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Phenomenological Nuclear Interaction Derived from Binding Energies*

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An attempt is made to clarify the basis of a formula for binding energies derived in the framework of the *j-j* coupling shell model. Studies are made of the effect of three-body correlations, by introducing a density dependent term. It is seen that the *j-j* coupling calculation can be suitably corrected for the many body effects. The hard-core influence is introduced as a pseudopotential. An attempt is made to fit the binding energies with a phenomenological central potential, including a density-dependent term. The best exchange mixture obtained is not far from a Serber type but includes a considerable amount of Bartlett force. When the hard core is included, the best exchange mixture becomes of the Serber type. With this potential a good fit is obtained over a large range of configurations.

I. INTRODUCTION

We want to study in this paper some recent results concerning calculations in the shell model.[1–3] In this model an inert core is assumed which corresponds to the closed-shell nucleus. If we look at a nucleus in which there is one single nucleon outside closed shells, we observe levels obtained by exciting the extra nucleon to higher orbits. However, if there are more extra nucleons in definite orbits (a given configuration), there are many levels obtained by the various modes of coupling of the angular momenta of the extra nucleons. The energy in the central field is the same for all these levels. This degeneracy will be removed if a perturbing interaction between the particles outside the closed shell is introduced.

The basis of the shell model has been much solidified by recent advances in the theory of many-particle systems.[4] Brueckner and co-workers have shown that instead of using the real correlated wave function, ψ, and the two-body, free-particle force, one can introduce model wave functions and an appropriate effective interaction operator. Then one can write

$$(\psi|v_{ij}|\psi)=(\phi|K_{ij}|\phi)+(\phi|\sum K_{ij}GK_{jl}GK_{il}|\phi)+\cdots, \quad (1)$$

K_{ij} is related to v_{ij} by a nonlinear integral equation. The G in the last expression is a particle propagator. This last term and the higher one in this expansion are the so-called cluster terms. It was shown[5] that for nuclear matter such terms may be neglected. In adopting Eq. (1) for the finite nucleus one has to look upon K_{ij} as a function of density,[6] ρ. If we expand $K_{ij}(\rho)$ around the mean nuclear density ρ_0, we can write to first order

$$K_{ij}(\rho)=K_{ij}(\rho_0)+(\rho-\rho_0)K_{ij}'(\rho_0). \quad (2)$$

We thus interpret $K_{ij}(\rho_0)$ as the effective two-body force which has to be used in shell model calculations. In later sections we will study this term, and try to estimate it from experiment.

It has seemed useful in earlier work to avoid at first any discussion of the effective two-body force, and to see to what extent the shell model, and especially the pure *j-j* coupling shell model, can give a good approximation to energy levels regardless of the exact form of the effective interaction. In these studies it was also implicitly assumed that a simple two-body force acted between particles, and hence that density-dependent terms were absent. We shall see that this assumption, even in the framework of these earlier calculations, is not strictly necessary, and further that improved agreement with experiment may be obtained if density-dependent interaction terms are included.

A procedure was adopted by Talmi and co-workers[1–3] for the light nuclei which did not involve the detailed knowledge of the interaction between nucleons or the wave functions. The following assumptions were made:

I. The wave function describing the nucleus is a pure *j-j* coupling shell model wave function, the single-particle wave functions entering the complete wave function being independent of the number of nucleons in the shell.

II. The residual interaction between nucleons is a two-body charge-independent interaction (this may include central forces, any mutual spin-orbit interactions, tensor forces, etc.).

It should be remarked here that seniority was considered a good quantum number. Later calculations[7] for the $d_{\frac{5}{2}}$ configuration showed that for the ground states this was a reasonable assumption (to within a few percent).

The application of the shell model requires the calculation of the expectation value of the two-body interaction among nucleons in one shell. This expectation value can be expressed as a linear combination of the energies in a two-nucleon configuration. This method is

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[1] I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956).

[2] R. Thieberger and A. de-Shalit, Phys. Rev. **108**, 378 (1957).

[3] S. Goldstein and I. Talmi, Phys. Rev. **105**, 995 (1957).

[4] A comprehensive list of references is given by Brueckner, Gammel, and Weitzner, Phys. Rev. **110**, 431 (1958).

[5] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).

[6] K. A. Brueckner and W. Wada, Phys. Rev. **103**, 1008 (1956).

[7] R. Thieberger, Ph.D. thesis, Jerusalem, 1958 (unpublished).

due to Racah.[8] We describe it here again for the sake of completeness.

