PHYSICAL REVIEW VOLUME 116, NUMBER 3 NOVEMBER 1, 1959

Analytic Properties of Partial Amplitudes in Meson-Nucleon Scattering*

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(Received January 26, 1959)

The analytic properties of partial wave amplitudes in meson-nucleon scattering are investigated on the basis of the Mandelstam representation and an integral representation is set up for them which explicitly exhibits those properties.

INTRODUCTION

A METHOD for the separation of partial waves from nonforward relativistic dispersion relations has been put forward by Capps and Takeda.[1] The scattering amplitudes in the integrals are expanded in terms of Legendre polynomials which are analytically continued into the unphysical region. Each partial amplitude is then related to two integrals involving infinite series of partial amplitudes, which are convergent at low energies.

We have considered a different approach to this problem which consists in deducing the analytic properties of the partial amplitudes and establishing a representation for them, based on those properties. In order to investigate the analytic properties of the partial amplitudes it is necessary to know the analytic structure of the scattering amplitudes as functions of two variables, the energy and momentum transfer. A full representation of these amplitudes has been proposed by Mandelstam[2] and is used here. The main result is that the partial amplitudes are analytic functions of the energy throughout the complex plane except for cuts along the real axis and a cut along a circle with its center on the negative real axis.

Integral representations are obtained by applying Cauchy's theorem to suitable combinations of pairs of partial amplitudes with the same total angular momentum but opposite parities. The integrals along the real axis depend on the imaginary part of the function considered and one can apply the unitarity condition in the physical region. The meaning of the unphysical region as well as the integral along the circle are discussed.

1. THE COVARIANT AMPLITUDES. MANDELSTAM'S REPRESENTATION

The matrix element of the S-matrix for meson-nucleon scattering may be written in the form[3]:

$$S_{fi}=\delta_{fi}-(2\pi)^4 i\delta^4(p_2+q_2-p_1-q_1)\left(\frac{M^2}{4q_{10}q_{20}p_{10}p_{20}}\right)^{\frac{1}{2}} \times F_{sr}(p_2q_2,p_1q_1). \quad (1.1)$$

where p_1, q_1 are the initial momenta of the nucleon and the meson, respectively, p_2, q_2 the final momenta and r,s the spin states of the incoming and outgoing nucleon. The isotopic spin coordinates of the nucleons and the mesons have been omitted.

Due to conservation of total momentum, out of the four momenta there will be only three independent vectors which determine the kinematics of the process. We choose the combinations:

$$P=\tfrac{1}{2}(p_1+p_2),\quad Q=\tfrac{1}{2}(q_1+q_2),$$
$$K=\tfrac{1}{2}(p_2-p_1)=\tfrac{1}{2}(q_1-q_2). \quad (1.2)$$

With these vectors one can form the invariants:

$$K^2=-\tfrac{1}{4}t,\quad P^2=-M^2-K^2,\quad Q^2=-m^2-K^2,$$
$$P\cdot Q=-\nu,\quad P\cdot K=Q\cdot K=0, \quad (1.3)$$

so that there are only two independent invariants.

We introduce an invariant matrix $\mathfrak{M}$ in spin space related to the Feynman amplitude $F_{sr}(p_2q_2,p_1q_1)$ by

$$F_{sr}(p_2q_2,p_1q_1)=\bar{u}_s(p_2)\mathfrak{M}u_r(p_1), \quad (1.4)$$

where $u_r(p_1)$ and $\bar{u}_s(p_2)$ are Dirac spinors.

Assuming parity conservation and making use of the Dirac equation, one can reduce $\mathfrak{M}$ to the form[4]:

$$\mathfrak{M}=-U+\mathbf{Q}V, \quad (1.5)$$

where U and V are functions of the invariants ν and t, and $\mathbf{Q}\equiv\gamma_\mu Q_\mu$.

