Ground-State Energy and Excitation Spectrum of a System of Interacting Bosons

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(Received May 4, 1959)

In this paper properties of a boson gas at zero temperature are investigated by means of field-theoretic methods. Difficulties arising from the depletion of the ground state are resolved in a simple way by the elimination of the zero-momentum state. The result of this procedure when applied to the calculation of the Green's functions of the system is identical to that of Beliaev. It is then shown generally that for a repulsive interaction the energy $E(\mathbf{k})$ of a phonon of momentum $\mathbf{k}$, which is found as the pole of a one-particle Green's function, approaches zero for zero momentum, which means that the phonon spectrum does not exhibit an energy gap.

The Green's function method is applied to the calculation of the properties of a low-density boson gas. The next order term beyond that calculated by Lee and Yang, and Beliaev for the ground-state energy is obtained and the general form of the series expansion is found to be

$$(E_0/\Omega)=\tfrac{1}{2}n^2f_0[1+a(nf_0^3)^{\frac{1}{2}}+b(nf_0^3)\ \ln nf_0^3+c(nf_0^3)+d(nf_0^3)^{\frac{1}{2}}\ln(nf_0^3)+\cdots],$$

where n is the density and f_0 is the scattering length for the assumed two-body interaction between the bosons. The coefficients a and b are independent of the shape of the interaction, and are the only terms thus far calculated. The coefficient b is in agreement with the hard-sphere gas calculations of Wu and of Sawada.

A discussion is given of the intermediate-density calculation of Brueckner and Sawada, and certain possible improvements in the method of summing a selected set of higher-order terms are proposed.

1. INTRODUCTION

THE realization that there exists a great formal similarity between the quantum theory of a large number of interacting Fermi particles and quantum field theory has led in recent years to the development of new methods for the treatment of such a fermion gas,[1] in particular at zero temperature.

The application of similar methods to a system of particles obeying Bose statistics gives rise to two difficulties of a different nature. The first difficulty has to do with the particular role played by the large number of particles of momentum zero. In the noninteracting system all particles have zero momentum. In the interacting system the zero-momentum state likewise contains very many particles, since only a finite fraction of these is excited as a consequence of the interaction. The fraction of particles of nonzero momentum in the ground state of the interacting system is a function of the density and is very small for low density. Hence for low density this so-called "depletion" of the ground state can be neglected, as in the work of Bogoliubov[2] and in the pseudopotential method of Lee, Huang, and Yang.[3] However, for calculations of the energies of the ground state and of low-lying excited states going beyond the extreme low density case, the depletion effect must be taken into account.

Another difficulty, which also is absent in the fermion case, is the fact that even for a regular, repulsive interaction perturbation theory diverges Bogoliubov has shown how in the case of very weak interaction and low density the divergences can be removed by means of a canonical transformation. This same procedure was also used by Lee, Huang, and Yang in their pseudopotential method.

Recently Beliaev[4] developed a method which enables one to take into account the depletion effect rigorously and which furthermore leads to a formulation which to all orders is free of divergences. In this method essential use is made of the Green's functions, which are well-known in field theory. From these functions both the phonon spectrum and the energy E_0 of the ground state can be obtained.

In the present paper we present in the first place another, and to our opinion, simpler and more transparent treatment of the depletion effect, in which we do not make use of any form of perturbation theory. This forms the content of Sec. 3, which follows a rather extensive discussion of the difficulties in Sec. 2.

In Sec. 4 we introduce the one-particle Green's functions. We follow Beliaev and obtain a closed expression for the Green's functions in terms of two functions Σ_{11} and Σ_{20} which are the analog of the proper self-energy parts in field theory. This procedure involves a partial summation of the perturbation series expansion of the Green's functions and is sufficient to remove all low-momentum divergences. One then has a consistent scheme where both difficulties, mentioned above, have been resolved. We use this scheme in Sec. 6 to derive a quite general relationship between the chemical potential $\mu=dE_0/dN$ and the functions Σ_{11} and Σ_{20}, both for zero momentum and energy. This relation permits one to prove that the phonon energy is equal to zero for zero momentum. That some calculations give rise to an

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[1] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957). N. M. Hugenholtz, Physica **23**, 481 (1957). V. M. Galitskii and A. B. Migdal, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 139 (1958) [translation: Soviet Phys. JETP **7**, 96 (1958)]. See also A. Klein and R. Prange (to be published).

[2] N. N. Bogoliubov, J. Phys. U.S.S.R. **9**, 23 (1947).

[3] Lee, Huang, and Yang, Phys. Rev. **106**, 1135 (1957).

[4] S. T. Beliaev, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 417 (1958) [translation: Soviet Phys. JETP **7**, 289 (1958)].

energy gap in the phonon spectrum[5] is due either to an incorrect treatment of the depletion effect, or to an inconsistent treatment of some of the terms in the interaction.

In Sec. 5 we give a short discussion of more general Green's functions and their significance for treating the scattering of neutrons by a boson gas.

Sections 7 and 8 are devoted to the calculation of the properties of the low-density boson gas. In Sec. 8 it is shown that, in agreement with Sawada[6] and Wu,[7] there exists a term in the expansion for E_0/N of the form $n^3 f_0^4 \ln(nf_0^3)$, the coefficient of which is calculated. Finally in Sec. 9, we discuss the general form of the series expansion, and, somewhat briefly, the intermediate-density theory of the hard-sphere gas due to Brueckner and Sawada.[9]

2. DIFFICULTIES WITH DEPLETION OF THE GROUND STATE

The system under consideration consists of N interacting particles enclosed in a cubic box of volume Ω. We assume these particles to obey Bose statistics. The Hamiltonian can be written as $H=H_0+V$, in which the kinetic energy H_0 and the interaction V have the usual form in second quantization[9]:

$$H_0=\sum_{\mathbf{k}} \tfrac{1}{2}k^2 a_{\mathbf{k}}^* a_{\mathbf{k}},$$
$$V=\tfrac{1}{4}\Omega^{-1}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}[v(\mathbf{k}_1-\mathbf{k}_3)+v(\mathbf{k}_1-\mathbf{k}_4)] \times\delta_{\mathrm{Kr}}(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* a_{\mathbf{k}_3}a_{\mathbf{k}_4}. \quad (2.1)$$

The operators $a_{\mathbf{k}}^*$ and $a_{\mathbf{k}}$ are creation and annihilation operators, satisfying the commutation relations

$$[a_{\mathbf{k}},a_l]=[a_{\mathbf{k}}^*,a_l^*]=0;\quad [a_{\mathbf{k}},a_l^*]=\delta_{\mathbf{k}l}.$$

The function $v(\mathbf{k})$ is the Fourier transform of the central two-body interaction

$$v(\mathbf{k})=\int d^3x\, v(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

The Kronecker symbol δ_{Kr} is equal to one if the argument is zero, and zero otherwise.

Since we are interested in the limiting case in which both N and Ω are infinite, with a finite particle density $n=N/\Omega$, we find it convenient to use another notation, which is more suitable for that case. We define

$$\int_{\mathbf{k}}=(2\pi)^3\Omega^{-1}\sum_{\mathbf{k}},\quad \delta^3(\mathbf{k}-\mathbf{k}')=(2\pi)^{-3}\Omega\delta_{\mathrm{Kr}}(\mathbf{k}-\mathbf{k}'),$$
$$\xi_{\mathbf{k}}=\Omega^{\frac{1}{2}}(2\pi)^{-\frac{3}{2}}a_{\mathbf{k}}.$$

The commutation relations for the ξ-operators are

$$[\xi_{\mathbf{k}},\xi_l]=[\xi_{\mathbf{k}}^*,\xi_l^*]=0;\quad [\xi_{\mathbf{k}},\xi_l^*]=\delta^3(\mathbf{k}-\mathbf{l}).$$

In the limit of an infinite system, $\delta^3(\mathbf{k}-\mathbf{l})$ is the Dirac δ-function and the symbol $\sum_{\mathbf{k}}$ is replaced by the integration sign $\int d^3k$. In our new notation the Hamiltonian becomes

$$H_0=\int_{\mathbf{k}}\tfrac{1}{2}k^2\xi_{\mathbf{k}}^*\xi_{\mathbf{k}},$$

and

$$V=\tfrac{1}{4}(2\pi)^{-3}\int_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}[v(\mathbf{k}_1-\mathbf{k}_3)+v(\mathbf{k}_1-\mathbf{k}_4)] \times\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\xi_{\mathbf{k}_1}^*\xi_{\mathbf{k}_2}^*\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4}.$$