We first wish to show that the interaction in a configuration j^n can be expressed as a linear combination of the energies in the configuration j^2. We consider any two-body operator t_{ij}. In a configuration of n equivalent nucleons in the state j, coupled to total angular momentum J, the expectation value will be

$$(j^nJ|\sum_{i<j}^{n} t_{ij}|j^nJ)=\tfrac{1}{2}n(n-1)(j^nJ|t_{12}|j^nJ), \qquad (3)$$

where the equality holds because of the equivalence of the nucleons. This expectation value can be further simplified by noting that t_{12} operates on particles 1 and 2 only and therefore

$$(j''J|t_{12}|j''J)=\sum_{J_{12}} a(J_{12},J)(j^2J_{12}|t_{12}|j^2J_{12}), \qquad (4)$$

where the sum extends over the N possible values of J_{12}. The positive coefficients $a(J_{12},J)$ depend only on the nature of the state $|j^nJ)$ and not on the operators t_{ij}, and are simply related to the fractional parentage coefficients. Most methods of calculation devote much effort to the calculation of these parentage coefficients. In Racah's method this is not necessary, and we continue as follows. Since the number N of states in a configuration j^2 is finite, there exist N independent operators $t_{ij}^{(1)}, \cdots, t_{ij}^{(N)}$ such that for a given j and every J_{12} one will have

$$(j^2J_{12}|V_{12}|j^2J_{12})=\sum_{k=1}^{N} \alpha_k(j^2J_{12}|t_{12}^{(k)}|j^2J_{12}), \qquad (5)$$

where V_{12} is an effective two-body interaction. As the number of such equations is N (the number of different J_{12}), one can solve this system of equations for the N unknowns α_k, which will depend on V, j, and the special choice of $t_{ij}^{(k)}$. Using (3), (4), and (5), one now finds

$$\begin{aligned}(j^nJ|\sum_{i<j} V_{ij}|j^nJ)&=\tfrac{1}{2}n(n-1)\\ &\quad\times\sum_{J_{12}} a(J_{12},J)(j^2J_{12}|V_{12}|j^2J_{12})\\ &=\sum_k \alpha_k(j^nJ|\sum_{i<j} t_{ij}^{(k)}|j^nJ). \qquad (6)\end{aligned}$$

The whole method depends on whether it is possible to find enough such operators $t_{ij}^{(k)}$, whose expectation values can be easily evaluated. If this is possible then Eq. (6) gives us the desired energy levels. Unfortunately, the number of such known simple operators is not large enough. One has therefore to limit oneself in the following manner. Instead of looking at all the levels of the configuration j^2, we look at the mean energies:

$$\bar{E}(J_1, \cdots, J_k)=\sum_{n=1}^{k} (2J_n+1)E(J_n)\Big/\sum_{n=1}^{k} (2J_n+1), \qquad (7)$$

which belong to the same eigenvalues of the known operators.

At our disposal are three operators: 1, the unit operator; (t_1t_2), the isotopic spin; and (e_1e_2) Casimir's operator, which is defined by[8]

$$e_1\cdot e_2=4_j\sum_{\rho,\sigma} g^{\rho\sigma}e_\rho{}^{(1)}e_\sigma{}^{(2)},$$

the $g^{\rho\sigma}$ being the elements of the inverse of Cartan's matrix,

$$g_{\rho\sigma}=\sum C_{\rho\mu}{}^{\nu}C_{\sigma\nu}{}^{\mu},$$

where $C_{\rho\mu}{}^{\nu}$ are the structure constants and $e_\sigma{}^{(i)}$ are the infinitesimal operators of the symplectic group in $2j+1$ dimensions. Every irreducible representation of this group is described by $j+\frac{1}{2}$ numbers $W(w_1, \cdots, w_{j+\frac{1}{2}})$ and to each such representation there corresponds an eigenvalue,

$$\begin{aligned}g(W)=w_1(w_1+2j_{+1})+w_1(w_1+2_{j-1})&\\ +\cdots+w_{j+\frac{1}{2}}(w_{j+\frac{1}{2}}+2)&,\end{aligned}$$

of Casimir's operator.

We return now to the energy levels. The j^2 levels belong just to three different W's:

$T=1,\quad J=0:\quad W=(000\cdots),$
$T=1,\quad J=2, 4, \cdots:\quad W=(110\cdots),$
$T=0,\quad J=1, 3, \cdots:\quad W=(200\cdots).$

Therefore, we have to arrange our energies in the three groups $E(0)$, $\bar{E}(2, 4, \cdots)$, $\bar{E}(1, 3, \cdots)$. These three expressions can be included in the one equation

$$\bar{V}=a+2b(t_1\cdot t_2)+c[g(W)-4(j+1)]; \qquad (8)$$

a, b, and c are to be chosen so that the three equations for $\bar{V}$ equal to $E(0)$, $\bar{E}(2, 4, \cdots)$, and $\bar{E}(1, 3, \cdots)$ will be satisfied.

It can be shown[9] by group-theoretical methods that, upon taking these averages $\bar{V}$, the same result as obtained in Eq. (6) can be written for the averages of the energies belonging to the same eigenvalue of Casimir's operator.

