the spin deviation operator discussed by Holstein and Primakoff.[11] We have assumed periodic boundary conditions and the components of the wave number vector are integer multiples of $2\pi/L$, where L is the length of the periodicity cube. In the absence of dipolar interaction the variables s_k would be the normal coordinates of the problem; i.e., for small amplitudes the equations of motion would be separated if expressed in terms of these variables. The presence of dipolar interaction makes necessary an additional linear transformation to new variables u_k.

$$u_k = \lambda_k s_k + \mu_k s_{-k}^*. \tag{5}$$

This transformation was introduced by Holstein and Primakoff[11] and has also been used by Suhl.[3] The coefficients λ_k and μ_k are given in reference 3. The equations of motion now have the very simple form

$$\dot{u}_k = i\partial\mathcal{H}/\partial u_k^*, \tag{6}$$

where the asterisk denotes the complex conjugate and where

$$\mathcal{H} = (2\gamma/ML^3)\int E d^3r \tag{7}$$

is the Hamiltonian. E is the energy density, γ the gyromagnetic ratio, and M the saturation magnetization. The integration in Eq. (7) extends over the periodicity volume L^3.

The Hamiltonian including Zeeman, exchange, and dipolar energy can be calculated without major difficulty. It can be expanded in powers of the amplitudes u_k. The significant contributions to the Hamiltonian are

$$\mathcal{H} = \sum_k \omega_k u_k^* u_k + \tfrac{1}{2}\sum_{k\neq 0} g_k(u_k^* u_{-k}^* u_0^2 + \text{c.c.}) + \tfrac{1}{2}\sum_{kk'} j_{kk'} u_k^* u_k u_{k'}^* u_{k'} + \tfrac{1}{2}\sum_{\substack{k,k'\neq 0\\ k\neq \pm k'}} l_{kk'} u_k^* u_{-k}^* u_{k'} u_{-k'} + \sum_{kk'} P_{kk'} u_k^* u_{k'} - \gamma(h u_0^* + \text{c.c.}), \tag{8}$$

[11] T. Holstein and H. Primakoff, Phys. Rev. 58, 1098 (1940).

where c.c. denotes the complex conjugate of the expression preceding it.

The first part on the right of Eq. (8) gives rise to the usual linear equations of motion and the ω_k are the spin-wave frequencies

$$\omega_k = \gamma\{[H+H_{\text{ex}}(ak)^2+4\pi M\sin^2\theta_k][H+H_{\text{ex}}(ak)^2]\}^{\frac{1}{2}}. \tag{9}$$

Here H is the internal (demagnetized) magnetic field and H_{ex} a phenomenological constant characterizing the strength of the exchange coupling. "a" is the lattice constant and θ_k the angle between the propagation direction and the dc magnetic field.

The second term in Eq. (8) represents the nonlinear interaction between the uniform mode and the spin waves. It can be shown that this term gives rise to unstable growth of certain spin-wave pairs if the amplitude of the uniform mode u_0 is sufficiently large. It should be pointed out that in general a term involving $u_k^* u_{-k}^* u_0$ is also present in the Hamiltonian. This term gives rise to instability of spin waves with half the signal frequency. It is neglected in Eq. (8) because under the conditions of the experiment to be discussed all spin waves have frequencies appreciably higher than half the signal frequency so that this process is effectively forbidden. The "coupling constant" g_k is to a good approximation given by

$$g_k = \tfrac{1}{2}\omega_M(1-\tfrac{3}{2}\sin^2\theta_k), \tag{10}$$

where $\omega_M = \gamma 4\pi M$.

The third and fourth terms of Eq. (8) represent the important parts of the nonlinear interaction between the excited spin waves. For the present calculation only the coefficients involving nearly z-directed spin waves are required, because only these spin waves are excited. In this case the coefficients are approximately

$$\begin{aligned} j_{kk'} &= \tfrac{1}{4}\omega_M(2N_z+1),\\ j_{0k} &= j_{k0} = \tfrac{1}{2}\omega_M(N_z-N_\perp+1),\\ j_{00} &= \tfrac{1}{2}\omega_M(N_z-N_\perp),\\ l_{kk'} &= \tfrac{1}{4}\omega_M(2N_\perp+1), \end{aligned} \tag{11}$$

where it is assumed that the sample is a spheroid magnetized along its axis and where N_z and $N_\perp$ are the longitudinal and transverse demagnetizing factors (both equal to $\frac{1}{3}$ for a sphere).

The fifth term of Eq. (8) represents an interaction between spin waves (including the uniform mode) which is due to inhomogeneities. Such inhomogeneities can arise on a microscopic scale from the availability of various sites to the magnetic ions in the spinel lattice.[8] They are also present in polycrystals because of the anisotropy of the individual single-crystal grains.[9,10] Surface roughness will also contribute to this part of the Hamiltonian. The coefficients $P_{kk'}$ form a Hermitian matrix. They are related to the Fourier components of certain random functions which describe the inhomogeneities present in the sample. For a more complete discussion of the matrix $P_{kk'}$ for the special case of polycrystals the reader is referred to reference 10. In the present context this part of the Hamiltonian plays only a minor role, and it will not be necessary to specify $P_{kk'}$ in great detail.