To fix our ideas consider the processes:

$$N_1+K_1\to N_2+K_2,\quad \text{(I)}$$
$$N_1+\bar{K}_2\to N_2+\bar{K}_1,\quad \text{(II)}$$
$$N_1+N_2\to K_2+\bar{K}_1.\quad \text{(III)}$$

The Feynman amplitudes for these processes are obtained from the same Green's function taken in different and nonoverlapping domains of the variables ν and t, which of course have different physical meanings for each of them. The physical regions for the respective

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[1] R. H. Capps and G. Takeda, Phys. Rev. **103**, 1877 (1956).
[2] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[3] For an expression of the Feynman amplitudes in terms of Heisenberg operators see F. E. Low, Phys. Rev. **97**, 1392 (1955); M. L. Goldberger, Phys. Rev. **97**, 508 (1955); and Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955).
[4] A. Salam, Nuovo cimento **3**, 427 (1956).

processes are

$$\nu>\nu_T, \quad t<0, \quad \text{(I)}$$
$$\nu<-\nu_T, \quad t<0, \quad \text{(II)}$$
$$\nu^2<\nu_T^2, \quad t>4M^2, \quad \text{(III)}$$

where

$$\nu_T=[(M^2-t/4)(m^2-t/4)]^{\frac{1}{2}}.$$

Mandelstam assumes that the Green's function is analytic in both variables except for poles on the real axis and cuts along certain hyperplanes.[2] The poles arise from the bound states of the system, and the location of the cuts is determined by the threshold energies for the allowed virtual transitions. It is then convenient to introduce the new variables s and $\bar{s}$ which are the square of the energy in the center-of-mass system for the processes I and II, respectively. They are related to ν and t by

$$\tfrac{1}{2}(s-M^2-m^2)=\nu-\tfrac{1}{4}t, \tag{1.6}$$

$$\tfrac{1}{2}(\bar{s}-M^2-m^2)=-\nu-\tfrac{1}{4}t. \tag{1.7}$$

The corresponding variable for process III is t and there exists the relation:

$$s+\bar{s}+t=2(M^2+m^2). \tag{1.8}$$

The square of the momentum transfer between the mesons or nucleons in the first two processes is $-t$; in process III it is $-\bar{s}$ between the nucleon N_1 and the meson K_2, and $-s$ between N_1 and $\bar{K}_1$.

On the basis of Mandelstam's assumptions one obtains for each of the covariant amplitudes a representation of the form:

$$A=\sum_{Y=\Lambda,\Sigma}\frac{\alpha_Y g_Y^2}{M_Y^2-\bar{s}}+\frac{1}{\pi^2}\int_{(M+m)^2}^{\infty}ds'\int_{(M_Y+\mu)^2}^{\infty}d\bar{s}'\frac{A_{12}(s',\bar{s}')}{(s'-s)(\bar{s}'-\bar{s})}+\frac{1}{\pi^2}\int_{(M+m)^2}^{\infty}ds'\int_{(2\mu)^2}^{\infty}dt'\frac{A_{13}(s',t')}{(s'-s)(t'-t)}+\frac{1}{\pi^2}\int_{(M_Y+\mu)^2}^{\infty}d\bar{s}'\int_{(2\mu)^2}^{\infty}dt'\frac{A_{23}(\bar{s}',t')}{(\bar{s}'-\bar{s})(t'-t)}+\frac{1}{\pi}\int_{(M_Y+\mu)^2}^{\infty}d\bar{s}'\frac{a_2(\bar{s}')}{\bar{s}'-\bar{s}}. \tag{1.9}$$

In this case the conservation laws and in particular the conservation of strangeness and parity allow for bound states only in process II, corresponding to the Λ and Σ particles. The energy thresholds for the three processes correspond to transitions into the virtual intermediate states $(K+N)$, $(Y+\pi)$, $(\pi+\pi)$, respectively. The bound-state contributions exactly coincide with the first Born approximation in the conventional perturbation theory.