Fortunately, inasmuch as ground states are characterized by seniority 0 or 1, they are the only members of their group (as we have seen for $J=0$ in j^2) and therefore the energies of ground states can be obtained directly with Racah's method. Thus for the configuration j^n the following formula has been obtained[1]:

$$\begin{aligned}E=nA+\tfrac{1}{2}n(n-1)a+[T(T+1)-\tfrac{3}{4}n]b&\\ +[g(W)-2n(j+1)]c+\text{coulomb energy}.&\qquad (9)\end{aligned}$$

Here E is the binding energy of n particles with angular momentum j outside a doubly magic shell, A is the

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[8] G. Racah, *Farkas Memorial Volume* (Research Council of Israel, Jerusalem, 1952).

[9] G. Racah, Phys. Rev. **76**, 1352 (1949).

single nucleon energy (the sum of its kinetic energy and its energy of interaction with the closed shells), while the other terms are those which represent the mutual interaction. T is the isotopic spin, and the fourth term is essentially the pairing energy in even-even and odd-even nuclei. Its meaning for odd-odd nuclei is less simple and has been explained earlier.

In the study of the binding energies of light nuclei all the available energies, E, in a definite shell were fitted to the linear combination (9). The best values of the coefficients were determined by a least-squares fit, and very good agreement was obtained between the theoretical formula and the observed energies.

In this paper we shall attempt to clarify the basis of Eq. (9), particularly to see to what extent the typical density-dependent interaction effects of Eq. (2) are already partially included. We also wish to investigate the modifications of Eq. (9) due to the effects of three-body interactions to see if improved agreement with experiment may so be obtained. Our final objective will be to see if the parameters a, b, c of Eq. (9) can be suitably corrected for these many-body effects obtained directly from a reasonable two-body interaction.

II. DENSITY CHANGE AND THREE-BODY CORRELATIONS

Density-dependent effects in the effective interaction are in first approximation equivalent to three-body correlation effects. This is seen easily if we recall the origin of the density dependence in the exclusion effect and in the variation of the self-consistent nuclear field. If a third particle is added to a partially filled orbit, then its effect on the motion of particle pairs already present appears largely through these two effects.

We include such possible correlation effects first in the simplest way by allowing for possible changes in Eq. (9) in the interaction parameter a which determines the effective pair interaction strength. As more particles are added, the exclusion effects increase and we expect a decrease in the effective particle-particle force. We shall assume that the change in a gives rise to a new term proportional to the number of triplets, so that Eq. (9) now becomes

$$nA+\tfrac{1}{2}n(n-1)a+\binom{n}{3}d+[T(T+1)-\tfrac{3}{4}n]b$$
$$+[g(W)-2n(j+1)]c. \quad (10)$$

This additional term is equivalent to a linear n-dependence in the effective two-body parameter a. This is seen immediately if we re-write $\frac{1}{2}n(n-1)a+\binom{n}{3}d$ as $\frac{1}{2}n(n-1)(a-\frac{2}{3}d)+\frac{1}{2}n(n-1)(n\times\frac{1}{3}d)$.[10]

TABLE I. Experimental and calculated binding energy (in Mev), using Eqs. (2), (3), with the modification explained in the text.

Nucleus	Exp	Calc	Nucleus	Exp	Calc
Ca^{42}	19.83	20.13	Ti^{48}	76.50	76.59
Ca^{43}	27.75	28.07	Ti^{49}	84.62	84.56
Ca^{44}	38.89	39.32	Ti^{50}	95.54	95.57
Ca^{45}	46.31	46.67	V^{47}	61.41	60.89
Ca^{46}	56.72	57.23	V^{49}	83.23	83.26
Ca^{47}	63.87	63.81	V^{51}	103.44	103.46
Ca^{48}	73.95	73.50	Cr^{50}	92.87	93.10
Sc^{43}	24.79	24.85	Cr^{51}	102.02	102.30
Sc^{45}	45.85	45.80	Cr^{52}	114.25	114.44
Sc^{47}	65.23	65.28	Mn^{53}	120.42	120.61
Sc^{49}	83.27	82.95	Fe^{53}	116.17	116.67
Ti^{45}	43.03	42.71	Fe^{54}	129.76	129.78
Ti^{46}	56.32	55.62	Co^{55}	134.63	134.05
Ti^{47}	64.84	64.54	Ni^{56}	141.19	141.22

Nuclear parameters:

$a=0.397,\quad d=-0.044,\quad b=-1.173,\quad 2(j+1)c=-1.788.$

Coulomb parameters:

$\alpha=-0.212,\quad \gamma=0.348.$

We have carried out calculations for the configuration $f_{7/2}{}^{n}$ using Eq. (10). The procedure adopted was a little different than the one used in reference 1. We did not assume, for the Coulomb forces, a harmonic oscillator but wrote[2,11]

$$zB+\tfrac{1}{2}z(z-1)\alpha+[\tfrac{1}{2}z]\gamma. \quad (11)$$

Here B, α, and γ are free parameters. B represents the Coulomb interaction with the closed shells, α is the Coulomb interaction between z equivalent protons, and $[\frac{1}{2}z]=\frac{1}{2}z$ or $\frac{1}{2}(z-1)$, whichever is integral. The last term represents the pairing energy resulting from the fact that two protons have a larger probability of being found close together if their spins are oppositely directed.