It is our purpose to calculate the energies of the ground state and of low-lying excited states of this system, all at zero temperature. Since practically all available methods for the treatment of such problems are based on some form of perturbation theory (in which the interaction between the particles is considered as the perturbation) let us first consider the noninteracting system. Here we notice a marked difference with the fermion gas, a difference which, as we shall see, gives rise to definite complications in any treatment of the boson gas. In the ground state $|\phi_0\rangle$ of the noninteracting system all particles have zero momentum. Consequently in any form of perturbation theory this zero-momentum state will play a role different from the other single-particle states. It is therefore convenient to rewrite the interaction Hamiltonian in a form in which all terms with one or more $\mathbf{k}$'s equal to zero are written separately. When we do this, we find

$$H=H_0+V_a+V_b+V_c+V_d+V_e+V_f+V_g, \quad (2.2)$$

where

$$V_a=\tfrac{1}{4}(2\pi)^{-3}{\int_{\mathbf{k}_1\cdots\mathbf{k}_4}}'[v(\mathbf{k}_1-\mathbf{k}_3)+v(\mathbf{k}_1-\mathbf{k}_4)] \times\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\xi_{\mathbf{k}_1}^*\xi_{\mathbf{k}_2}^*\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4},$$

$$V_b=\tfrac{1}{2}(2\pi)^{-3}a_0(2\pi)^{\frac{3}{2}}\Omega^{-\frac{1}{2}}{\int_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}}'[v(\mathbf{k}_1)+v(\mathbf{k}_2)] \times\delta^3(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)\xi_{\mathbf{k}_1}^*\xi_{\mathbf{k}_2}^*\xi_{\mathbf{k}_3},$$

$$V_c=\tfrac{1}{2}(2\pi)^{-3}a_0^*(2\pi)^{\frac{3}{2}}\Omega^{-\frac{1}{2}}{\int_{\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}}'[v(\mathbf{k}_3)+v(\mathbf{k}_4)] \times\delta^3(\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\xi_{\mathbf{k}_2}^*\xi_{\mathbf{k}_3}\xi\mathbf{k}_4,$$

$$V_d=\tfrac{1}{4}(2\pi)^{-3}a_0^2(2\pi)^3\Omega^{-1}{\int_{\mathbf{k}_1\mathbf{k}_2}}'[v(\mathbf{k}_1)+v(\mathbf{k}_2)] \times\delta^3(\mathbf{k}_1+\mathbf{k}_2)\xi_{\mathbf{k}_1}^*\xi_{\mathbf{k}_2}^*, \quad (2.3)$$

$$V_e=\tfrac{1}{4}(2\pi)^{-3}a_0^{*2}(2\pi)^3\Omega^{-1}{\int_{\mathbf{k}_3\mathbf{k}_4}}'[v(\mathbf{k}_3)+v(\mathbf{k}_4)] \times\delta^3(\mathbf{k}_3+\mathbf{k}_4)\xi_{\mathbf{k}_3}\xi_{\mathbf{k}_4},$$

$$V_f=(2\pi)^{-3}a_0^*a_0(2\pi)^3\Omega^{-1}{\int_{\mathbf{k}_2\mathbf{k}_3}}'[v(\mathbf{k}_2)+v(0)] \times\delta^3(\mathbf{k}_2-\mathbf{k}_3)\xi_{\mathbf{k}_2}^*\xi_{\mathbf{k}_3},$$

$$V_g=\tfrac{1}{2}(2\pi)^{-3}a_0^{*2}a_0^2(2\pi)^6\Omega^{-2}\delta^3(0)v(0).$$

[5] M. Girardeau and R. Arnowitt, Phys. Rev. **113**, 755 (1959). See also S. Butler and J. Valatin, Nuovo cimento **10**, 37 (1958).
[6] K. Sawada (to be published).
[7] T. T. Wu (to be published).
[8] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957).
[9] We choose such units that $\hbar=M=1$.

Here the primed summation symbols mean that the summation is extended over all $\mathbf{k}\neq 0$.

The operators a_0^* and a_0 always appear together with a factor $\Omega^{-\frac{1}{2}}$ in the Hamiltonian (2.2). At first sight one might suppose therefore that the operators $a_0^*\Omega^{-\frac{1}{2}}$ and $a_0\Omega^{-\frac{1}{2}}$ could be neglected. However, because the state of zero momentum contains a large number of particles, this is not the case; thus $a_0^*a_0\Omega^{-1}|\phi_0\rangle=n|\phi_0\rangle$, if $|\phi_0\rangle$ is the unperturbed state of the system, corresponding to N particles with momentum zero.

Let us now see the way in which the operators a_0^* and a_0 prevent the immediate application to the boson problem of methods which have been very successful in the case of fermions. These methods possess the common feature that the ground state of the system is considered as the analog of the vacuum in field theory; that is, the ground state of the noninteracting system is then defined by the condition that all annihilation operators applied to that state give zero. In the case of the Fermi gas this means that neither holes nor additional particles are present. For the Bose gas a similar situation does not exist. Although in the unperturbed ground state there are no particles of momentum $\mathbf{k}\neq 0$, a large number N of particles has momentum zero. Hence the noninteracting ground state cannot be considered as vacuum with respect to the operators a_0^* and a_0.

This leads to serious difficulties, which become apparent as soon as one starts calculating the ground-state energy, using perturbation theory. The energy of a gas of interacting Fermi particles can be expressed as a power series in the interaction V, and the various contributions can be represented in terms of diagrams. As shown by Goldstone,[1] an expression for the ground-state energy may be derived, in which only connected diagrams appear. The derivation depends upon a rule which makes it possible to express the contributions of disconnected diagrams in terms of their connected parts.

For the boson gas one can likewise represent the various terms arising from the power series expansion of V in terms of diagrams. The difference lies in the fact that, when one calculates the contribution from a given diagram, the resulting expression is multiplied by the expectation value for the unperturbed ground state of products of $a_0\Omega^{-\frac{1}{2}}$ and $a_0^*\Omega^{-\frac{1}{2}}$, arising from the various terms in V. For instance, for the fourth-order diagram of Fig. 1(a), where the dashed lines represent particles of zero momentum, this factor is equal to $\Omega^{-3}\langle\phi_0|a_0^{*3}a_0^3|\phi_0\rangle$ $=\Omega^{-3}N(N-1)(N-2)$. For any connected diagram the contribution for large systems is asymptotically proportional to Ω, so that in the foregoing expression one may replace the factors $N-1$ and $N-2$ by N, the neglected terms being of relative order N^{-1}. This approximation amounts to replacing the operators a_0^* and a_0 by the c-number $N^{\frac{1}{2}}$.

In contributions from disconnected diagrams this replacement would lead to incorrect results. Consider the disconnected fourth-order diagrams of Fig. 1(b) and (c).

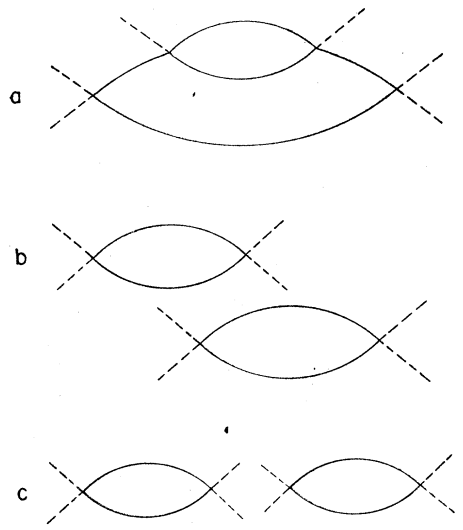

FIG. 1. Three fourth-order ground-state diagrams; diagram a is connected, b and c disconnected. The dashed lines refer to annihilation and creation of particles with zero momentum.

They give contributions of order Ω^2;[10] therefore a term of relative order N^{-1} can no longer be neglected. Thus the diagrams b and c differ not only in the fact that the energies of the intermediate states which appear are different (as is the case for analogous diagrams in the fermion problem) but also in that the operators a_0 and a_0^* appear in different orders, contributing in the one case a factor $N(N-1)(N-2)(N-3)$ and in the other $N^2(N-1)^2$. It is this latter difference which renders invalid the theorem on the contribution of disconnected diagrams and makes a linked cluster expansion impossible for the boson gas.

For the case of extreme low density, the correction terms we have discussed may be neglected, since they lead to higher powers of the density $n=N/\Omega$. In that case one is justified in replacing the operators a_0 and a_0^* by the c-number $N^{\frac{1}{2}}$, a procedure well known from the work of Bogoliubov.[2] Then also the theorem on the disconnected diagrams and hence the Goldstone formula are valid. But for cases where n is not small, the depletion of the ground state spoils the validity of both.

3. TREATMENT OF THE DEPLETION EFFECT

Two essentially different ways are open to resolve the difficulties related to the depletion of the zero-momentum state discussed in the preceding section. One possibility is to carry through all necessary calculations, treating the operators a_0^* and a_0 exactly, until a stage has been reached in which only Ω-independent expressions appear. In such expressions one is justified in neglecting terms of order Ω^{-1}, a procedure which leads to great simplifications. This is the basis for the Green's function approach of Beliaev.[4] Recently Sawada[6] has handled the problem along these lines, with the aid of ordinary time-independent perturbation theory. In both

[10] See, for instance, N. M. Hugenholtz, reference 1.

methods one runs into considerable complications in the derivation of Ω-independent expressions.

We propose another and simpler method of dealing with the depletion of the zero-momentum state; we use a Lagrangian multiplier technique to eliminate this state at the outset. Our method amounts to a generalization of the original argument of Bogoliubov[2] concerning the role played by a_0 and a_0^*, and forms the natural extension of his argument to finite densities.

We remark that, no matter what the density is, the commutator of the operators $a_0^*\Omega^{-\frac{1}{2}}$ and $a_0\Omega^{-\frac{1}{2}}$ is equal to Ω^{-1}, and therefore vanishes for an infinitely large system. Furthermore, these operators commute with all other operators in the problem, so that in this limit they can be considered as c-numbers. The operator $a_0^*a_0/\Omega$, the density n_0 of particles of zero momentum, is then likewise a c-number. Thus it seems natural to replace the operators $a_0\Omega^{-\frac{1}{2}}$ and $a_0^*\Omega^{-\frac{1}{2}}$ by a c-number $n_0^{\frac{1}{2}}$. The variable n_0 is to be determined by the properties of the interacting system, in a way which will be discussed below. It will turn out that for low densities, n_0 is approximately equal to n, the particle density, so that the foregoing procedure then reduces to that of Bogoliubov.