From the reality of the absorptive parts of the amplitudes for the physical processes I–III, it follows that all the weight functions are real in the respective domains of integration. Although it is irrelevant for our present purpose, we must point out that the representation (1.9) as it stands is incorrect for the amplitude U. In fact, from perturbation theory one obtains that for large s' and $\bar{s}'$ and fixed t', A_{13} and A_{23} do not approach zero as required for the convergence of the integrals. Therefore one has to make subtractions which give rise to another single integral of the form:

$$\frac{1}{\pi}\int_{(2\mu)^2}^{\infty}dt'\frac{a_3(t')}{t'-t}.$$

2. THE SCATTERING MATRIX IN THE CENTER-OF-MASS SYSTEM

The covariant transition matrix is related in the c.m. system to the scattering matrix by

$$\bar{u}_s(k')\mathfrak{M}u_r(k)=4\pi\frac{W}{M}u_s^\dagger(0)fu_r(0), \tag{2.1}$$

where $u_r(0)$ are orthonormal eigenstates of γ_0 $[\gamma_0 u_r(0)=u_r(0)]$ and

$$u_r(k)=\frac{\boldsymbol{k}+M}{[2M(E+M)]^{\frac{1}{2}}}u_r(0). \tag{2.2}$$

Here $(kE,\boldsymbol{k})$ and $k'(E,\boldsymbol{k}')$ are the four-momenta of the incoming and outgoing nucleons in the c.m. system and $W=\sqrt{s}$ is the total energy.

The matrix f may be written in the form:

$$f=f_1+(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}')(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})f_2, \tag{2.3}$$

where $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\varepsilon}$ are unit vectors in the direction of $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively, and

$$f_1=\sum(f_l^+P_{l+1}'-f_l^-P_{l-1}'),\quad f_2=\sum(f_l^--f_l^+)P_l'. \tag{2.4}$$

The $f_l^\pm$ are amplitudes for transitions in given states of total angular momentum $j=l\pm\frac{1}{2}$ and orbital angular momentum l. For K-nucleon scattering below the threshold for pion production the amplitudes $f_l^\pm$ have the form

$$f_l^\pm=\exp(i\delta_l^\pm)\sin\delta_l^\pm/k,$$

where the phases $\delta_l^\pm$ are real functions of k. For $\bar{K}$-nucleon scattering as well as for K-nucleon scattering above that threshold the phases are complex.

From (1.5), (2.1), and (2.2) one obtains the following relations between the covariant amplitudes U, V and f_1, f_2[5]:

$$4\pi f_1=\frac{E+M}{2W}[U+(W-M)V], \tag{2.5}$$

$$4\pi f_2=\frac{E-M}{2W}[-U+(W+M)V], \tag{2.6}$$

[5] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1377 (1957).

and hence

$$\frac{1}{4\pi}U=\frac{W+M}{E+M}f_1-\frac{W-M}{E-M}f_2, \tag{2.7}$$

$$\frac{1}{4\pi}V=\frac{1}{E+M}f_1+\frac{1}{E-M}f_2. \tag{2.8}$$

One can see that

$$f_1(-W)=-f_2(W), \tag{2.9}$$

which is a consequence of invariance under Schwinger's space-time reflection.

The partial wave amplitudes may be projected out of f_1 and f_2. One obtains

$$f_l^{\pm}=\frac{1}{2}\int_{-1}^{+1}(f_1P_l+f_2P_{l\pm1})dz. \tag{2.10}$$

Then from the relation (2.9) it follows that

$$f_l^{+}(-W)=-f_{l+1}^{-}(W) \tag{2.11}$$

3. THE ANALYTIC PROPERTIES OF THE PARTIAL AMPLITUDES

We investigate now the analytic properties of the partial amplitudes as defined by (2.10) as functions of the variable

$$\omega=\tfrac{1}{2}(s-M^2-m^2). \tag{3.1}$$

They are derived from the representation (1.9) for U and V, setting

$$t=-2k^2(1-z), \tag{3.2}$$

and varying z from -1 to $+1$.