The last part of Eq. (8) represents the contribution of the rf magnetic field to the Hamiltonian. Here

$$h=h_x+ih_y, \tag{12}$$

and it has been assumed that the rf field is uniform over the sample volume.

The Hamiltonian of Eq. (8) is approximate. It contains only those terms that are believed to be important in the present context. The neglected terms include:

1. The third order terms previously mentioned.
2. All terms of higher than fourth order in the amplitudes.
3. Fourth order terms that do not involve the amplitudes and their conjugates two times each.
4. Fourth order terms like $u_k{}^*u_{k'}{}^*u_{k''}u_{k'''}$ except those explicitly retained in Eq. (8).

The neglect of these terms can be justified with varying degrees of rigor.

The equations of motion can now easily be derived from Eqs. (6) and (8). Spin-lattice relaxation can be taken into account by assigning positive imaginary parts to all spin-wave frequencies. We shall assume that the driving field is circularly polarized, i.e., $h\sim e^{i\omega t}$. The stationary solution in the presence of this driving field is such that all spin-wave amplitudes (including the uniform mode) have the same time dependence $\sim e^{i\omega t}$. The amplitudes are related by a complicated set of nonlinear algebraic equations. This set of equations can be written in the form

$$(\bar{\Omega}_k-\omega)u_k+\bar{g}_k u_0^2 u_{-k}{}^*+p_k u_0=0, \qquad \text{for } k\neq 0, \tag{13}$$

$$(\bar{\Omega}_0-\omega)u_0+u_0{}^*\sum_{k\neq 0} g_k u_k u_{-k} +\sum_{k\neq 0} P_{0k}u_k=\gamma h, \qquad \text{for } k=0. \tag{14}$$

Here $\bar{\Omega}_k$ is an effective spin-wave frequency

$$\bar{\Omega}_k=\omega_k+\textstyle\sum_{k'} j_{kk'}|u_{k'}|^2+i\eta_k, \tag{15}$$

where η_k is a phenomenological damping constant. Similarly $\bar{g}_k$ is an effective coupling constant.

$$\bar{g}_k=g_k+\sum_{k'\neq 0,k,-k} l_{kk'}u_{k'}u_{-k'}/u_0^2. \tag{16}$$

The coefficients p_k are related to the inhomogeneity interaction

$$p_k=P_{k0}+\sum_{k'\neq 0} P_{kk'}u_{k'}/u_0. \tag{17}$$

The formal solution of the equations of motion can now be achieved by combining Eq. (13) with its adjoint (the complex conjugate equation with k replaced by $-k$). Since $\bar{\Omega}_k=\bar{\Omega}_{-k}$ and $\bar{g}_k=\bar{g}_{-k}$, one obtains after elimination of $u_{-k}{}^*$

$$\frac{u_k}{u_0}=\frac{-p_k(\bar{\Omega}_k{}^*-\omega)+\bar{g}_k p_{-k}{}^*|u_0|^2}{|\bar{\Omega}_k-\omega|^2-|\bar{g}_k|^2|u_0|^4}. \tag{18}$$

Insertion of Eq. (18) into Eqs. (16) and (17) leads to two sets of self-consistency conditions for $\bar{g}_k$ and p_k which involve only the absolute squares of the spin-wave amplitudes.

In its present general form the problem posed by the nonlinear equations of motion is obviously not amenable to a simple solution. We shall, therefore, now introduce two further approximations which will greatly simplify the problem. The approximations consist in replacing: (a) p_k by P_{k0} and (b) $\bar{g}_k$ by g_k. Physically the approximation (a) means that in the inhomogeneity interaction only that part is retained which involves the uniform mode. This is probably a good approximation in the range of small signal powers and it has generally been adopted in previous calculations.[8,10] In the present case the approximation is not as well justified since spin waves other than the uniform mode are also excited to a comparatively high level. The neglected interaction may be expected to contribute to the decay constants of the spin waves in the same way as the retained interaction contributes to the decay constant of the uniform mode. Since the decay constants are treated as adjustable phenomenological parameters this part of the interaction is in fact implicitly taken into account in the theory.

The physical significance of approximation (b) is the neglect of the nonlinear interaction between excited spin waves (with $k\neq 0$) except for a rather trivial part which leads to a shift of the spin-wave frequencies with power level. This approximation cannot generally be justified except as a device to simplify the mathematical problem. In introducing this approximation one probably looses a large part of the physically significant information. It will be shown, however, that the fine structure is essentially unaffected by this approximation.