Let us see what happens when one replaces $a_0\Omega^{-\frac{1}{2}}$ and $a_0^*\Omega^{-\frac{1}{2}}$ by $n_0^{\frac{1}{2}}$. The zero-momentum state then simply disappears from the problem. The new Hamiltonian is

$$H(n_0)=H_0+V(n_0). \tag{3.1}$$

Now the number of particles is no longer conserved, since $V(n_0)$ contains terms which do not commute with the operator $N'=\Sigma' a_k^* a_k$. However N' is still approximately a good quantum number, in that it commutes with H to order Ω^{-1}. Thus

$$N'H=HN'[1+O(\Omega^{-1})].$$

Our original problem was to determine the ground state of the system of N interacting Bose particles, that is, that eigenstate of the total Hamiltonian H which has the lowest eigenvalue E_0, subject to the condition that the number of particles is equal to N. In the modified problem, in which the momentum-state zero has been eliminated, we must therefore impose the subsidiary condition that

$$\langle N'\rangle = N-n_0\Omega. \tag{3.2}$$

The variable n_0 must be determined in such a way that the energy we find is minimal.

In a theory in which N' commutes rigorously with H, the subsidiary condition (3.2) could be satisfied most easily by imposing it on the unperturbed wave functions. It would then be automatically satisfied for the true wave function, since N' would then also commute with the operator e^{-iHt}, which enters when one wants to describe the transition from the unperturbed to the perturbed wave function. In the present case, N' commutes with H only to order Ω^{-1}; since e^{-iHt} has matrix elements which contain arbitrarily high powers of the volume Ω (due to disconnected diagrams) a correction term of order Ω^{-1} cannot be neglected. Hence we cannot satisfy the subsidiary condition by imposing it on the unperturbed states and we must turn to another method for satisfying (3.2). To that purpose we use the method of the undetermined multiplier.

We first remark that the ground state of the Hamiltonian (3.1) with the subsidiary condition (3.2) is also the ground state of the Hamiltonian

$$H'=H(n_0)-\mu N', \tag{3.3}$$

without any subsidiary condition. Clearly now the ground-state wave function $|\psi_0(n_0,\mu)\rangle$, and thus also the expectation values $E_0'(n_0,\mu)$, $E_0(n_0,\mu)$ and $N'(n_0,\mu)$ of H', H, and N', respectively, depend on the parameter μ, which is determined by the condition

$$n'(n_0,\mu)=n-n_0. \tag{3.4}$$

This relation expresses, for fixed n, the parameter μ in terms of n_0. As said before n_0 is determined by the condition that, again for n fixed,

$$\frac{d}{dn_0}\left(\frac{E_0}{\Omega}\right)=0. \tag{3.5}$$

Using (3.4) and (3.5) we may derive two useful equations for μ. From the observation that $|\psi_0(n_0,\mu)\rangle$ is the ground-state wave function of H' in (3.3) we conclude that the expectation value of H' for the wave function $|\psi_0(n_0',\mu')\rangle$ has a minimum for $n_0'=n_0$ and $\mu'=\mu$, and hence

$$\frac{\partial}{\partial n_0}\left(\frac{E_0'}{\Omega}\right)=\left\langle\psi_0\left|\frac{d}{dn_0}\left(\frac{V}{\Omega}\right)\right|\psi_0\right\rangle, \tag{3.6}$$

and

$$\frac{\partial}{\partial\mu}\left(\frac{E_0}{\Omega}\right)-\mu\frac{\partial n'}{\partial\mu}=0. \tag{3.7}$$

Keeping n fixed and using (3.4) and (3.7), one finds easily

$$\frac{d}{dn_0}\left(\frac{E_0}{\Omega}\right)=\frac{\partial}{\partial n_0}\left(\frac{E_0'}{\Omega}\right)-\mu,$$

so that

$$\mu=\frac{\partial}{\partial n_0}\left(\frac{E_0'}{\Omega}\right). \tag{3.8}$$

The second equation we obtain by noticing that E_0/Ω is a function of n, since μ, which is a solution of (3.4), is a function of n_0 and n. By virtue of (3.5) we have

$$\frac{d}{dn}\left(\frac{E_0}{\Omega}\right)=\frac{\partial}{\partial\mu}\left(\frac{E_0}{\Omega}\right)\cdot\left(\frac{\partial\mu}{\partial n}\right)_{n_0}=-\frac{\partial}{\partial\mu}\left(\frac{E_0}{\Omega}\right)\Big/\frac{\partial n'}{\partial\mu},$$

which with (3.7) reduces to

$$\frac{d}{dn}\left(\frac{E_0}{\Omega}\right)=\mu. \tag{3.9}$$

We thus arrive at the following procedure. Instead of the Hamiltonian $H(n_0)$ we consider H', given by Eq. (3.3), in which the kinetic energies of the particles $k^2/2$ are replaced by $k^2/2-\mu$, with $\mu=dE_0/dN$ playing the role of a potential. We then calculate the ground-state energy $E_0'(n_0,\mu)$ of H', the parameter n_0 being determined by (3.4).

Our original problem of calculating the ground-state energy of a system of N interacting boson is thereby reduced to the mathematically simpler problem of finding the smallest eigenvalue of the Hamiltonian $H'=H_0'+V(n_0)$, in which

$$H_0'=\int_{\mathrm{k}} \xi_{\mathrm{k}}^*\xi_{\mathrm{k}}(\tfrac{1}{2}k^2-\mu),$$

and $V(n_0)$ is a sum of terms, which are obtained from (2.3) by replacing the operators $a_0\Omega^{-\frac{1}{2}}$ and $a_0^*\Omega^{-\frac{1}{2}}$ by $n_0^{\frac{1}{2}}$.

It should be emphasized that here all difficulties connected with the zero-momentum state are absent, since this state has been eliminated. Hence, in this respect, we are now free to use the same methods, which are applied successfully to a gas of fermions, or to field theory. In particular, one can now use the linked cluster expansion of Goldstone[1]:

$$E_0'=\left\langle 0\left|\left[V-V\frac{1}{H_0'}V+V\frac{1}{H_0'}V\frac{1}{H_0'}V-\cdots\right]_C\right|0\right\rangle. \quad (3.10)$$

Here $|0\rangle$ is the vacuum state, which now replaces the state $|\phi_0\rangle$ with N particles of zero momentum in the original problem. The subscript C means that only connected diagrams contribute to the expression.

We will end this section by making a comment on the number of zero-momentum particles in the ground state of the interacting system. It is very easy to show that, as long as perturbation methods are applicable, a finite fraction of particles always occupies the zero-momentum state. Suppose namely that the interaction be such that n_0, the density of particles of zero momentum, is equal to zero. In that case the only nonvanishing term in the interaction (2.3) is V_a, which describes the interaction of two particles in excited states. With such an interaction as a perturbation, it is clearly impossible to get the perturbed ground state, if one starts from an unperturbed state in which all particles have zero momentum.

4. THE GREEN'S FUNCTIONS

There is a further complication in the boson problem which arises from the fact that the interaction allows for the creation of pairs of equal and opposite momenta. As a result one has terms in which two or more intermediate states consist of pairs of particles of the same energy, as shown in Fig. 2. This leads to divergent integrals for low momenta.

One way of handling this problem is to transform to a new set of variables, as proposed by Bogoliubov[2]:

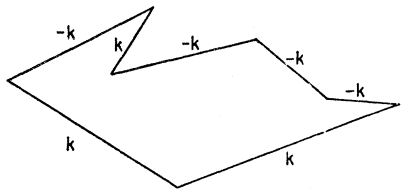

FIG. 2. A ground-state diagram leading to an integral which is highly divergent for small k.

$$a_{\mathrm{k}}=u_k'b_{\mathrm{k}}-v_k'b_{-\mathrm{k}}^*;\quad a_{\mathrm{k}}^*=u_k'b_{\mathrm{k}}^*-v_k'b_{-\mathrm{k}}, \quad (4.1)$$

where u_k' and v_k' are real, and satisfy $u_k'^2-v_k'^2=1$. One can then determine the coefficients of the transformation by imposing the principle of the compensation of dangerous diagrams,[11] and work with the linked cluster expansion (3.10) in terms of the new variables b_{k} and b_{k}^*.

We shall here follow another method, due to Beliaev,[4] which is closely related to the Green's function methods of quantum field theory. The relationship between the Green's function method and the Bogoliubov transformation will be discussed briefly in Sec. 7. In the Green's function method the divergences are removed by making partial summations over classes of diagrams. Our presentation differs from Beliaev's mainly through the fact that in our work the zero-momentum state has already been removed. The results turn out to be completely equivalent to those obtained by Beliaev.

We define a one-particle Green's function by[12]

$$G(\mathbf{x}_1-\mathbf{x}_2,\, t_1-t_2)=-i\langle\psi_0|\,T\psi(\mathbf{x}_1t_1)\psi^*(\mathbf{x}_2t_2)\,|\psi_0\rangle, \quad (4.2)$$

in which $|\psi_0\rangle$ is the ground-state wave function of the interacting system and the $\psi(\mathbf{x}t)$ are field operators in Heisenberg representation. They are related to the creation and annihilation operators ξ_{k}^* and ξ_{k} by

$$\psi(\mathbf{x}t)=(2\pi)^{-\frac{3}{2}}\int_{\mathrm{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\xi_{\mathrm{k}}(t).$$

Defining the Fourier transform of $G(\mathbf{x},t)$ by

$$G(\mathbf{x},t)=(2\pi)^{-3}\int d^3p\; e^{i\mathbf{p}\cdot\mathbf{x}}G(\mathbf{p},t),$$

one finds

$$G(\mathbf{p},\,t-t')\delta^3(\mathbf{p}-\mathbf{p}')=-i\langle\psi_0|\,T\xi_{\mathrm{p}}(t)\xi_{\mathrm{p}'}^*(t')\,|\psi_0\rangle. \quad (4.3)$$

For an extensive discussion of the use of Green's functions in the theory of many-particle systems we refer to the work of Beliaev and also to Migdal and Galitskii,[1] who studied the fermion problem in this way.

The one-particle Green's functions, as defined above, are appropriate tools to describe single-particle excitations. In particular, the analytical behavior near the

[11] N. N. Bogoliubov, Nuovo cimento **7**, 794 (1958).