Let us first consider the Born approximation, $f_l^{\pm(1)}$. In the K-nucleon scattering the poles of U and V give rise to branch lines of $f_l^{\pm(1)}(\omega)$ in the intervals of the real axis along which

$$\omega_Y+\omega-k^2(1-z)=0,$$

that is $[-\infty,\ -\frac{1}{2}(M^2+m^2)]$ and $(-\omega_Y',\ -\omega_Y)$, where

$$\omega_Y=\tfrac{1}{2}(M_Y^2-M^2-m^2),$$
$$-\omega_Y'=\tfrac{1}{2}\{[(M^2-m^2)^2/M_Y^2]-M^2-m^2\}.$$

On the other hand, in $\bar{K}$-nucleon scattering the Born approximation vanishes for all amplitudes except the S and $P_{\frac{1}{2}}$ states in which there remain poles at $\omega=\omega_Y$.

Let us now consider the singularities of $f_l^{\pm(2)}=f_l^{\pm}-f_l^{\pm(1)}$. They lie on the lines of the complex ω-plane defined by

$$s'-s=0, \tag{3.3}$$

$$\bar{s}'-\bar{s}=0, \tag{3.4}$$

$$t'-t=0, \tag{3.5}$$

where s', $\bar{s}'$, and t' are parameters assuming values within the intervals of integration in (1.9).

For K-nucleon scattering $\bar{s}$ and t are expressed in terms of ω and z by means of (1.8), (3.1), and (3.2).

Equation (3.3) defines a cut on the positive real axis in the interval $(mM,\ +\infty)$.

The solutions of Eq. (3.4) for a fixed value of $\bar{s}'$ and $-1\leqslant z\leqslant +1$ are real and lie in the intervals $[-\infty,\ -\frac{1}{2}(M^2+m^2)]$ and $(-\bar{\omega}'',\ -\bar{\omega}')$, where

$$\bar{\omega}'=\tfrac{1}{2}(\bar{s}'-M^2-m^2),$$

$$-\bar{\omega}''=\frac{1}{2}\left[\frac{(M^2-m^2)^2}{\bar{s}'}-M^2-m^2\right].$$

As $\bar{s}'$ varies from $(M_Y+\mu)^2$ to $+\infty$ these intervals cover a branch line from $-\infty$ to

$$\bar{\omega}_0=\tfrac{1}{2}[(M_Y+\mu)^2-M^2-m^2]. \tag{3.6}$$

Therefore Eq. (3.4) defines a cut on the negative real axis in the interval $(-\infty,\ \bar{\omega}_0)$.

Equation (3.5) is equivalent to

$$k^2=-t'/2(1-z),$$

where the right-hand side varies from $-\infty$ to $-\mu^2$. As k^2 varies from $-\infty$ to $-M^2$, ω describes two branch lines on the real axis along $(-\infty,\ -M^2)$ and $[-\frac{1}{2}(M^2+m^2),\ -M^2]$; as k^2 varies from $-M^2$ to $-m^2$, ω goes from $-M^2$ to $-m^2$ along two branches (above and below the real axis) of a circle in the complex plane, with center at $\omega_c=-\frac{1}{2}(M^2+m^2)$ and radius $\frac{1}{2}(M^2-m^2)$; as k varies from $-m^2$ to $-\mu^2$, ω describes two branch lines on the real axis along $(-m^2,\ -\omega_2)$, $(-m^2,\ \omega_2)$ where $-\omega_1$ and ω_2 are the roots $k^2+\mu^2=0$ (see Fig. 1).