Using these approximations and the fact that

$$P_{-k0}{}^*=P_{k0}=P_{0k}{}^*, \tag{19}$$

one obtains from Eqs. (14) and (18)

$$(\Omega_{0\text{eff}}-\omega)u_0=\gamma h, \tag{20}$$

where $\Omega_{0\text{eff}}$ is an effective resonance frequency for the uniform mode

$$\Omega_{0\text{eff}}=\bar{\Omega}_0+|u_0|^2\sum_{k\neq 0} g_k|P_{0k}|^2\frac{(\bar{\Omega}_k{}^*-\omega-g_k|u_0|^2)^2}{(|\bar{\Omega}_k-\omega|^2-g_k{}^2|u_0|^4)^2} -\sum_{k\neq 0}|P_{0k}|^2\frac{\bar{\Omega}_k{}^*-\omega-g_k|u_0|^2}{|\bar{\Omega}_k-\omega|^2-g_k{}^2|u_0|^4}, \tag{21}$$

and

$$\bar{\Omega}_0=\omega_0+\sum_k j_{0k}|u_k|^2+i\eta_0. \quad (21a)$$

It will be recognized that in the limit as $|u_0|\rightarrow 0$ the last term on the right of Eq. (21) reduces to the inhomogeneity contribution to the complex resonance frequency that has been discussed previously.[10]

Equation (21) shows that one of the important consequences of the excitation of the uniform mode may be described as a reduction of the loss parameter of the spin waves. To make this more apparent it is convenient to redefine the complex spin-wave frequencies as

$$\tilde{\Omega}_k=\bar{\omega}_k+i\bar{\eta}_k, \quad (22)$$

where

$$\bar{\omega}_k=\omega_k+\sum_{k'} j_{kk'}|u_{k'}|^2, \quad (22a)$$

$$\bar{\eta}_k=(\eta_k^2-g_k^2|u_0|^4)^{\frac{1}{2}}. \quad (22b)$$

With this convention the denominators in Eq. (21) are proportional to $|\tilde{\Omega}_k-\omega|^{-4}$ and $|\tilde{\Omega}_k-\omega|^{-2}$, respectively.

The physical reason for the reduction of the damping constants $\bar{\eta}_k$ with increasing power level, resides in the time-varying coupling between spin-wave pairs (**k** and $-$**k**) which arises from the large excitation of the uniform mode. This coupling transfers energy from the uniform modes to the spin waves. It has been shown by Suhl[3] that it leads to unstable growth of certain spin waves if the amplitude of the uniform mode exceeds a threshold value related to the low power damping constants of the potentially unstable modes. The threshold is reached as the smallest of the effective damping constants given by Eq. (22b) approaches zero. It was pointed out in the introduction that at low power levels the separation of adjacent spin-wave modes is much smaller than the inverse of their relaxation times. The spin-wave spectrum is, therefore, quasi-continuous and no fine structure would occur. It is very important in the present context that at high power levels the damping constants of z directed spin waves are reduced to a value smaller than their frequency separation.

In order to evaluate the sums in Eq. (21) we note that the functions $|\tilde{\Omega}_k-\omega|^{-4}$ and $|\tilde{\Omega}_k-\omega|^{-2}$ have sharp peaks at the position $\bar{\omega}_k-\omega=0$. It is easily shown that these functions approach δ functions in the limit of vanishing damping constant.

$$\frac{1}{|\tilde{\Omega}_k-\omega|^2}\rightarrow\frac{\pi}{\bar{\eta}_k}\delta(\bar{\omega}_k-\omega),$$
$$\frac{1}{|\tilde{\Omega}_k-\omega|^4}\rightarrow\frac{\pi}{2\bar{\eta}_k^3}\delta(\bar{\omega}_k-\omega). \quad (23)$$

Similarly

$$\frac{\bar{\omega}_k-\omega}{|\tilde{\Omega}_k-\omega|^4}\rightarrow-\frac{\pi}{2\bar{\eta}_k}\delta'(\bar{\omega}_k-\omega), \quad (24)$$

where δ' is the derivative of the δ function.

In the usual resonance experiments at high signal power the dc field is adjusted for resonance at each power level. The real part of the nonlinear contribution to $\Omega_{0\mathrm{eff}}$ is, therefore, irrelevant as far as the magnitude of the absorption is concerned since it is compensated by a change in the dc magnetic field. The imaginary part of $\Omega_{0\mathrm{eff}}$ is, according to Eqs. (21) and (21a), (23), and (24),

$$\mathrm{Im}\Omega_{0\mathrm{eff}}=\eta_0+\pi|u_0|^4\sum_{k\neq0}|P_{0k}|^2\frac{g_k^2\eta_k}{\bar{\eta}_k^3}\delta(\bar{\omega}_k-\omega)$$
$$+\pi|u_0|^2\sum_{k\neq0}|P_{0k}|^2\frac{g_k\eta_k}{\bar{\eta}_k}\delta'(\bar{\omega}_k-\omega)$$
$$+\pi\sum_{k\neq0}|P_{0k}|^2\frac{\eta_k}{\bar{\eta}_k}\delta(\bar{\omega}_k-\omega). \quad (25)$$

The sums over k in Eq. (25) will later be evaluated as sums of integrals, where each integral contains the contribution of a layer of spin waves whose wave numbers have the same z component.