[12] The sign agrees with that of reference 4. In field theory the opposite sign is customary.

real ϵ-axis of the function $G(\mathbf{p},\epsilon)$, defined by

$$G(\mathbf{p},\epsilon)=\int_{-\infty}^{+\infty}G(\mathbf{p},t)e^{i\epsilon t}dt, \tag{4.4}$$

tells us the energy and the life-time of a single-particle excitation of momentum $\mathbf{p}$.[13]

It is also possible to derive a formula expressing the total energy of the system in terms of the one-particle Green's function. One proceeds as follows. Using (4.3) for $t<t'$ and taking the derivative with respect to t, one has

$$\begin{aligned}&\frac{d}{dt}G(\mathbf{p},t-t')\delta^3(\mathbf{p}-\mathbf{p}')\\&=\langle\psi_0|\,\xi_{\mathbf{p}'}{}^*(t')[H'(t),\xi_{\mathbf{p}}(t)]|\psi_0\rangle\\&=-i(\tfrac{1}{2}p^2-\mu)G(\mathbf{p},t-t')\delta^3(\mathbf{p}-\mathbf{p}')\\&\qquad+\langle\psi_0|\,\xi_{\mathbf{p}'}{}^*(t')[V(t),\xi_{\mathbf{p}}(t)]|\psi_0\rangle.\end{aligned}$$

Here $H'(t)$ and $V(t)$ are the total and interaction Hamiltonian in Heisenberg representation, and one should bear in mind the fact that the unperturbed energies are shifted from $\frac{1}{2}p^2$ to $\frac{1}{2}p^2-\mu$. Using (4.4), taking the limit $t\to t'$ (always keeping $t<t'$), and summing both over $\mathbf{p}$ and $\mathbf{p}'$, one gets

$$\begin{aligned}&\Omega^{-1}\int d^3p\langle\psi_0|\,\xi_{\mathbf{p}}{}^*[V,\xi_{\mathbf{p}}]|\psi_0\rangle\\&\qquad=\frac{-i}{(2\pi)^4}\int d^3p\int_C d\epsilon(\epsilon-\tfrac{1}{2}p^2+\mu)G(\mathbf{p},\epsilon).\end{aligned}\tag{4.5}$$

The path of integration C is a contour consisting of the real axis from $-\infty$ to $+\infty$, together with a semicircle in the upper half plane. Making use of (2.3) for V, in which $a_0\Omega^{-\frac{1}{2}}$ and $a_0{}^*\Omega^{-\frac{1}{2}}$ must be replaced by $n_0{}^{\frac{1}{2}}$, one easily finds that the left-hand side of Eq. (4.5) can be written

$$\Omega^{-1}\left[-2\langle\psi_0|\,V\,|\psi_0\rangle+n_0\left\langle\psi_0\left|\frac{d}{dn_0}V\right|\psi_0\right\rangle\right]. \tag{4.6}$$

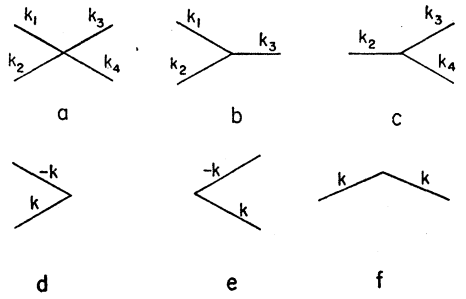

FIG. 3. The vertices a, b, $\cdots$, f correspond to the terms V_a, V_b, $\cdots$, V_f of Eq. (2.3).

[13] The function $G(p,\epsilon)$ plays a role very similar to that of the function $D_p(z)$ of Van Hove and Hugenholtz. See, e.g., Physica **24**, 363 (1958).

The ground-state expectation value of the kinetic energy H_0 is

$$\langle\psi_0|\,H_0\,|\psi_0\rangle=\Omega(2\pi)^{-3}\int d^3p\,\tfrac{1}{2}p^2\langle\psi_0|\,N_{\mathbf{p}}\,|\psi_0\rangle. \tag{4.7}$$

The expectation value $\langle\psi_0|\,N_{\mathbf{p}}\,|\psi_0\rangle$ for the number of particles of momentum $\mathbf{p}$ can be expressed in terms of the one-particle Green's function

$$\langle\psi_0|\,N_{\mathbf{p}}\,|\psi_0\rangle=iG(\mathbf{p},-0)=\frac{i}{2\pi}\int_C d\epsilon\,G(\mathbf{p},\epsilon). \tag{4.8}$$

Equations (4.5), (4.6), (4.7), and (4.8), together with (3.6), and (3.8) lead to

$$\frac{E_0}{\Omega}-\tfrac{1}{2}n_0\mu=\frac{i}{(2\pi)^4}\int d^3p\int_C d\epsilon\,\tfrac{1}{2}(\epsilon+\tfrac{1}{2}p^2+\mu)G(\mathbf{p},\epsilon),$$

or with

$$n'=i(2\pi)^{-4}\int d^3p\int_C d\epsilon\,G(\mathbf{p},\epsilon), \tag{4.9}$$

$$\frac{E_0}{\Omega}-\tfrac{1}{2}n\mu=\frac{i}{(2\pi)^4}\int d^3p\int_C d\epsilon\,\tfrac{1}{2}(\epsilon+\tfrac{1}{2}p^2)G(\mathbf{p},\epsilon). \tag{4.10}$$

This is actually a rather complicated differential equation in E_0/Ω as a function of n, since $\mu=(d/dn)(E_0/\Omega)$ appears not only at the left-hand side, but also in $G(\mathbf{p},\epsilon)$. In the low-density case the situation is, in fact, much less complicated. If one uses Eq. (4.10) to calculate E_0/Ω to a certain accuracy, one can use for μ on the right-hand side an expression of lower order. One therefore finds a simple linear first order differential equation for E_0/Ω which leaves the term $\sim n^2$ undetermined. This is however the first term in the expansion for E_0/Ω, which can easily be calculated by other methods; for example, by the formula (6.2) for μ derived in Sec. 6.

Having seen that both single particle properties and the ground-state energy can be derived from the one-particle Green's function, our task is to derive a useful expression for $G(\mathbf{p},\epsilon)$. Here we use the method of Beliaev.[4] Since we will use the same ideas in Sec. 8 to obtain a new result, we will have to repeat some of his arguments.

Using the interaction representation we have

$$G(\mathbf{p},t-t')\delta^3(\mathbf{p}-\mathbf{p}')=-i\langle 0|\,T\xi_{\mathbf{p}}(t)\xi_{\mathbf{p}'}{}^*(t')S\,|0\rangle, \tag{4.11}$$

where $|0\rangle$ is the vacuum state (no particles of momentum $|\mathbf{k}|\neq 0$) and $\xi_{\mathbf{p}}(t)=e^{iH_0t}\xi_{\mathbf{p}}e^{-iH_0t}$. The S-matrix can be expanded in powers of the interaction V:

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1\cdots\int_{-\infty}^{+\infty}dt_n\,TV(t_1)\cdots V(t_n).$$

If one substitutes this expression in (4.11), one obtains an infinite series, in which each term contains a vacuum expectation value of a time-ordered product of ξ and ξ^*

operators. Using the method of Wick, one can write such a vacuum expectation value as a sum of terms, which are found by forming pairs of creation and annihilation operators in all possible ways and taking the product of the vacuum expectation values of the time-ordered product of each such pair. From the definition (4.3) of the Green's function we see that each such pair carries a factor

$$\langle 0|T\xi_{\mathbf{k}}(t)\xi_{\mathbf{k}'}^*(t')|0\rangle = iG_0(\mathbf{k}, t-t')\delta^3(\mathbf{k}-\mathbf{k}'),$$

G_0 being the Green's function of the noninteracting system.

As usual in field theory, we represent the various ways in which such pairs of creation and annihilation operators can be formed by Feynman diagrams. In that way one obtains a one-to-one correspondence between diagrams and terms of our Green's function. It is now a simple matter to find the following rules for calculating the contribution of each diagram to $G(\mathbf{p},\epsilon)$. (a) For each line, either internal or external, of momentum $\mathbf{k}$ and energy ϵ one has a factor $G_0(\mathbf{k},\epsilon)\equiv(\epsilon-\frac{1}{2}k^2+\mu+i\delta)^{-1}$. (b) For each vertex one has a factor $\delta^3(\sum_i \mathbf{k}_i)\,\delta(\sum_i \epsilon_i)$ for conservation of momentum and energy. (c) For each vertex of type a, b, c, d, e, f of Fig. 3, which correspond to the terms V_a, V_b, V_c, V_d, V_e, and V_f of Eq. (2.3) a factor $v(\mathbf{k}_1-\mathbf{k}_3)+v(\mathbf{k}_1-\mathbf{k}_4)$, $v(\mathbf{k}_1)+v(\mathbf{k}_2)$, $v(\mathbf{k}_3)+v(\mathbf{k}_4)$, $v(\mathbf{k})$, $v(\mathbf{k})$, $v(\mathbf{k})+v(0)$, respectively. (d) For each pair of equivalent lines (i.e., two lines connecting the same pair of vertices) a factor $\frac{1}{2}$. (e) For each incomplete vertex (i.e., a vertex with one or two missing lines) a factor $n_0^{\frac{1}{2}}$ or n_0 as the case may be. (f) A numerical factor $1/r(2\pi)^{-4m+4s}\cdot i^{m-s}$, where m is the order of the diagram and s is the number of factors n_0, due to the incomplete vertices; r is the number of ways in which the vertices can be permuted without changing the diagram. In addition to $G(\mathbf{p},\epsilon)$ it is advantageous to introduce two similar functions

$$\tilde{G}(\mathbf{p}, t-t')\delta^3(\mathbf{p}+\mathbf{p}') = -i\langle\psi_0|T\xi_{\mathbf{p}}(t)\xi_{\mathbf{p}'}(t')|\psi_0\rangle,$$

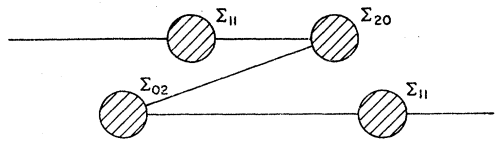

FIG. 4. The general form of the diagrams contributing to $G(p,\epsilon)$.

and

$$\bar{G}(\mathbf{p}, t-t')\delta^3(\mathbf{p}+\mathbf{p}') = -i\langle\psi_0|T\xi_{\mathbf{p}}^*(t)\xi_{\mathbf{p}'}^*(t')|\psi_0\rangle,$$

which are represented by diagrams with two outgoing lines (i.e., external lines running to the left) and two ingoing lines (i.e., lines running to the right), respectively. These functions obviously have no counterpart in the unperturbed system.