In addition to the change in handling the Coulomb energy, we have reduced the number of free parameters by taking A [Eq. (10)] and B [Eq. (11)] from Ca^{41} and Sc^{41}. The results of the computation are given in Table I. The experimental energies are given in relation to Ca^{40}. All the binding energies of the Ca isotopes are known and are included. In the other nuclei, unfortunately, the situation is not so complete. No set of energy values of odd-odd nuclei (belonging to the same W) are known completely, so that their mean value [Eq. (7)] cannot be included in the table. If we compare the results in Table I to those of reference 1, we see considerably better agreement resulting from the n^3 term. Instead of a root mean square deviation of 750 kev, we obtain now 360 kev. Observing the table more closely and comparing it to reference 1, we see that the main improvement is at the beginning and the end of the shell. This could be expected because by not including the parameter d one obtains some mean value of a, corresponding to the nuclei in the middle of

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[10] It has already been shown in a previous investigation (reference 7), that in the case of $d_{5/2}$ such a cubed term improved the agreement remarkably, the root mean square deviation becoming less than half its former value.

[11] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).

the shell. Comparing the parameters in Table I and reference 1, one sees that the main difference is that using Eq. (10) instead of (9) enlarges a by a certain amount—in our case by about 0.250 Mev.

We next consider briefly a further improvement in our treatment of the many-body correlation effects. Thus far we have ignored the possible dependence of the effects on the type of nucleons interacting. Not only is there some spin dependence in the interactions but also the exclusion effects depend strongly on the nature of the interacting triplet of particles. Three protons or three neutrons would not, because of Pauli's exclusion principle, exhibit such an effect in their effective interactions as long as one assumes short-range forces. To look for such an effect, we have repeated our calculation by inserting in Eq. (10), instead of the term $\binom{n}{3}d$ a term $\left[\binom{n}{3}-\binom{N}{3}-\binom{z}{3}\right]d'$ where $n=N+z$. The result improves but by a nonsignificant amount. We conclude therefore, that we cannot distinguish between these possibilities. This difficulty arises because n is not large enough, even for $f_{7/2}$, for such a delicate comparison. It may be remarked however that an argument in favor of the d' expression is given by the nice results obtained for Ca isotopes without any cubed term.[12] For $j>\frac{7}{2}$, the protons and neutrons fill different shells and so it seems that we cannot in this way distinguish between the two possibilities.

The appearance of the term discussed here cannot be confined solely to particles outside a closed shell. We have to expect the same sort of change in the interaction with the closed shells, which has been represented up to this point only by the coefficient A. The additional extra core nucleons will alter the core energies, the effect appearing as a term quadratic in n. Therefore Eq. (9) and Eq. (10) would not look different if such a term were introduced. We would only have to understand that the a appearing in these equations is composed of two terms:

$$a=a_1+a_2. \tag{12}$$

Here a_1 represents the expression appearing from the interaction of the particles outside the closed shells, the expression previously thought of as comprising the whole of a. a_2 represents the change in the single-nucleon energy due to the effect of density changes on the core energies.

In the following analysis we will show the two a's to be about equal in magnitude and opposite in sign. We turn now to the calculation of the parameters a_1, b, and c.

III. THE WAVE FUNCTION

The usual way of treating energy levels in atomic spectroscopy was developed by Slater. The two-body potential is expanded in a series of Legendre polynomials. The integration over the angular variables of the nucleons is carried out, and the final result is a sum of radial integrals. The form of these radial integrals for the case we are considering here is

$$F^k=\int\int f_k(r_1,r_2)R_a{}^2(r_1)R_b{}^2(r_2)dr_1dr_2, \tag{13}$$

where $R_i(r)$ are the radial parts of the wave functions, r_1 and r_2 are the coordinates of the first and second particle, $f_k(r_1,r_2)$ are the coefficients in the expansion of the two-body potential in Legendre polynomials:

$$f_k(r_1,r_2)=\frac{2k+1}{2}\int_{-1}^{1}V(\mathbf{r}_2-\mathbf{r}_1)P_k(\cos\omega_{12})d\cos\omega_{12},$$

where ω_{12} is the angle between r_1 and r_2.