If the amplitude of the uniform mode is very small the third sum on the right of Eq. (25) is much larger than the first and second. The third sum in fact reduces to the well-known inhomogeneity contribution to the line width in the limit of zero excitation (when $|u_0|\rightarrow 0$ and $\bar{\eta}_k\rightarrow\eta_k$). If one is interested in the range of fairly low signal powers it is convenient to expand the right-hand side of Eq. (25) in powers of $|u_0|^2$. The third sum contributes to the constant term of this expansion and only the second sum contributes to the linear term. It can easily be shown from this that in the range of fairly low signal powers the susceptibility should vary linearly with the square of the rf magnetic field.[7]

In the present context, the extreme nonlinear range is of particular interest. Under these conditions $|u_0|^2$ is very close to the minimum possible value of η_k/g_k. Hence $\bar{\eta}_k\ll\eta_k$ for some spin waves and the first sum in Eq. (25) may be expected to outweigh the second and third sum. We shall therefore not consider the last two sums in detail. Their contribution to the line width, however, as far as it is independent of the power level, is, of course, taken into account.

The coupling constants g_k are largest for z-directed spin waves. The region corresponding to nearly z-directed spin waves therefore contributes most strongly to the first sum of Eq. (25). It is thus permissible in the evaluation of this sum to use approximate expressions valid in this region for the dependence of the spin-wave frequencies and the coupling constants on the wave number vector. It was mentioned before that the frequency separation of adjacent modes, whose wave number vectors have the same z component, is much smaller than the frequency separation of adjacent modes, whose wave number vectors have different z components. It is, therefore, reasonable to evaluate the sum in Eq. (25) by integration over layers corresponding to fixed values of k_z and summation of these integrals.

This calculation is described in the Appendix. For practical purposes the summation of the integrals representing the contributions of individual layers can in part be approximated by an integral. Only the first few and largest terms in this sum must be added separately. In this way one obtains for the effective damping constant (i.e., the imaginary part of the complex resonance frequency)

$$\mathrm{Im}\Omega_{0\mathrm{eff}}=\bar{\eta}_0[1+cF(U_0)]. \tag{26}$$

Here $\bar{\eta}_0$ is the damping constant of the uniform mode at low power levels. It contains a contribution arising from spin-lattice relaxation (η_0) and a contribution arising from the inhomogeneity interaction. c is a dimensionless constant which is calculated as

$$c=(4\pi^2|P_{0k}|^2/3\omega_M N_\perp\bar{\eta}_0)(L/\lambda)^3, \tag{27}$$

where λ is the wavelength of the unstable spin waves and P_{0k} the matrix element of the inhomogeneity interaction, which connects the uniform mode and the unstable (z-directed) spin waves. It should be noticed that c is independent of the volume since $|P_{0k}|^2$ is proportional to L^{-3} (see reference 10).

The function $F(U_0)$ describes the dependence of the effective loss parameter on the amplitude of the uniform mode. It is convenient to express it in terms of a reduced amplitude U_0 which approaches unity (or values slightly in excess of unity) at high power levels.

$$U_0=|u_0|(\omega_M/2\eta_k)^{\frac{1}{2}}. \tag{28}$$

Here the damping parameter η_k relates to the unstable spin waves. The function $F(U_0)$ has been calculated explicitly for the two cases in which the surface of constant frequency a or b in Fig. 1 either intersects a layer in the immediate vicinity of the z axis or just barely misses it. Using approximations described in the Appendix one obtains

$$F(U_0)=F_1(U_0)=\frac{1-\beta}{[1-(1-\beta)^2U_0^4]^{\frac{1}{2}}}-\frac{\sin^{-1}[(1-\beta)U_0^2]}{U_0^2}, \tag{29}$$

if the first intersection is far away from the z axis. Similarly

$$F(U_0)=F_2(U_0)=F_1(U_0)+\frac{\beta U_0^4}{(1-U_0^4)^{\frac{3}{2}}}, \tag{30}$$

if the first intersection is very close to the z axis. Here

$$\beta=\frac{3}{2}\frac{N_\perp}{N_\perp+\frac{1}{2}}\frac{\lambda}{L}. \tag{31}$$

Since the effective spin-wave frequencies $\bar{\omega}_k$ increase with increasing power level the two limiting cases characterized by Eqs. (29) and (30) are realized alternately. For instance, the first intersection might occur far away from the z axis at low power levels. Under those conditions Eq. (29) is applicable. As the power level is raised the first intersection moves towards the z axis until finally Eq. (30) becomes applicable. If the power level is raised further, the surface of constant frequency will cease to intersect the layer, which previously contained the first intersection. Thus the function $F(U_0)$ changes abruptly from F_2 to F_1. It is obvious that a series of such transitions will occur as the power level is raised.