The general structure of the diagrams contributing to $G(\mathbf{p},\epsilon)$ is shown in Fig. 4. It simply forms a chain consisting of three types of proper parts, connected by a single line. These proper parts will be called $\Sigma_{11}(\mathbf{p},\epsilon)$, $\Sigma_{02}(\mathbf{p},\epsilon)$, or $\Sigma_{20}(\mathbf{p},\epsilon)$ depending on whether they have one ingoing and one outgoing line, two ingoing lines or two outgoing lines, respectively. One sees easily that Σ_{02} and Σ_{20} are equal. Clearly $\tilde{G}$ and $\bar{G}$ have diagrams of the same general structure as G.

It is now very easy to express $G(p,\epsilon)$, $\tilde{G}(p,\epsilon)$, and $\bar{G}(p,\epsilon)$ in terms of these three quantities Σ_{11}, Σ_{02}, and Σ_{20}. One can immediately write down the equations

$$G(p,\epsilon) = G_0(p,\epsilon)+G(p,\epsilon)\Sigma_{11}(p,\epsilon)G_0(p,\epsilon) + \tilde{G}(p,\epsilon)\Sigma_{20}(p,\epsilon)G_0(p,\epsilon),$$

$$\tilde{G}(p,\epsilon) = \tilde{G}(p,\epsilon)\Sigma_{11}(p,-\epsilon)G_0(p,-\epsilon) + G(p,\epsilon)\Sigma_{02}(p,\epsilon)G_0(p,-\epsilon),$$

$$\bar{G}(p,\epsilon) = \bar{G}(p,\epsilon)\Sigma_{11}(p,-\epsilon)G_0(p,-\epsilon) + G(p,\epsilon)\Sigma_{02}(p,\epsilon)G_0(p,-\epsilon).$$

These equations are represented graphically by Fig. 5, where the thick lines are exact Green's functions and the thin lines the unperturbed Green's functions. These three algebraic equations can be solved and one finds the expressions

$$G(p,\epsilon) = \frac{\epsilon+\frac{1}{2}p^2-\mu+\Sigma_{11}^-}{[\epsilon-\frac{1}{2}(\Sigma_{11}^+-\Sigma_{11}^-)]^2-[\frac{1}{2}p^2-\mu+\frac{1}{2}(\Sigma_{11}^++\Sigma_{11}^-)]^2+\Sigma_{02}^2}, \tag{4.12}$$

$$\bar{G}(p,\epsilon) = \tilde{G}(p,\epsilon) = \frac{-\Sigma_{02}}{[\epsilon-\frac{1}{2}(\Sigma_{11}^+-\Sigma_{11}^-)]^2-[\frac{1}{2}p^2-\mu+\frac{1}{2}(\Sigma_{11}^++\Sigma_{11}^-)]^2+\Sigma_{02}^2}, \tag{4.13}$$

as derived by Beliaev. Here $\Sigma_{11}^+=\Sigma_{11}(p,\epsilon)$ and $\Sigma_{11}^-=\Sigma_{11}(p,-\epsilon)$. By calculating these explicit expressions of the Green's functions in terms of Σ_{11} and Σ_{02}, one has performed the partial summation necessary to remove the divergences from the theory.

5. THE CORRELATION FUNCTION

One is also interested in more general Green's functions than the one-particle Green's functions defined in the preceding sections, namely those which characterize the interaction of the many-boson system with external fields. For instance, the only way in which the elementary excitation spectrum of liquid helium may be directly measured is through the inelastic scattering of slow neutrons. As has been shown by Van Hove,[14] the probability per unit time that a slow neutron give up energy ω and momentum $\mathbf{k}$ to a boson gas in its ground

[14] L. Van Hove, Phys. Rev. **95**, 249 (1954).

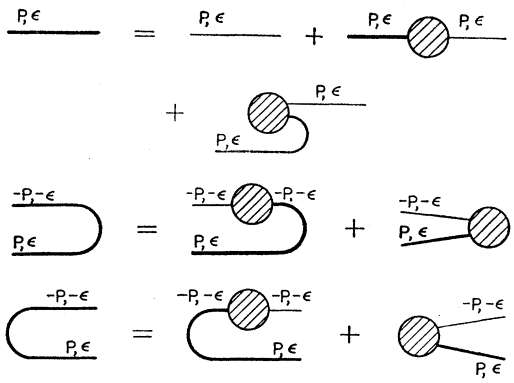

FIG. 5. The graphical representation of the algebraic relations between $G(p,\epsilon)$, $\tilde{G}(p,\epsilon)$ and $\bar{G}(p,\epsilon)$.

state may be written in the Born approximation as

$$w(\mathbf{k},\omega)=AS(\mathbf{k},\omega). \tag{5.1}$$

A is a constant which characterizes the neutron-boson interaction, and $S(k,\omega)$ characterizes the elementary excitation spectrum of the boson system according to

$$S(\mathbf{k},\omega)=\sum_n(\rho_\mathbf{k})_{n0}{}^2\delta(\omega-\omega_{n0}). \tag{5.2}$$

Here $\rho_\mathbf{k}$ is the density fluctuation of momentum $\mathbf{k}$,

$$\rho_\mathbf{k}=\int n(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x=\sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i},$$

which may easily be written in second quantization as $\rho_\mathbf{k}=\sum_\mathbf{q} a_\mathbf{q}{}^* a_{\mathbf{q}+\mathbf{k}}$. $(\rho_\mathbf{k})_{n0}$ denotes the matrix element between the exact wave functions $|\psi_0\rangle$ of the ground state and $|\psi_n\rangle$ of the excited state, which correspond to the exact eigenvalues E_0 and E_n, so that $\omega_{n0}=E_n-E_0$ is the exact energy of the excitation produced by the neutron.

$S(\mathbf{k},\omega)$ is the Fourier-transform in space and time of the pair distribution function. It is simply related to the following function:

$$iF_\mathbf{k}(t-t')=\langle 0|T\{\rho_\mathbf{k}(t)\rho_{-\mathbf{k}}(t')\}|0\rangle. \tag{5.3}$$

This relationship can be seen if we define the Fourier transform of $F_k(t-t')$ by

$$F_\mathbf{k}(t-t')=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, F(\mathbf{k},\omega)e^{-i\omega(t-t')}. \tag{5.4}$$

It then follows that

$$F(\mathbf{k},\omega)=\sum_n(\rho_\mathbf{k})_{n0}{}^2\left\{\frac{1}{\omega-\omega_{n0}+i\delta}-\frac{1}{\omega+\omega_{n0}-i\delta}\right\}, \tag{5.5}$$

from which one immediately finds

$$\mathrm{Im}F(\mathbf{k},\omega)=\pi S(\mathbf{k},\omega). \tag{5.6}$$

We further remark that the structure factor, $S(\mathbf{k})$, which is defined by

$$S(\mathbf{k})=\langle 0|\rho_\mathbf{k}{}^*\rho_\mathbf{k}|0\rangle/n, \tag{5.7}$$

is given then by

$$S(\mathbf{k})=\frac{1}{n}\int_C d\omega\,\frac{i}{2\pi}F(\mathbf{k},\omega), \tag{5.8}$$

where the contour may be closed either above or below the real axis (since $\rho_\mathbf{k}$ and $\rho_{-\mathbf{k}}$ commute).

If we now eliminate the condensed state operators, a_0 and $a_0{}^*$, according to the prescription of the preceding sections, we see that there are three distinct contributions to $iF_\mathbf{k}(t-t')$, corresponding to diagrams with two, three, and four external lines, respectively. Thus we may write

$$iF_\mathbf{k}(t-t')=i\{F_\mathbf{k}{}^a(t-t')+F_\mathbf{k}{}^b(t-t')+F_\mathbf{k}{}^c(t-t')\}, \tag{5.9}$$

where

$$F_\mathbf{k}{}^a(t-t')=n_0\langle\psi_0|T\{[a_{-\mathbf{k}}{}^*(t)+a_\mathbf{k}(t)]\times[a_\mathbf{k}{}^*(t')+a_\mathbf{k}(t')]\}|\psi_0\rangle, \tag{5.10a}$$

$$F_\mathbf{k}{}^b(t-t')=2(n_0)^{\frac{1}{2}}\langle\psi_0|T\{[a_{-\mathbf{k}}{}^*(t)+a_\mathbf{k}(t)]\times[\textstyle\sum_\mathbf{q} a_{\mathbf{q}+\mathbf{k}}{}^*(t')a_\mathbf{q}(t')]\}|\psi_0\rangle, \tag{5.10b}$$

$$F_\mathbf{k}{}^c(t-t')=\langle\psi_0|T\{[\textstyle\sum_\mathbf{q} a_{\mathbf{q}-\mathbf{k}}{}^*(t)a_\mathbf{q}(t)]\times[\sum_{\mathbf{q}'} a_{\mathbf{q}'+\mathbf{k}}{}^*(t')a_{\mathbf{q}'}(t')]\}|\psi_0\rangle. \tag{5.10c}$$