F^k can be evaluated simply if we use a special form of the single-nucleon wave function, the harmonic oscillator wave functions.[13,14] These wave functions are determined from the single-particle potential,

$$U(r)=\tfrac{1}{2}m\omega^2r^2=\hbar\omega\nu r^2, \tag{14}$$

and are

$$\psi(r,\vartheta,\varphi)=[R(r)/r]Y_l{}^m(\vartheta,\varphi),$$

where the radial part is

$$R_{nl}(r)=N_{nl}r^{l+\frac{1}{2}}\exp(-\nu r^2)L_{n+l+\frac{1}{2}}{}^{l+\frac{1}{2}}(2\nu r^2).$$

N_{nl} is a normalization factor and $L_{n+l+\frac{1}{2}}{}^{l+\frac{1}{2}}(2\nu r^2)$ is an associated Laguerre polynomial.

For the case of the harmonic oscillator wave function one can perform a separation of the variables: $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$. One can then integrate over the angle between $\mathbf{r}$ and $\mathbf{R}$, and over $\mathbf{R}$, and one is left with integrals (Talmi's integrals) of the form:

$$I_l(\nu/2)=\int_0^\infty R_l{}^2(r)V(r)dr,$$

where $R_l=R_{0l}$, and $I_l(\nu/2)$ means that one has to substitute in R_l, $\nu/2$ instead of ν.[15] In our calculations we have equivalent nucleons, i.e., $a=b$. For this case we can always use the harmonic oscillator wave functions, where the correction to the self-consistent shell model wave functions will enter into the effective interaction. We can write $R_a(r_1)=c(r_1)R_{nl}(r_1)$ and we obtain $f_k{}^{(\mathrm{new})}=f_k{}^{(\mathrm{old})}c^2(r_1)c^2(r_2)$, which defines a new interaction:

$$V^{(\mathrm{new})}(\mathbf{r}_2-\mathbf{r}_1)=\sum_{k=0}^{\infty}f_k{}^{(\mathrm{new})}P_k(\cos\omega_{12}). \tag{15}$$

To determine the wave function we have to choose ν, the spring constant. Although $A^{\frac{1}{3}}$ may vary quite considerably over the range of configurations with which

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[12] I. Talmi, Phys. Rev. **107**, 326 (1957).

[13] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

[14] R. Thieberger, Nuclear Phys. **2**, 533 (1956/7).

[15] See reference 14, p. 539, the definition of λ.

we will be dealing, $\sqrt{\nu}$ changes much less. It has been shown[11] that the mean square radius is given by $\langle r^2\rangle_{\rm Av} = [\eta(A)/4\nu]$, where $\eta(A)$ is the average of $(4n+2l+3)$ over the particles. As later particles have higher quantum numbers, $\eta(A)$ is a monotonically increasing function of A. As $(\langle r^2\rangle_{\rm Av})^{\frac{1}{2}}$ is proportional to $A^{\frac{1}{3}}$, we get that $\sqrt{\nu}$ does not decrease as quickly as $A^{\frac{1}{3}}$. We will assume ν to be constant for each configuration and its value chosen from Coulomb energies.[1,11]

IV. THE HARD CORE

When a hard core is present in the potential, Eq. (13) presents a difficulty since the integral runs over the core region also, and in the core region $V(r)=\infty$. The integral can be finite only if the wave function vanishes in the core region. According to Brueckner and Gammel,[16] and in the notation we use in the previous section, we make the replacement:

$$c^2(r)V_{\rm core}(r)=c\delta(r-r_c)/R_0^2(r).$$

The argument which will follow is independent of whether we choose R_0 or any other R_l. We obtain therefore

$$I_l=c\frac{\nu^l 2^l}{(2l+1)!!}\int r^{2l}\delta(r-r_c)dr=c\frac{2^l}{(2l+1)!!}(\nu r_c^2)^l,$$

where $(2l+1)!!\equiv 1\times 3\times 5\times\cdots\times(2l+1)$. This could be approximated with an appropriate Gaussian potential.[17] Assuming for the hard core $r_c=0.5\times 10^{-13}$ cm, we obtain with the value of ν from the last section $\nu r_c^2=0.034$. Therefore the I_l with $l\neq 0$ can be neglected in comparison to I_0.

A δ-force at the origin is characterized by having $I_0\neq 0$ and $I_l=0$ for $l\neq 0$. Therefore a good approximation for the hard core is represented by a δ force. This result was obtained previously.[18] There it is shown that to a first approximation the strength of the δ force is $4\pi(\hbar^2 r_c/m)$. More generally, the effect of the core is enhanced both by many-body exclusion effects and by the presence of the strongly attractive potential. We write our force in the form $[\delta(r)/r^2\nu^{\frac{3}{2}}]\alpha V_c$, the α being chosen so that the value of I_0 should coincide with the one obtained for the radial part of the central potential chosen. For the case taken in the next section $\alpha=2\lambda[\lambda^2/(1+\lambda^2)]$, where $\lambda=r_0\nu^{\frac{1}{2}}$. We will leave the strength V_c as a free parameter adjustable to give the best results.