Consider now the circular susceptibility as a function of the power level. Its negative imaginary part at resonance (where the real part of $\Omega_{0\mathrm{eff}}-\omega$ vanishes) is

$$\chi''=M|u_0|/h. \tag{32}$$

By introducing the susceptibility appropriate for low power levels

$$\chi_0''=\gamma M/\bar{\eta}_0, \tag{33}$$

and the reduced amplitude of the uniform mode [Eq. (28)], Eq. (32) may also be expressed as

$$\frac{\chi''}{\chi_0''}=\frac{\bar{\eta}_0}{\gamma h}\left(\frac{2\eta_k}{\omega_M}\right)^{\frac{1}{2}}U_0. \tag{34}$$

From Eq. (26), on the other hand,

$$\frac{\chi''}{\chi_0''}=\frac{1}{1+cF(U_0)}. \tag{35}$$

By elimination of U_0 between the last two equations one obtains the susceptibility as a function of the rf field strength.

Following Suhl[3] we shall discuss the solution of Eqs. (34) and (35) by a graphical method. If χ''/χ_0'' is plotted as a function of U_0, the first equation is represented by a set of straight lines through the origin, whose slopes are inversely proportional to the rf field strength. The second equation, on the other hand, is represented by a curve with a horizontal tangent at $U_0\ll 1$, which intersects the abscissa in the vicinity of $U_0=1$. Figure 2 shows this construction. The two dashed, curved lines correspond to the two limiting cases in which the surface of constant frequency (a or b in Fig. 1) either intersects a layer in the immediate vicinity of the k_z axis or just barely misses it. The actual curve representing $[1+cF(U_0)]^{-1}$ must lie between these limits. It is represented by the full zigzag line in Fig. 2. This curve has not been calculated in detail. The separation of subsequent kinks, however, will be discussed below. From Fig. 2 it is obvious that the susceptibility should be approximately constant at low power levels. It will then decrease and finally drop off as $1/h$ at very high powers, where U_0 is approximately constant. Superimposed on this general decrease there will be a fine structure which arises from the discrete nature of the spin-wave spectrum.

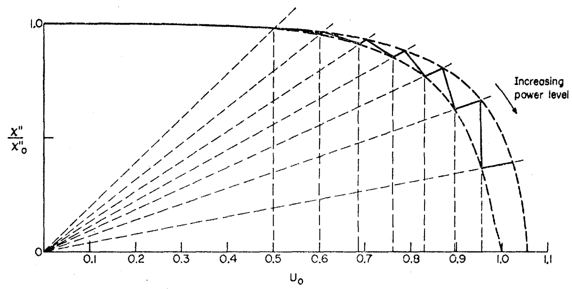

FIG. 2. Graphical solution of Eqs. (34) and (35). The dashed curved lines represent the two limiting cases in which the surface of constant frequency either intersects a layer of spin-wave modes in the immediate vicinity of the k_z axis or just barely misses it. These curves are calculated from Eqs. (35), (29), and (30) using $c=1$, $\beta=0.1$. Under the usual experimental conditions both constants are appreciably smaller. The fine structure is then less conspicuous.

We shall now calculate the separation of subsequent kinks. In this connection it is important to remember that the resonance frequencies of *all* spin waves including the uniform mode are changed with increasing excitation. In the conventional experiments the change in the resonance frequency of the uniform mode is compensated by a change of the dc magnetic field. One thus has to calculate the shift in the spin-wave frequencies subject to the side condition that the effective frequency of the uniform mode is constant.

In this calculation we shall neglect the frequency shift produced by the real parts of the two sums in Eq. (21). This appears to be a good approximation if the inhomogeneity interaction is not too large. Thus, the effective resonance frequencies of the uniform mode and of nearly z-directed spin waves are according to Eq. (22a)

$$\bar{\omega}_0=\omega_0+j_{00}|u_0|^2+j_{0k}\sum_{k\neq0}|u_k|^2,$$

$$\bar{\omega}_k=\omega_k+j_{k0}|u_0|^2+j_{kk'}\sum_{k\neq0}|u_k|^2. \qquad (36)$$

Here we have used the fact that the important spin waves are confined to a fairly small region of k space and that the coefficients j_{0k} and $j_{kk'}$ are approximately constant over this region. If the sample is a spheroid with the dc field applied along the axis of rotation, ω_k and ω_0 are both linearly dependent on the dc magnetic field with the same factor of proportionality. Thus, the dependence of the effective spin-wave frequency $\bar{\omega}_k$ on $|u_0|^2$ and $\sum|u_k|^2$ subject to the side condition $\bar{\omega}_0-\omega=0$ is given by

$$\bar{\omega}_k|_{\bar{\omega}_0-\omega=0}=\text{const}+(j_{k0}-j_{00})|u_0|^2 -(j_{0k}-j_{kk'})\sum_{k\neq0}|u_k|. \qquad (37)$$