$F_\mathbf{k}{}^a(t-t')$ may be expressed in terms of G and $\bar{G}$ as

$$F_\mathbf{k}{}^a(t-t')=n_0\{G_\mathbf{k}(t-t')+G_\mathbf{k}(t'-t)+\bar{G}_\mathbf{k}(t-t')+\tilde{G}_\mathbf{k}(t'-t)\}. \tag{5.11}$$

With the aid of (4.12) and (4.13), we may write its Fourier transform, $F^a(k,\omega)$, as

$$F^a(k,\omega)=n_0\frac{k^2+\Sigma_{11}{}^+ +\Sigma_{11}{}^- -2\mu-2\Sigma_{02}}{[\epsilon-\frac{1}{2}(\Sigma_{11}{}^+-\Sigma_{11}{}^-)]^2-[\frac{1}{2}p^2-\mu+\frac{1}{2}(\Sigma_{11}{}^++\Sigma_{11}{}^-)]^2+\Sigma_{02}{}^2}. \tag{5.12}$$

6. THEOREM ON THE PHONON SPECTRUM

The expression for $G(p,\epsilon)$, derived in Sec. 4 can be used to calculate the energy $E(k)$ of a single particle excitation as a function of its momentum $\mathbf{k}$. In the low-density approximation it appears that for small momenta $E(k)$ is proportional to k, so that in particular $E(0)=0$.

It is the purpose of this section to show generally, i.e., to all orders in the interaction, that the phonon energy is equal to zero for zero momentum. The proof is based on a simple relationship we shall establish between the chemical potential μ and the functions $\Sigma_{11}(0,0)$ and $\Sigma_{02}(0,0)$ for $\mathbf{p}$ and ϵ equal to zero.

In order to derive this relationship we shall start with

the well-known expansion

$$U(t-t')\equiv e^{-iH(t-t')}=e^{-iH_0t}\sum_{n=0}^{\infty}(-i)^n\int_{t'}^{t}dt_1 \times\int_{t'}^{t_1}dt_2\cdots\int_{t'}^{t_{n-1}}dt_n\,V(t_1)\cdots V(t_n)e^{iH_0t'},$$

where, as before, $V(t)$ is the interaction in interaction representation. Introducing the time-ordering operator T, one can easily write

$$U(t-t')=e^{-iH_0t}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!} \times\int_{t'}^{t}dt_1\cdots\int_{t'}^{t}dt_n\,TV(t_1)\cdots V(t_n)e^{iH_0t'}. \quad (6.1)$$

Since the asymptotic behavior of $\langle 0|U(t)|0\rangle$ for large t is of the form $N_0\exp(-iE_0t)$,[15] one can use the diagonal element $\langle 0|U(t)|0\rangle$ to derive a convenient expression for the total energy E_0 of the system in its ground state. Using diagrams to represent the various terms of the expansion (6.1), one finds that $\langle 0|U(t)|0\rangle$ can be expressed in terms of connected diagrams only, by the formula[15]

$$\langle 0|U(t)|0\rangle=\exp(\langle 0|\bar{U}(t)|0\rangle),$$

where $\langle 0|\bar{U}(t)|0\rangle$ is defined by

$$\langle 0|\bar{U}(t)|0\rangle=\sum_n\frac{(-i)^n}{n!}\int_{-\frac{1}{2}t}^{\frac{1}{2}t}dt_1\cdots \times\int_{-\frac{1}{2}t}^{\frac{1}{2}t}dt_n\langle 0|[TV(t_1)\cdots V(t_n)]_C|0\rangle.$$

The subscript C means that only connected ground-state diagrams contribute to $\langle 0|\bar{U}(t)|0\rangle$. In exactly the same way as in the case of the Green's function, the time-ordered product can be expressed in terms of normal products. We now study this function in the limit $t\to\infty$. As before, the integrations over t_i lead to factors $\delta(\sum_j\epsilon_j)$ in each vertex, saying that the sum of the ϵ_j's, with appropriate signs, must be zero for each interaction. However, in the case without external lines, one of these relations is identically satisfied as soon as the others are fulfilled. This means that the last integration simply leads to a factor t, expressing the fact that $\langle 0|\bar{U}(t)|0\rangle$ is asymptotically proportional to t. The proportionality factor must clearly be $-iE_0$. It is now very easy to establish that E_0/Ω can be calculated as the sum of the contributions of all connected ground state diagrams. The contribution of each diagram must be calculated according to the rules given in Sec. 4, with the following modifications: (b'). For each vertex except one, there is a factor $\delta^3(\sum_i\mathbf{k}_i)\delta(\sum_i\epsilon_i)$. ($f'$). A numerical factor $1/r(2\pi)^{-4m+4s-4}i^{m-s+1}$.

[15] Cl. Bloch, Nuclear Phys. 7, 451 (1958).

We shall now use this expansion of E_0 to establish the following equation involving the chemical potential μ, Σ_{11}, and Σ_{02}:

$$\mu=\Sigma_{11}(0,0)-\Sigma_{02}(0,0). \quad (6.2)$$

To prove this equation we remark that all diagrams of $\Sigma_{11}(0,0)$ can be obtained in a unique fashion from the connected ground-state diagrams by attaching in all possible ways one ingoing and one outgoing line of momentum and energy zero to one or two incomplete vertices; to obtain $\Sigma_{02}(0,0)$ one attaches two ingoing lines in a similar fashion. Let us consider an arbitrary connected ground-state diagram, which is built up from n_a vertices of type a of Fig. 3, n_b of type b, etc., and denote its value by $\{n_a,n_b,n_c,n_d,n_e,n_f\}$. Obviously $n_b+2n_d=n_c+2n_e$. The number s of factors n_0 is given by

$$s=\tfrac{1}{2}n_b+\tfrac{1}{2}n_c+n_d+n_e+n_f=n_b+2n_d+n_f. \quad (6.3)$$

With the Eq. (3.8) derived in Sec. 3, the value of μ arising from this diagram is found to be

$$\mu^{(d)}=sn_0^{-1}\{n_a,\cdots,n_f\}. \quad (6.4)$$

We now calculate the sum of all terms of $\Sigma_{11}(0,0)$ and $\Sigma_{02}(0,0)$ which can be obtained from this ground-state diagram. The process of attaching two external lines diminishes the number of factors n_0 by one. Hence, in the numerical factor for Σ_{11} or Σ_{02}, as given in Sec. 4 under f, we must replace s by $s-1$, which makes this factor identical to the corresponding one for $n_0^{-1}E_0/\Omega$.

Let us now start adding the two external lines, one by one. An ingoing line can only be added to vertices b, d, and f, transforming them into a, b, and c, respectively. The transitions $b\to a$ and $f\to c$ do not change the value of the diagram, since the vertex-functions $v(\mathbf{k}_1)+v(\mathbf{k}_2)$ and $v(\mathbf{k}_1-\mathbf{p})+v(\mathbf{k}_2-\mathbf{p})$ are equal for $\mathbf{p}=0$, and similarly for $v(\mathbf{k})+v(0)$ and $v(\mathbf{k})+v(\mathbf{p})$. However the transition $d\to b$ leads to a factor of 2.

If we denote by $I\{n_a,n_b,n_c,n_d,n_e,n_f\}$ the sum of terms one gets by adding one ingoing line in all possible ways, one finds

$$I\{n_a,n_b,\cdots,n_f\} =n_b\{n_a+1,\,n_b-1,\,n_c,\,n_d,\,n_e,\,n_f\} +n_d\{n_a,\,n_b+1,\,n_c,\,n_d-1,\,n_e,\,n_f\} +n_f\{n_a,\,n_b,\,n_c+1,\,n_d,\,n_e,\,n_f-1\},$$

where one should remember that the different bracket expressions, representing different diagrams, have all equal values, except for a factor of 2 for each missing vertex of type d or e. Adding one more ingoing line in exactly the same manner, one finds

$$\Sigma_{02}^{(d)}(0,0)=II\{n_a\cdots n_f\}=[n_b(n_b-1+2n_d+n_f) +2n_d(n_b+1+2n_d-2+n_f) +n_f(n_b+2n_d+n_f-1)]n_0^{-1}E_0/\Omega,$$

which, by virtue of (6.3), gives

$$\Sigma_{02}^{(d)}(0,0)=(s^2-s)n_0^{-1}E_0/\Omega. \quad (6.5)$$

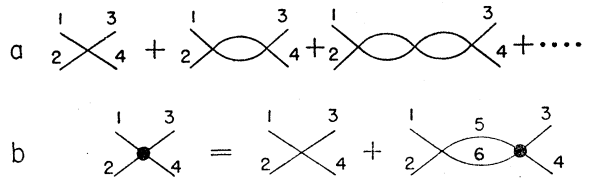

FIG. 6. (a) The multiple scattering terms which for low density are all of equal importance; (b) the graphical representation of Eq. (7.3) for the scattering matrix t.

Similarly

$$\Sigma_{11}^{(d)}(0,0)=OI\{n_a\cdots n_f\}=s^2n_0^{-1}E_0/\Omega, \qquad (6.6)$$

in which O is the operation of adding an outgoing line. Equations (6.4), (6.5), and (6.6) give immediately

$$\mu^{(d)}=\Sigma_{11}^{(d)}(0,0)-\Sigma_{02}^{(d)}(0,0).$$

Since we proved this relation for an arbitrary diagram of order $n>1$, and since for $n=1$ this relation is also satisfied, we have proved Eq. (6.2) generally.