We should remark at this point, and it will be seen from the results, that the hard-core term has many characteristics of enlarging the range of the two-body force. A larger range will have as a result large I_l, $l\neq 0$, compared to I_0. This same result is brought about by the hard core because it diminishes I_0 (as the hard core is repulsive compared to the otherwise attractive force) but does not influence the higher I_l, and so I_l, $l\neq 0$, will become relatively larger.

We put all the hard-core effects into the δ-force and take for the rest of the potential the integrals from zero and not r_c. According to the same argument as given here, this is allowed and will result in a modification of the δ-force strength.

V. THE CALCULATION OF THE PARAMETERS

Our task is to calculate the parameters a, b, and c. For this purpose we calculate the energy levels of the j^2 configuration concerned, in terms of the Slater integrals. To calculate the Wigner force a formula is available.[19] For the Majorana force we perform the calculation in L-S coupling and then transform[20] the result to j-j coupling. This can then be used to obtain the Bartlett force expression.

We have not considered noncentral forces. There seems to be no conclusive evidence, in shell model calculations, that it is necessary to include in the effective force a tensor force term. For simplification, therefore, we have ignored such a term. The spin-orbit forces are mostly included by our assumption of j-j coupling.

As we are assuming harmonic oscillator wave functions, we have to calculate Talmi's integrals Eq. (15). For this purpose we need a specific assumption of the radial dependence of the two-body force. The question arises of limitations imposed by taking a specifically chosen radial dependence. As the Talmi integrals of higher l have a higher power of r in the integrand, one obtains that essentially $I_{l+1}/I_l\sim r_0\nu^{\frac{1}{2}}=\lambda$, as long as λ is considerably smaller than one. (This argument was also used in connection with the hard core in Sec. IV.) As our λ will be around 0.5 we see that the contribution of integrals higher than I_2 is very small. We have a few parameters determining our potential, the strength, the strength of the hard-core term, and to a smaller extent λ. The hard core determines I_0; the strength determines, we can say, I_1; I_2 will remain in certain limits because λ was chosen to correspond to similar shell-model calculations. Every potential chosen would have its own set of strength, hard-core strength, and λ. So the main integrals, I_0, I_1, and I_2 would be determined in the same way for different radial dependence. We obtain the result that the choice of the radial dependence does not play an important part in determining the over-all picture. According to this argument a simple radial dependence was chosen:

$$V(r)=V\frac{\exp(-r^2/r_0^2)}{r/r_0}.$$

We assumed the same range, r_0, for the different ex-

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[16] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[17] R. D. Amado, Phys. Rev. **111**, 548 (1958).

[18] M. Bauer and M. Moshinsky, Nuclear Phys. **4**, 615 (1957).

[19] G. Racah, Phys. Rev. **62**, 438 (1942).

[20] G. Racah, Physica **16**, 651 (1950).

change forces. This is a somewhat arbitrary constant, but as explained previously is not very decisive. By this procedure we obtain the energy levels as functions of the potential strength V, or more precisely, as a function of four sorts of potential strengths, the one corresponding to a Wigner force, Majorana force, Bartlett force, and the δ-force strengths. We call them V_w, V_m, V_b, and V_c, respectively.

It was mentioned in the introduction that we can express $E(0)$, $\bar{E}(\text{even})$, and $\bar{E}(\text{odd})$ in terms of the parameters a_1, b, and c, according to formula (8). One can then solve these three inhomogeneous equations for the three unknowns a_1, b, and c in terms of the energy combinations $E(0)$, $\bar{E}(\text{even})$, $\bar{E}(\text{odd})$.[8] As mentioned in the last paragraph, these levels can be expressed in terms of V_w, V_m, V_b, and V_c. Thus we obtain a_1, b, and c as functions of V_w, V_m, V_b, and V_c.

We have considered the configurations $0p_{1/2}$, $0d_{5/2}$, $1s_{1/2}$, $0d_{3/2}$, and $0f_{7/2}$. We have omitted the configuration $0p_{3/2}$ as it seems to us that the regularities seen in later configurations do not necessarily appear in this case.

We have now described all the procedure necessary to calculate the two-body interaction parameters of Eq. (10). The equations relating the configuration-dependent energy parameters and the parameters of the Wigner, Majorana, Bartlett, and core terms in the potential are, for the ith configuration:

$$\begin{aligned} a^{(i)} &= a_2^{(i)} + a_W^{(i)} V_W + a_M^{(i)} V_M + a_B^{(i)} V_B + a_c^{(i)} V_c, \\ b^{(i)} &= b_W^{(i)} V_W + b_M^{(i)} V_M + b_B^{(i)} V_B + b_c^{(i)} V_c, \qquad (16) \\ c^{(i)} &= c_W^{(i)} V_W + c_M^{(i)} V_M + c_B^{(i)} V_B + c_c^{(i)} V_c. \end{aligned}$$

TABLE II. The relation between the parameters and Slater integrals.