The frequency separation of adjacent z-directed spin waves is calculated in the Appendix. For spheroids magnetized parallel to the axis of rotation this separation is in the range of interest equal to $\omega_M N_\perp\lambda/L$, where L is the length of the sample in the direction of the dc magnetic field. Between subsequent kinks the quantity on the right of Eq. (37) has increased by an amount equivalent to the frequency separation of adjacent z-directed spin waves. Thus

$$\left(|u_0|^2-\frac{j_{0k}-j_{kk'}}{j_{k0}-j_{00}}\sum_{k\neq0}|u_k|^2\right)_n =\text{const}+\frac{\omega_M N_\perp}{j_{k0}-j_{00}}\frac{\lambda}{L}n. \qquad (38)$$

Here the quantity on the left should be taken at the nth kink. Using the coefficients given in Eq. (11), Eq. (38) may also be written as

$$\left(|u_0|^2-(\tfrac{1}{2}-N_\perp)\sum_{k\neq0}|u_k|^2\right)_n=\text{const}+2N_\perp\frac{\lambda}{L}n. \qquad (39)$$

A comparison with experimental data has indicated that the second term in the parenthesis is usually much smaller than the first term as long as the power level is not too high.

In order to test Eq. (39) quantitatively one has to infer $|u_0|^2$ and $\sum_{k\neq0}|u_k|^2$ from the experimentally accessible quantities such as χ'', h, and M. $|u_0|$ is easily obtained from Eq. (32), but the determination of $\sum_{k\neq0}|u_k|^2$ is not as straightforward. For a rough test of the theory it is sufficient to plot the values of $|u_0|^2$ at the various kinks *versus* the order n. These points should fall on a straight line. For each material the slope of this line should be inversely proportional to the diameter of the sample. It should be noticed that it is immaterial which kink is assigned the index $n=0$.

For a more sophisticated comparison between theory and experiment which extends to fairly high power levels the second term in the parenthesis of Eq. (39) must be taken into account. Thus one has to infer $\sum_{k\neq0}|u_k|^2$ from the experimentally accessible quantities. This can be achieved if one uses the fact that under stationary conditions the dissipated power equals the power absorbed. One can show from the general equations of motion that under stationary conditions

$$\sum_k \eta_k|u_k|^2=\gamma\,\text{Im}(hu_0^*). \qquad (40)$$

The validity of this relation is not limited to the case in which the Hamiltonian has the particular form given under Eq. (8). It is valid as long as the nonlinearities do not generate frequencies different from the signal frequency.

The left-hand side of Eq. (40) can now be split up into two contributions. One arises from the uniform mode and from spin waves excited through their inhomogeneity interaction with the uniform mode. The other part arises from spin waves excited through the nonlinearity. At resonance (where χ is purely imaginary), one then obtains

$$1+\frac{\eta_k}{\bar{\eta}_0}\frac{\sum_k|u_k|^2}{|u_0|^2}=\frac{\chi_0''}{\chi''}. \tag{41}$$

Here the k summation extends only over those spin waves which are excited through the nonlinear terms in the equations of motion, and it has been assumed that the damping constant η_k is essentially the same for all these modes.

Using Eq. (41) one can obviously infer $\sum|u_k|^2$ from experimentally accessible quantities if the damping constant η_k is known. The damping constant in turn can be obtained from Suhl's theory and measurements of the onset of saturation effects. At the present time, the correct interpretation of such experiments is not completely settled because the inhomogeneity interaction, which was neglected in previous calculations, apparently plays a rather important role.[12] For the purposes of applying the correction in Eq. (39) it is probably justified to assume that η_k equals $\bar{\eta}_0$.

IV. DISCUSSION

In this section the validity of various approximations introduced in the course of the paper will be reviewed. Finally, the physical information obtainable from an interpretation of experimental data in terms of the present theory will be discussed.

The unqualified use of periodic boundary conditions can be very misleading in the present context. Since the discrete nature of the spin-wave spectrum plays a major role, it is obviously very desirable to consider the exact normal modes. Mathematical difficulties have so far rendered this approach impractical. In the present paper the periodic boundary conditions are not used indiscriminately. In determining the frequency separation of adjacent z-directed spin waves it was assumed that these modes are standing waves containing an integer number of half-wavelengths over the thickness of the sample. This appears to be very reasonable although slight modifications may be expected if the exact physical boundary conditions are taken into account.

In Eq. (25) the last two sums were not considered in detail. It is clear that these sums will modify the function $F(U_0)$ of Eq. (26) which describes the dependence of the effective damping constant on the excitation of the uniform mode. It is equally clear, however, that the location of the kinks if expressed in terms of Eq. (39) is not affected by the inclusion of the two neglected sums. This approximation is, therefore, well justified in the present context.