One can use these analytic properties to obtain a representation for the partial amplitudes in the form of dispersion relations. It is convenient to introduce the combinations:

$$\varphi_l^{+}=\frac{1}{W}(f_l^{+}+f_{l+1}^{-})$$
$$=\frac{1}{2W}\int_{-1}^{+1}(f_1+f_2)(P_l+P_{l+1})dz, \tag{3.7}$$

$$\varphi_l^{-}=(f_l^{+}-f_{l+1}^{-})=\frac{1}{2}\int_{-1}^{+1}(f_1-f_2)(P_l-P_{l+1})dz,$$

which are symmetric functions of W. Recalling that $\varphi_l^{\pm}$ behaves like $k^{2l}\sim(\omega^2-m^2M^2)^l$ as $k^2\to 0$, and applying Cauchy's theorem to $\varphi_l^{\pm}(\omega')/(\omega'-\omega)$

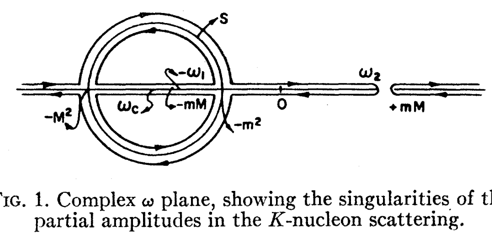

FIG. 1. Complex ω plane, showing the singularities of the partial amplitudes in the K-nucleon scattering.

$\times(\omega'^2-m^2M^2)^l$ for the contour shown in Fig. 1, one obtains

$$\varphi_l^{\pm}(\omega)=\varphi_l^{\pm(1)}(\omega)+\frac{1}{\pi}\int_{mM}^{\infty}\operatorname{Im}\varphi_l^{\pm}(\omega')\left(\frac{\omega^2-m^2M^2}{\omega'^2-m^2M^2}\right)^l\frac{d\omega'}{\omega'-\omega}+\frac{1}{\pi}\int_{-\infty}^{\omega_2}\operatorname{Im}\varphi_l^{\pm(2)}(\omega')\left(\frac{\omega^2-m^2M^2}{\omega'^2-m^2M^2}\right)^l\frac{d\omega'}{\omega'-\omega}+\frac{1}{\pi}\int_S\phi_l^{\pm}(\omega')\left(\frac{\omega^2-m^2M^2}{\omega'^2-m^2M^2}\right)^l\frac{d\omega'}{\omega'-\omega},\quad(3.8)$$

where $\varphi_l^{(1)}(\omega)$ is the Born approximation corresponding to the first term in (1.9) and $\varphi_l^{(2)}=\varphi_l-\varphi_l^{(1)}$.

From the representation (1.9) one obtains, for ω in the interval $(-\infty,\omega_2)$

$$\operatorname{Im}\int_{-1}^{+1}A(\omega,z)P_l(z)dz=\operatorname{Re}[I_{l2}(\omega)+I_{l3}(\omega)],\quad(3.9)$$

where:

$$I_{l2}(\omega)=-\int_{z_0}^{1}A_2(\bar{s},t)P_l(z)dz,\qquad(-\infty<\omega<-\bar{\omega}_0)\quad(3.10)$$

$$I_{l3}(\omega)=\pm\int_{-1}^{(1+2\mu^2/k^2)}A_3(t,s)P_l(z)dz,\qquad\begin{cases}-\infty<\omega<\omega_c,\\-\omega_1<\omega<\omega_2.\end{cases}\quad(3.11)$$

The lower limit of integration in (3.10) is $z_0=-1$ when

$$\omega_c<\omega<\frac{1}{2}\left(\frac{(M^2-m^2)^2}{(M_Y+\mu)^2}-M^2-m^2\right),$$

and $z_0=1-(\bar{\omega}_0+\omega)/k^2$, for ω outside that interval. The + sign in (3.11) is taken in the region outside the circle where $dk^2/d\omega>0$ and the − sign is taken inside it, where $dk^2/d\omega<0$. All these limits and intervals can be better visualized by an inspection of Fig. 2.