This equation makes it very easy to prove our assertion concerning the phonon spectrum. Indeed, it follows immediately from (4.12) that for μ satisfying (6.2) both poles of $G(p,\epsilon)$ coincide at $\epsilon=0$. Hence there can be no energy gap in the phonon spectrum.

We therefore conclude that for those theories of the boson gas for which an energy gap has appeared in the elementary excitation spectrum, the cause of the apparent gap is to be found in an inconsistent treatment of the vertices Σ_{11} and Σ_{02}, or the depletion effect, in the higher order terms of the perturbation-theoretic expansion.[5]

A few remarks should still be made concerning the conditions under which Eq. (6.2) is valid. We made use of a power series expansion for E_0 which, as can be proved easily, is equivalent to the linked cluster expansion (3.10). We obtained E_0 from the limiting process

$$E_0=i\lim_{t\to\infty}\frac{d}{dt}\langle 0|\,\tilde{U}(t)\,|0\rangle. \qquad (6.7)$$

It is clear that this expansion in powers of V does not converge in the boson case. In fact, many of the terms are infinite, due to the divergence of integrals for small momenta. For actual calculations this expansion is therefore not very useful and we prefer (4.10). However, in our proof we implicitly used a cutoff for small momenta, knowing that the result will not depend on the cutoff, provided the limit in (6.7) exists. A criterion for this existence is not known at present. We believe that, at least for repulsive interactions, this condition is fulfilled.

7. THE LOW-DENSITY LIMIT

In this and the succeeding section we will be concerned with the calculation of the properties of a dilute boson gas at zero temperature. Our goal is to calculate the first few terms in the expansion of these properties in an ascending series in n; for instance, the ground-state energy may be written as

$$E_0=E_0^{(1)}+E_0^{(2)}+E_0^{(3)}+\cdots,$$

where in the low-density limit $E_0^{(3)}\ll E_0^{(2)}\ll E_0^{(1)}$. As discussed in Sec. 4, the calculation of the properties of a given system in the present method begins with the calculation of the effective potentials, Σ_{11} and Σ_{02}. Once these are determined, in a given order, say, the Green's function $G(p,\epsilon)$ is obtained from (4.12). The poles of $G(p,\epsilon)$ then yield the low-lying elementary excitations, while the ground-state energy, E_0, may be obtained by a suitable integration over $\mathbf{p}$ and ϵ, according to (4.10).

It should be emphasized that the Green's function method differs markedly from a conventional perturbation-theoretic approach, in that a first-order determination of the G's (and the system properties deriving therefrom) already corresponds to the summation of an infinite sequence of terms in a perturbation-theory approach. As an example, calculating Σ_{11} and Σ_{02} in first order, one finds

$$\Sigma_{11}=n_0(V_0+V_p), \qquad (7.1)$$

$$\Sigma_{02}=n_0V_p, \qquad (7.2)$$

where Σ_{11} represents the sum of a direct and an exchange term. As Beliaev has remarked, the results (7.1), (7.2) when combined with (4.12) already contain the classic result of Bogoliubov for the excitation spectrum of a dilute gas of weakly interacting bosons.[2] We may further remark that a calculation of the ground-state energy based on (7.1) and (7.2) is formally equivalent to the high-density electron gas calculation of Gell-Mann and Brueckner.[16] It represents a sum of all ground-state diagrams which are topologically equivalent to a continuous line, punctuated by dots to represent the interactions, as illustrated in Fig. 2. The use, then, of (7.1) and (7.2) for the Σ's is likewise equivalent to the random-phase approximation introduced by Bohm and Pines for the electron gas.[17]

The first-order calculation above is not sufficiently accurate to describe the properties of the system in the low-density limit. The reason is that there is, in fact, a whole sequence of contributions to the Σ's which are of equal importance in this limit. These correspond to the repeated scatterings of a given pair of particles.[8,4] Consider the scattering of particles of momentum $\mathbf{p}_1$ and $\mathbf{p}_2$ to $\mathbf{p}_3$ and $\mathbf{p}_4$. Then, as shown in Fig. 6(a), not only is the first-order scattering of importance, but also all the additional multiple scatterings which are indicated there. Thus all the terms of Fig. 6(a) contribute to the Σ's in the same order of n_0. From Fig. 6(b), it is clear that this infinite sequence of terms may be summed with the aid of an integral equation which may be

[16] M. Gell-Mann and K. A. Brueckner, Phys. Rev. **106**, 364 (1957).

[17] D. Bohm and D. Pines, Phys. Rev. **92**, 608 (1953). See also P. Nozières and D. Pines, Nuovo cimento **9**, 470 (1958).

written symbolically as

$$t_{12;34}=v_{12;34}+v_{12;56}G_5^0G_6^0t_{56;34}. \tag{7.3}$$

Henceforth, in all diagrams, we shall assume the vertex to be given by t according to (7.3), so that the point representing a given vertex is in reality a sum over an infinite sequence of diagrams. The class of diagrams which must then be considered is correspondingly considerably reduced.

The solutions of (7.3) may be expressed in terms of the scattering amplitude for two particles in a vacuum. The particular vertices which are of interest to us here have been calculated by Beliaev, who finds

$$t(00\mathbf{p}-\mathbf{p})=f^*(\mathbf{p},0)+\int\frac{d^3q}{(2\pi)^3}f(0,\mathbf{q})f^*(\mathbf{p},\mathbf{q})\times\left\{\frac{1}{2\mu-q^2+i\delta}+\frac{1}{q^2}\right\}, \tag{7.4}$$

$$t(0\mathbf{pp}0)+t(0\mathbf{p}0\mathbf{p})=2f_s(\tfrac{1}{2}\mathbf{p},\tfrac{1}{2}\mathbf{p})+2\int\frac{d^3q}{(2\pi)^3}|f_s(\tfrac{1}{2}\mathbf{p},\mathbf{q})|^2\times\left\{\frac{1}{\epsilon+2\mu-\frac{1}{4}p^2-q^2+i\delta}+\frac{1}{q^2-\frac{1}{4}p^2-i\delta}\right\}, \tag{7.5}$$

where $f_s(\mathbf{p}',\mathbf{p})=\{f(\mathbf{p}',\mathbf{p})+f(-\mathbf{p}',\mathbf{p})\}/2$, $f(\mathbf{p}',\mathbf{p})$ is the scattering amplitude defined by

$$f(\mathbf{p}',\mathbf{p})=\int d^3x\, v(\mathbf{x})e^{-i\mathbf{p}'\cdot\mathbf{x}}\Psi_{\mathbf{p}}(\mathbf{x}), \tag{7.6}$$

and $\Psi_{\mathbf{p}}(\mathbf{x})$ is the eigenfunction for a particle moving in a potential $v(\mathbf{x})$ which behaves at infinity like a plane wave of momentum $\mathbf{p}$ plus an outgoing spherical wave.

It is not necessary to know the complete solutions of (7.4) and (7.5) in order to determine the leading terms in the low-density expansion of the ground-state energy and excitation spectrum. As we shall see these are completely determined by the properties of the Σ's for momentum transfers which are small compared to the inverse of the zero-momentum scattering amplitude, f_0. In this limit, we have

$$f_s(\tfrac{1}{2}\mathbf{p},\tfrac{1}{2}\mathbf{p})\cong f(\mathbf{p},0)\cong f(0,0)=f_0,\quad (p\ll f_0^{-1}). \tag{7.7}$$

Further, the integrals in (7.4) and (7.5) give rise to terms of order $f_0(n_0f_0^3)^{\frac{1}{2}}$. As we shall see, the expansion parameter for the low-density hard sphere gas is just $(n_0f_0^3)^{\frac{1}{2}}$, so that contributions arising from the integrals may properly be regarded as giving rise to second-order corrections to the first-order Σ's formed from the t's. The latter are, therefore,

$$\Sigma_{02}^{(1)}=n_0f_0, \tag{7.8a}$$

$$\Sigma_{11}^{(1)}=2n_0f_0; \tag{7.8b}$$

while the first-order chemical potential is, according to (6.2),

$$\mu^{(1)}=\Sigma_{11}(0)-\Sigma_{02}(0)=n_0f_0. \tag{7.8c}$$

Let us now consider the properties of the system in first and second orders. The first-order ground-state energy $E_0^{(1)}$ is determined from the "zeroth" order Green's function, $(\Sigma_{11}^{(0)}=\Sigma_{02}^{(0)}=0)$,

$$G^0(p,\epsilon)=1/(\epsilon-\tfrac{1}{2}p^2+i\delta),$$

and the first-order chemical potential, $\mu^{(1)}$. According to (4.10) we have

$$E_0^{(1)}/\Omega=\tfrac{1}{2}n\mu^{(1)}=\tfrac{1}{2}n^2f_0, \tag{7.9}$$

since there is no contribution from the integral in (4.10). The energy spectrum of the elementary excitations derived from $G^0(p,\epsilon)$ is of course that of a gas of free particles.

The first-order excitation spectrum and the second-order ground-state energy are obtained from the first-order Green's functions, which are, according to (4.12) and (7.8),

$$G^{(1)}(p,\epsilon)=(\epsilon+\tfrac{1}{2}p^2+n_0f_0)/(\epsilon^2-\omega_p{}^2+i\delta), \tag{7.10}$$

$$\tilde{G}^{(1)}(p,\epsilon)=\bar{G}^{(1)}(p,\epsilon)=-n_0f_0/(\epsilon^2-\omega_p{}^2+i\delta), \tag{7.11}$$

where the poles of G, and hence the energies of the low-lying elementary excitations, are given by

$$\omega_p{}^2=p^2n_0f_0+\tfrac{1}{4}p^4. \tag{7.12}$$

The dispersion relation, (7.12), for the excitation spectrum shows that in the low-momentum region $(p\ll(n_0f_0)^{\frac{1}{2}})$ the elementary excitations behave like sound waves with a constant velocity, $(n_0f_0)^{\frac{1}{2}}$. In the high-momentum region $(p\gg(n_0f_0)^{\frac{1}{2}})$, ω_p may be expanded in powers of n_0f_0;

$$\omega_p\cong\tfrac{1}{2}p^2+n_0f_0-(n_0^2f_0^2/p^2)+\cdots. \tag{7.13}$$

The elementary excitations then correspond to almost free particles moving in an "optical potential," n_0f_0.