Similar arguments apply to the approximations involved in replacing p_k by P_{k0} and $\bar{g}_k$ by g_k. These approximations also modify the decline of the absorption with increasing power level, but do not affect the fine structure.

In Sec. III the Hamiltonian (8) was simplified by restricting the term representing scattering to the lowest order (bilinear) in the spin-wave amplitudes. This approximation probably breaks down at very high power levels. The interpretation of the fine structure, however, is not affected by this approximation.

An interpretation of experimental results in terms of the present theory should yield information about the strength of the exchange coupling. With the help of Eq. (39) one can obtain the ratio of the wavelength of the unstable spin waves to the diameter of the sample. The exchange field occurring in Eq. (9) can then be inferred from the relation

$$H_{\text{ex}}(2\pi a/\lambda)^2=4\pi MN_\perp, \tag{42}$$

where "a" is the lattice constant.

The exchange field obtained in this way is proportional to the curvature of the spin-wave band in the region of small wave numbers. The same curvature determines also the magnetic contribution to the specific heat at low temperatures and the temperature dependence of the saturation magnetization at low temperatures. The exchange field of Eq. (42) is also related to the Curie temperature and various other magnetic properties, but those relations are not quite as direct. A comparison between the values of the exchange field obtained by the different methods should be very interesting. Experimental evidence in support of the present theory is described by Schlömann and Green.[13]

ACKNOWLEDGMENTS

The work reported in this paper was to a large extent stimulated by the experimental results of J. J. Green, who first observed a fine structure in the decline of the absorption with increasing power level. Several discussions with H. Suhl greatly helped to clarify the author's ideas on the subject.

APPENDIX. CALCULATION OF THE EFFECTIVE DAMPING CONSTANT

Let $\mathbf{k}_t$ be the transverse component of the wave number vector. ($\mathbf{k}_t$ is perpendicular to the z axis.) The spin-wave frequencies depend only on k_z and $k_t=|\mathbf{k}_t|$. One easily obtains from Eq. (9) for $k_t^2\ll k_z^2$

$$\omega_k=\gamma\{H+H_{\text{ex}}(ak_z)^2+[H_{\text{ex}}(ak_z)^2+2\pi M](k_t/k_z)^2\}. \tag{A1}$$

If we characterize the important layers by an index n $(=0, 1, 2\cdots)$ in the way indicated in Fig. 1 the dependence of the spin-wave frequencies on k_t in each layer is given by

$$\omega_k^{(n)}=\gamma[H+H_{\text{ex}}(ak_z^{(n)})^2]+\omega_M(N_\perp+\tfrac{1}{2})(k_t/k_z^{(0)}). \tag{A2}$$

[12] H. Suhl, J. Appl. Phys. (to be published).

[13] E. Schlömann and J. J. Green, Phys. Rev. Letters 3, 129 (1959).

In the second term on the right we have used the fact that $\gamma H_{\text{ex}}(ak_z)^2=\omega_M N_\perp$ for the important spin waves, and $k_z^{(n)}$ has been replaced by $k_z^{(0)}$. Similarly the coupling constant g_k for nearly z-directed spin waves is according to Eq. (10),

$$g_k=(\omega_M/2)[1-\tfrac{3}{2}(k_t/k_z^{(0)})^2]. \tag{A3}$$

The summation in Eq. (25) will now be carried out by means of an integration over k_t and subsequent summation over the layer index n. The summation sign in Eq. (25) can be replaced by

$$\sum_k \rightarrow 2\sum_n \left(\frac{L}{2\pi}\right)^2 \int dk_x dk_y = 2\pi \sum_n \left(\frac{L}{2\pi}\right)^2 \int dk_t^2. \tag{A4}$$

The factor 2 arises from the fact that two directions of propagation must be considered.

It is reasonable to expect that the damping constant η_k and the square of the matrix element $|P_{0k}|^2$ do not vary appreciably over the range of k values which contributes most strongly to the sum under consideration. The factor in front of the δ function in this sum is therefore proportional to

$$f(k_t)=\frac{[1-\frac{3}{2}(k_t/k_z^{(0)})^2]^2}{\{1-[1-\frac{3}{2}(k_t/k_z^{(0)})^2]^2 U_0^4\}^{\frac{3}{2}}}. \tag{A5}$$

Here we have used Eqs. (22b), (A3), and (28) and U_0 is the reduced amplitude of the uniform mode. The contributions of individual layers to the first sum of Eq. (25) are now according to Eq. (A2) proportional to

$$\left(\frac{L}{2\pi}\right)^2 \int dk_x dk_y f(k_t)\delta(\omega_k^{(n)}-\omega) = \frac{\pi}{\omega_M(N_\perp+\frac{1}{2})}\left(\frac{L}{\lambda}\right)^2 f(k_t^{(n)}). \tag{A6}$$

Here $k_t^{(n)}$ is the solution of $\omega_k-\omega=0$ in the nth layer, and $\lambda=2\pi/k_z^{(0)}$ is the wavelength of the unstable spin waves. The effective damping constant of the uniform mode thus becomes

$$\text{Im}\Omega_{0\text{eff}}=\bar{\eta}_0+CU_0^4 \textstyle\sum_n f(k_t^{(n)}), \tag{A7}$$

where

$$C=\frac{2\pi^2|P_{0k}|^2}{\omega_M(N_\perp+\frac{1}{2})}\left(\frac{L}{\lambda}\right)^2. \tag{A8}$$

The significance of $\bar{\eta}_0$ was described before in connection with Eq. (26).