The functions $A_2(\bar{s},t)$ and $A_3(t,s)$ are the absorptive amplitudes for processes II and III as defined in reference 2. The following discussion is based on the properties of these functions as described by Mandelstam.[2]

The functions $A_2(\bar{s},t)$ and $A_3(t,s)$ are real in the respective intervals $\omega_c<\omega<-\bar{\omega}_0$ and $-\omega_1<\omega<\omega_2$, where they coincide with the imaginary part of the amplitudes for processes II and III. For $\omega<\omega_c$ the branch lines arising from (3.4) and (3.5) overlap and these functions may become complex. The overlapping along the interval $(-\bar{\omega}_0,-\omega_1)$ is actually ficticious since it would occur from processes like those shown in Fig. 3, which are forbidden on account of conservation of strangeness and isotopic spin. The branch lines of

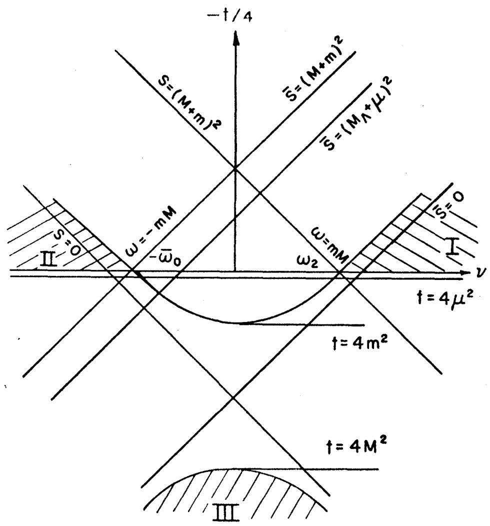

FIG. 2. Diagram of the (ν,K^2) plane. The hyperbola represents the curve $\nu^2=\nu_T^2$ or equivalently $k^2=K^2$. The physical regions for processes I, II, III are shaded. We have marked values of ω on the ν axis recalling that $\omega=\nu$ for $K^2=0$.

graphs with K-mesons in the intermediate states, already start outside that interval.

When ω is in the interval $(-\infty,\omega_2)$, $\operatorname{Im}\varphi_l^{\pm(2)}(\omega)$ is given by the right-hand side of (3.9), provided the following modifications are introduced in (3.10) and (3.11):

(i) Replace P_l by $\frac{1}{2}(P_l\pm P_{l+1})$;
(ii) replace A_i by

$$A_i^+=(1/W^2)H(-MU_i+\omega V_i),$$
$$A_i^-=(1/W)H[-EU_i+M(W-E)V_i].\quad(3.12)$$

At this point we assume charge independence and introduce the column matrices in isospin space:

$$U=\begin{pmatrix}U^0\\U^1\end{pmatrix},\quad V=\begin{pmatrix}V^0\\V^1\end{pmatrix},$$

whose elements correspond to transitions in states of isotopic spin $T=0$ and $T=1$, respectively. Isotopic spin has been taken into account in (3.12), where

$$H=\frac{1}{2}\begin{pmatrix}-1&3\\1&1\end{pmatrix}$$

is a matrix in isospin space.[6]

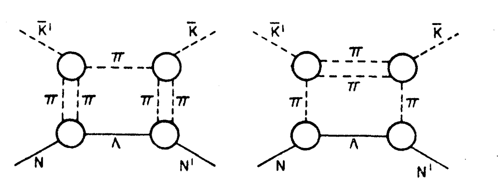

FIG. 3. The graphs shown here are forbidden because of conservation of strangeness.

[6] D. Amati and B. Vitale, Nuovo cimento 7, 190 (1958).

The function $\phi_l^{\pm}(\omega)$ in the last integral of (3.8) is also given by $I_{l3}^{\pm}$, where A_3 has been replaced by $A_3^{\pm}$. If the integration around the circle S follows the external lines in Fig. 1 the $+$ sign must be taken above and the $-$ sign below the real axis.

The method described in this section applies equally to process II and pion-nucleon scattering. But then the functions $\varphi_0^{\pm}$ (S and $P_{\frac{1}{2}}$ waves) have a pole at $W=0$, and an arbitrary parameter, the S-wave scattering length, is conveniently introduced by means of a subtraction at $\omega=mM$, where the $P_{\frac{1}{2}}$ scattering amplitude vanishes. The fortuitous absence of this pole in K-nucleon scattering is due to conservation of strangeness.