Configuration	Force	Parameter	F^0	F^2	F^4	F^6
$f_{7/2}$	Wigner	a	1	−0.04409	−0.02164	−0.01079
		b	0	−0.07054	−0.03463	−0.01727
		c	0	−0.01763	−0.00865	−0.00432
	Majorana	a	−0.21428	0.03897	0.02236	0.02020
		b	−0.57142	0.00141	−0.00577	−0.02097
		c	−0.03571	−0.00149	−0.00396	−0.01102
	Bartlett	a	0.5	−0.02204	−0.01082	−0.00540
		b	0	−0.03527	−0.01731	−0.00863
		c	0.03571	0.00149	0.00396	0.01102
$d_{5/2}$	Wigner	a	1	−0.06531	−0.02721	
		b	0	−0.09706	−0.04082	
		c	0	−0.02449	−0.01020	
	Majorana	a	−0.2	0.05715	0.03809	
		b	−0.6	0	−0.02857	
		c	−0.05	−0.00408	−0.01599	
	Bartlett	a	0.5	−0.03266	−0.01360	
		b	0	−0.04898	−0.02041	
		c	0.05	0.00408	0.01599	
$d_{3/2}$	Wigner	a	1	−0.12		
		b	0	−0.16		
		c	0	−0.04		
	Majorana	a	−0.22	0.08285	0.02285	
		b	−0.56	0.03428	−0.04571	
		c	−0.03	0.00714	−0.02285	
	Bartlett	a	0.5	−0.06	0	
		b	0	0.08	0	
		c	0.03	−0.00714	0.02285	
$p_{1/2}$	Wigner	a	1			
		b	0			
	Majorana	a	−0.11111	0.22222		
		b	−0.44444	0.08889		
	Bartlett	a	0.38889	−0.17778		
		b	−0.11111	−0.17778		
$s_{1/2}$	Wigner	a	1			
		b	0			
	Bartlett	a	−0.5			
		b	−1			

The configuration label "i" runs over the configurations $f_{7/2}$, $d_{5/2}$, $d_{3/2}$, $p_{1/2}$ and $s_{1/2}$. a_2 was defined in Eq. (12). The parameters $a_W^{(i)}$, $a_M^{(i)}$, $a_B^{(i)}$, $a_c^{(i)}$, etc., are independent of the potential strengths and fixed by the particle angular momentum, by the type and range of the two-body force, and by the assumed form of the radial wave functions.

In Table II we give the relations between the parameters and the Slater integrals.

In calculating the parameters $a_W^{(i)}$, $b_W^{(i)}$, $\cdots$, $c_c^{(i)}$, we assumed that ν changes according to the values derived from the experimental Coulomb energy in reference 1. We left $a_2^{(i)}$ as a free parameter for each configuration i.

We insert the parameters from Eq. (16) into Eq. (10). For simplicity we omitted the parameter A and the Coulomb energy, so that instead of considering E, we are considering $\bar{E} = E - nA -$ Coulomb energy. For the $p_{1/2}$ and $s_{1/2}$ configurations we took the A so that $\bar{E}(\mathrm{S}^{32}) = 2\bar{E}(\mathrm{P}^{31})$, $\bar{E}(\mathrm{O}^{16}) = 2\bar{E}(\mathrm{N}^{15})$ in the rest of the cases we took A from the nucleus with one nucleon outside a closed shell. The Coulomb energy was adjusted by use of the parameters obtained previously.

TABLE III. The two-body interaction potential.

	V_W	V_M	V_B	V_c	Rms
With the core	−49.0	−50.4	−7.7	21.1	0.40
Without the core	−34.0	−37.6	−10.5	···	0.41

VI. RESULTS AND CONCLUSIONS

Performing the least-squares fit to the sixty energy equations resulting from our available experimental data parameters, which are

$$a_2^{(i)},\ d(d_{5/2}),\ d(f_{7/2}),\ V_W,\ V_M,\ V_B,\ V_c,$$

we obtain the values of V_W, V_M, V_B, and V_c which give the best fit to the binding energies. The best values of the parameters, for a chosen range $r_0 = 1.40 \times 10^{-13}$ cm, are given in Table III. We also give the root mean square (rms) deviation. The rms deviation is defined in the usual way as

$$\Big[\sum_{i=1}^{N} \Delta_i^2 / N - K\Big]^{\frac{1}{2}},$$

where the Δ_i are differences between the experimental and calculated energies, N is the number of data and K is the number of parameters.

As is seen in Table III we obtain approximately a Serber exchange mixture, i.e., $V_M \approx V_W$, and a small Bartlett contribution. This is consistent with the two-body scattering potential.[21]

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[21] P. S. Signell and R. E. Marshak, Phys. Rev. **106**, 832 (1957); J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 and 1337 (1957).