Before the remaining summation over n can be carried out it is necessary to determine the value of $k_t^{(n)}$ for each layer. We note that for adjacent layers the z components of the propagation vectors differ by constant amounts inversely proportional to the diameter of the sample. In this connection the boundary conditions are quite important. Unfortunately it is very difficult to determine the correct normal modes, which satisfy the physical boundary conditions. Mercereau and Feynman[14] and Walker[15] have discussed this problem for the limiting case, in which the wavelength is very large, so that exchange effects can be neglected. Because of this approximation their analysis cannot be applied in the present case without major modification. One, therefore, has to rely on the use of fictitious boundary conditions introduced for the sake of their mathematical convenience. If periodic boundary conditions are imposed, the components of the permissible wave number vectors are integer multiples of $2\pi/L$. Another type of boundary condition which can be discussed without difficulty, consists in requiring the components of the rf magnetization to vanish along the surface of a cube (or more generally along the surface of a rectangular volume). In the latter case the components of the permissible wave number vectors are integer multiples of π/L. In the present paper we have used periodic boundary conditions. The calculation can be readily adapted, however, to the case of the second kind of (equally arbitrary) boundary condition. The only changes required consist of extending the **k** summations over the new set of permissible wave number vectors and imposing certain side conditions (like $u_{-k}=-u_k$) on the dynamic variables. The final results are essentially unchanged except for a change in the frequency separation of adjacent z-directed spin waves. Since the correct normal modes have the character of standing waves their frequency separation should be very close to that obtained with the second kind of boundary condition discussed above. In this case the values of k_z associated with adjacent layers differ by π/L

$$k_z^{(n)}-k_z^{(n+1)}=\pi/L. \tag{A9}$$

Hence

$$(k_z^{(n)})^2-(k_z^{(n+1)})^2\approx(k_z^{(n)})^2\lambda/L. \tag{A10}$$

Finally from Eqs. (A2) and (A10) for values of n which are not too large

$$\tfrac{3}{2}[k_t^{(n)}/k_z^{(0)}]^2\approx\alpha+\beta n, \tag{A11}$$

where

$$\beta=\tfrac{3}{2}[N_\perp/(N_\perp+\tfrac{1}{2})]\lambda/L, \tag{A12}$$

and

$$0\leq\alpha\leq\beta.$$

So far we have neglected the fact that the effective spin-wave frequencies increase with increasing excitation. Since the frequency shift is approximately the same for all the important spin waves it can formally be taken into account by assuming α to be dependent

[14] T. E. Mercereau and R. P. Feynman, Phys. Rev. **104**, 63 (1956).
[15] L. R. Walker, Phys. Rev. **105**, 390 (1957).

on the power level. With increasing power level α decreases continuously to zero, jumps abruptly to β, decreases again continuously to zero and so on. The effective damping constant considered as a function of the amplitude of the uniform mode U_0 thus varies between two limiting curves which correspond to the cases in which either $\alpha=0$ or $\alpha=\beta$.

Consider now the sum of the integrals $f(k_t^{(n)})$ as a function of α

$$S(\alpha)=\sum_n f(k_t^{(n)}). \tag{A13}$$

According to Eq. (A5) and (A11) the sums corresponding to $\alpha=0$ and $\alpha=\beta$ are almost identical, with $S(0)$ having one additional term.

$$S(0)=S(\beta)+1/(1-U_0^4)^{\frac{3}{2}}. \tag{A14}$$

In the evaluation of $S(\beta)$ we shall, for simplicity, approximate the summation by an integration, even though this is a rather poor approximation for small values of n. Thus

$$S(\beta)\approx\int_1^{n_0} dn\frac{(1-\beta n)^2}{[1-(1-\beta n)^2U_0^4]^{\frac{3}{2}}}. \tag{A15}$$

The upper integration limit n_0 is not very important since for $U_0\approx 1$ the major contribution to the integral comes from small values of n. For simplicity we will take $n_0=1/\beta$. One then obtains by standard integration techniques

$$S(\beta)=\frac{1}{\beta U_0^4}\left\{\frac{1-\beta}{[1-(1-\beta)^2U_0^4]^{\frac{1}{2}}} -\frac{\sin^{-1}[(1-\beta)U_0^2]}{U_0^2}\right\}. \tag{A16}$$

Equation (26) is now obtained from Eqs. (A7), (A8), (A13), (A14), and (A16).