FINAL REMARKS

The objective of setting up dispersion relations for partial amplitudes was to establish an integral representation of functions of a single variable $f_l^{\pm}(\omega)$, for which the unitarity condition takes on, in the physical region, the simple form:

$$\mathrm{Im} f_l(\omega)=\sum_n \rho_n |f_l^n|^2, \qquad (3.13)$$

where the index n refers to all allowed channels, compatible with conservation of energy and ρ_n is a phase space factor; in the elastic channel $\rho_n=k$.

In our integral representation there exist, however, regions of frequencies where the unitarity condition has no simple form. If we attempt to apply the unitarity condition for all ω' in order to obtain an integral equation, we find that the equations for partial amplitudes of different angular momentum and corresponding to processes I, II, and III, in different states of isotopic spin are all coupled. Moreover, in the region $\omega'<\omega_c$ and along the circle S the unitarity condition can only be obtained by means of analytical continuation.

In the interval $\omega_c<\omega<-mM$ the variables s and t in (3.10) are in the physical region for process II. One can re-express U_2 and V_2 in terms of the imaginary parts of the scattering amplitudes and expand in partial waves. This expansion is also valid in the unphysical region $-mM<\omega<-\bar{\omega}_0$, that is $(M_Y+\mu)^2<\bar{s}<(M+m)^2$. Only the virtual states $(Y+\pi)$ and $(\Lambda+2\pi)$ contribute to the amplitudes in this region, and we conjecture that the unitarity condition takes the form (3.13) except for a factor $(-1)^l$.

We remark that if all graphs in perturbation theory involving closed baryon loops and four-meson primary interactions are neglected, then from Eq. (3.5) only a cut along $(-\infty,\omega_c)$ survives. However, there is no reason to believe that those graphs are unimportant. It seems, therefore, that in order to make use of these relations as dynamical equations one has to introduce some approximations, replacing the integrals in the unphysical regions by simplified expressions.

ACKNOWLEDGMENTS

The author expresses his gratitude to Professor R. E. Peierls for his interest and continual support in this work. He is also very much indebted to L. Castillejo for enlightening dicussions and most helpful criticism.

PHYSICAL REVIEW VOLUME 116, NUMBER 3 NOVEMBER 1, 1959

Completion and Embedding of the Schwarzschild Solution

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(Received June 9, 1959)

An analytic manifold is found, the most important properties of which are that it is complete and that it contains the manifold of the Schwarzschild line element. It is thus the complete analytic extension of the latter. The manifold is represented as a Riemannian surface in a six-dimensional pseudo-Euclidean space. The subspace $d\varphi=d\vartheta=0$ is visualized as a two-dimensional Riemannian surface in a 3-dimensional hyperplane in the six-dimensional space. Although the manifold admits groups of motion isomorphic to the real 3-dimensional rotation group and the one-dimensional translation group, it is impossible to introduce a global time-coordinate in such a way that the latter is realized as translations in time. Hence in any global set of coordinates the gravitational field is nonstationary, although it can be made stationary for $r>1$ to any desired approximation. The question of what happens to small test bodies reaching the Schwarzschild critical radius is discussed.

STATEMENT OF THE PROBLEM

THE Schwarzschild line element[1] is

$$ds^2=\left(1-\frac{1}{r}\right)dt^2-\left(1-\frac{1}{r}\right)^{-1}dr^2-r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2). \qquad (1)$$

The coefficient of dr^2 becomes singular at $r=1$, but it has long been known that this is due only to the choice of coordinates. The proof of this statement may be based on the fact that the Petrov curvature scalars have no singularity at $r=1$,[2] but it is perhaps more convincing to refer to the fact that the equations for the geodesics show a singular behavior only at $r=0$. For a motion in

[1] For a derivation see, e.g., H. Weyl, *Space, Time, and Matter* (Dover Publications, New York, 1950), p. 252. We have put $2m\kappa=1$, where m is the mass and κ is the gravitational constant.

[2] D. Finkelstein, Phys. Rev. **110**, 965 (1959).