This result was expected since we believe that our mixture should not differ much from the free-nucleon interaction, since for small ranges Wigner and Majorana forces are similar (they become equal for a zero-range δ-force) and start to differ only for longer ranges, but at longer ranges the many-body correlation effects are much smaller.[4] Our calculations show that the Bartlett force also does not change much as a result of the correlations. This result can be seen from the expression:

$$V_{\text{triplet},\ l=0}/V_{\text{singlet},\ l=0}=1.22.$$

Table IV shows the density-dependent term, a_2, for the different configurations. This term is in a way a measure of the overlap of the wave function of the given configuration with nearby configurations. The results are consistent with this interpretation, as can be seen in the table. The configuration $f_{7/2}$, $d_{3/2}$, and to some extent $d_{5/2}$ are isolated and this is reflected in having smaller density terms for them. In the other cases there is considerable overlap: $p_{1/2}$ lies near $p_{3/2}$ and $s_{1/2}$ lies near $d_{5/2}$.

TABLE IV. The density-dependent parameters, a_2.

Configuration	a (obtained from the least-squares fit)	a_1 (calculated)	a_2 (resulting)
$f_{7/2}$	0.424	0.801	0.377
$d_{5/2}$	0.723	1.819	1.096
$d_{3/2}$	0.176	1.003	0.827
$p_{1/2}$	2.939	4.788	1.849
$s_{1/2}$	1.908	4.698	2.790

The d's obtained were $d(d_{5/2})=-0.0869$, $d(f_{7/2})=-0.0507$.

We also calculate the energies without a repulsive-core term. The over-all agreement worsens only a little but the exchange mixture obtained is different. It is still true that $V_W \cong V_M$, although the Majorana part is relatively somewhat larger, but the Bartlett force becomes larger. Table III also contains the constants for the case where the core term is omitted. We see that this term is not essential for the calculations of binding energies.

TABLE V. Comparison between experimental and calculated binding energies (in Mev).

Nucleus	Exp	Calc	Nucleus	Exp	Calc	Nucleus	Exp	Calc
Si^{30}	19.09	18.76	F^{19}	19.97	20.49	Sc^{45}	45.85	45.80
P^{30}	13.97	14.03	Ne^{21}	39.45	38.97	Sc^{47}	65.23	65.31
P^{31}	26.38	26.40	Ne^{22}	50.17	49.91	Sc^{49}	83.27	82.93
S^{32}	35.24	35.27	Ne^{23}	55.36	55.04	Ti^{45}	43.03	42.70
C^{14}	13.12	12.90	Ne^{24}	64.25	64.44	Ti^{46}	56.32	55.68
N^{14}	12.49	12.22	Na^{23}	58.52	58.10	Ti^{47}	64.84	64.55
N^{15}	23.32	23.36	Na^{25}	74.76	75.02	Ti^{48}	76.50	76.54
O^{16}	35.43	35.50	Mg^{24}	70.65	70.48	Ti^{49}	84.62	84.52
S^{34}	20.05	19.70	Mg^{25}	77.98	77.98	Ti^{50}	95.54	95.43
S^{35}	27.07	26.81	Mg^{26}	89.10	90.03	V^{47}	61.41	61.04
S^{36}	36.97	36.33	Al^{27}	97.35	97.18	V^{49}	83.23	83.35
Cl^{35}	26.46	26.21	Si^{28}	108.94	108.61	V^{51}	103.44	103.44
Cl^{37}	45.39	45.07	Ca^{42}	19.83	20.07	Cr^{50}	92.87	93.18
A^{36}	34.96	34.57	Ca^{43}	27.75	28.04	Cr^{51}	102.02	102.33
A^{37}	43.79	43.92	Ca^{44}	38.89	39.24	Cr^{52}	114.25	114.29
A^{38}	55.54	55.66	Ca^{45}	46.31	46.66	Mn^{53}	120.42	120.56
K^{39}	61.96	62.19	Ca^{46}	56.72	57.20	Fe^{53}	116.17	116.84
Ca^{40}	70.29	70.40	Ca^{47}	63.87	63.82	Fe^{54}	129.76	130.01
O^{18}	12.23	13.55	Ca^{48}	73.95	73.46	Co^{55}	134.63	134.03
O^{19}	16.34	16.45	Sc^{43}	24.79	24.84	Ni^{56}	141.19	141.12

The actual comparison between the calculations presented here and the experimental binding energies is given in Table V. We see a very good agreement, of the same order of magnitude as the one which was obtained with the free parameters.[1] The main errors lie in the $d_{5/2}$ configuration, where O^{18} and Mg^{26} have especially large errors. It should be remarked here that in this calculation we are assuming that the seniority is a good quantum number, and it was shown[7] for the $d_{5/2}$ configuration that this is not strictly the case. Still we would like to stress that even for these cases the agreement is quite good, the over-all error being 400 kev which is only 0.5% of the difference between the smallest and largest experimental values used in the least-squares fit.

We would like to point out that our results do not only justify to a large extent the main assumptions of the free-parameters method,[1] but show the possibility of calculating levels in many configurations together, where many of the parameters are constant over the whole range.

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