It is convenient to re-express the first-order Green's functions in the following form:

$$G^{(1)}(p+\mu)=\frac{u_p{}^2}{\epsilon-\omega_p+i\delta}-\frac{v_p{}^2}{\epsilon+\omega_p-i\delta}, \tag{7.14}$$

$$\tilde{G}^{(1)}(p+\mu)=\tilde{G}^{(1)}(p+\mu)=-u_pv_p\left\{\frac{1}{\epsilon-\omega_p+i\delta}-\frac{1}{\epsilon+\omega_p-i\delta}\right\}, \tag{7.15}$$

where

$$u_p{}^2=(\tfrac{1}{2}p^2+n_0f_0+\omega_p)/2\omega_p, \tag{7.16a}$$

$$v_p{}^2=(\tfrac{1}{2}p^2+n_0f_0-\omega_p)/2\omega_p, \tag{7.16b}$$

$$u_pv_p=n_0f_0/2\omega_p. \tag{7.16c}$$

We remark that the coefficients u_p and v_p may be regarded as coherence factors which, for a given momentum, measure the way in which the interaction be-

tween the particles influences the system properties. The role that these coherence factors play depends in turn on the relative size of the momentum $\mathbf{p}$, and the momentum which characterizes the strength of the interaction, $(n_0f_0)^{\frac{1}{2}}$. Two limiting cases are of interest:

Case 1: $p \ll (2n_0f_0)^{\frac{1}{2}}$,

$$u_p^2 \cong v_p^2 \cong u_p v_p \cong (n_0f_0)^{\frac{1}{2}}/2p,$$

Case 2: $p \gg (2n_0f_0)^{\frac{1}{2}}$,

$$u_p^2 \cong 1; \quad v_p^2 \cong -n_0^2f_0^2/p^2 \ll 1.$$

Thus for small momenta the coherence factors u_p and v_p are large and equal. The resulting Green's function differs markedly from its free-particle value, $G^{(0)}$, and the related properties of the system are determined by the sound-wave type excitations in a way which is not at all accessible to a conventional perturbation-theoretic treatment. On the other hand, for large momenta, the Green's function, $G^{(1)}$, approaches that of a free particle (as do the elementary excitations), $\bar{G}^{(1)}$ is negligible, and the interaction could easily be treated by ordinary perturbation theory, to which, in fact, the present treatment reduces.

The second-order ground-state energy is found from (4.10) and (7.14) to be

$$\frac{E_0^{(2)}}{\Omega} - \tfrac{1}{2}n\mu^{(2)} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int_C \frac{id\epsilon}{2\pi}(\epsilon + \tfrac{1}{2}p^2) \times \left\{\frac{u_p^2}{\epsilon - \omega_p + i\delta} - \frac{v_p^2}{\epsilon + \omega_p - i\delta}\right\}, \quad (7.17)$$

where the contour integral is to be closed in the upper half plane. We find, on carrying out the integration over ϵ,

$$\frac{E_0^{(2)}}{\Omega} - \tfrac{1}{2}n\mu^{(2)} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \times \frac{(\frac{1}{2}p^2 + n_0f_0 - \omega_p)(\frac{1}{2}p^2 - \omega_p)}{2\omega_p}. \quad (7.18)$$

According to (7.13), for large momenta ($p \gtrsim (2n_0f_0)^{\frac{1}{2}}$), the integrand on the right side of (7.18) is of order p^{-4}; the dominant contributions to the integral come from the low-momentum part ($p \lesssim (2n_0f_0)^{\frac{1}{2}}$). The integration is straightforward, and one finds

$$(E_0^{(2)}/\Omega) - \tfrac{1}{2}n\mu^{(2)} = -2(n_0f_0)^{\frac{5}{2}}/15\pi^2. \quad (7.19)$$

We remark that we can now see that the neglect of the dispersion in $f(\mathbf{p}'\mathbf{p})$ is justified in the calculation of the first-order ground-state energy: the contributions to (7.19) come from

$$p \lesssim (n_0f_0)^{\frac{1}{2}} \ll f_0^{-1},$$

since

$$(n_0f_0^3)^{\frac{1}{2}} \ll 1.$$

To complete the calculation, we consider (7.19) as a differential equation in E_0/Ω as a function of n, since $\mu = (d/dn)(E_0/\Omega)$. Putting in this order,

$$E_0^{(2)}/\Omega = \alpha(nf_0)^{\frac{5}{2}}, \quad (7.20)$$

one gets

$$\mu^{(2)} = \tfrac{5}{2}\alpha n^{\frac{3}{2}}f_0^{\frac{5}{2}}. \quad (7.21)$$

If we substitute (7.20) and (7.21) in (7.19) and bear in mind that in this order we may replace n_0 by n in the right-hand side of (7.19), we find $\alpha = 8/15\pi^2$, and thus

$$E_0^{(2)}/\Omega = \frac{8}{15\pi^2}n^2f_0(nf_0^3)^{\frac{1}{2}}, \quad (7.22)$$

in agreement with the results of Beliaev.[4] For the special case of a gas of hard spheres, the result (7.22) is at once seen to yield the result of Lee, Huang, and Yang,[3] since for this potential $f_0 = 4\pi a$, where a is the diameter of the spheres.

It is of interest to calculate the depletion of the ground state in this order. We have from (4.9) and (7.14)

$$n - n_0 = \int \frac{d^3p}{(2\pi)^3}\int_C d\epsilon \frac{i}{2\pi} \times \left\{\frac{u_p^2}{\epsilon - \omega_p + i\delta} - \frac{v_p^2}{\epsilon + \omega_p - i\delta}\right\}, \quad (7.23)$$

and hence

$$n = n_0 + \int \frac{d^3p}{(2\pi)^3}\frac{\frac{1}{2}p^2 + n_0f_0 - \omega_p}{2\omega_p} = n_0\left(1 + \frac{(n_0f_0^3)^{\frac{1}{2}}}{3\pi^2}\right). \quad (7.24)$$

As in (7.18) the main contributions to the integral come again from low momenta ($p \lesssim (2n_0f_0)^{\frac{1}{2}}$), since for large p, the integrand is of order p^{-4}. We see that the depletion of the ground state as a result of the interaction between the particles is of order $n_0^{\frac{3}{2}}f_0^{\frac{3}{2}}$.

We may also calculate the time-dependent correlation function $F(k,\omega)$ and the structure factor $S(k)$ defined in Sec. 5 in this order. It is straightforward to show that in lowest-order only the two-line part of $F(k,\omega)$ is of importance. One finds, on substituting Eqs. (7.8) in (5.12)

$$F^{(1)}(k,\omega) = \frac{nk^2}{(\epsilon - \omega_k + i\delta)(\epsilon + \omega_k - i\delta)}. \quad (7.25)$$

We then have, using (5.8)

$$S^{(1)}(k) = \frac{k^2}{2\omega_k} = \frac{k}{2(n_0f_0 + k^2/4)^{\frac{1}{2}}}. \quad (7.26)$$

We thus find from (7.26) that in this order the phonon excitation spectrum takes the form proposed by Feynman[18]

$$\omega_k = k^2/2S(k).$$

[18] R. P. Feynman, Phys. Rev. **91**, 1291 (1953).

We note that for large $k(k\gg(n_0f_0)^{\frac{1}{2}})$, $S(k)$ approaches unity, which is its free particle value. For small k $[k\ll(n_0f_0)^{\frac{1}{2}}]$ on the other hand, $S(k)$ differs greatly from unity, and in fact varies linearly with k. It is natural, then, to regard $(n_0f_0)^{-\frac{1}{2}}$ as the correlation length in the problem, i.e., the length over which correlations brought about by the particle interactions play an important role.

We see from (7.9), (7.22), and (7.24) that the parameter which characterizes the series expansion of the properties of the dilute boson gas is $(n_0f_0^3)^{\frac{1}{2}}$. That this parameter should enter (rather than, say, $f_0/n_0^{-\frac{1}{3}}$) is a direct consequence of the fact that it is the low momentum transfers which determine the properties of the system. Thus the interaction is weak when the scattering length f_0 is small compared to the correlation length, $(n_0f_0)^{-\frac{1}{2}}$, and it is this ratio which appears as the expansion parameter.

We conclude this section by remarking on the connection between the present method and that of Bogoliubov.[2] In the latter approach one obtains the first-order excitation spectrum and $E_0^{(2)}$ by keeping only the terms V_d, V_e, V_f, and V_g of (2.3) in the Hamiltonian (assuming one has first introduced an effective interaction by the pseudopotential method or by using [7.3]). The resulting Hamiltonian then may be diagonalized by means of a canonical transformation of the form (4.1). The condition that the new Hamiltonian be diagonal is simply

$$u_p'=u_p;\quad v_p'=v_p,$$

where u_p and v_p are defined by (7.16). In this fashion one may obtain a ground-state energy and excitation spectrum in accord with (7.22) and (7.12). It is interesting to note that the coefficient v_p which measures the admixture of the new creation operator in the old annihilation operator is likewise a measure of the strength of the negative frequency pole in our Green's function, (7.14), as might perhaps have been expected.

8. THE NEXT ORDER TERM IN THE GROUND-STATE ENERGY

We now carry out the calculation of the next term in the series expansion for the ground-state energy. To do this, we need to know the second-order effective potentials, $\Sigma_{11}^{(2)}$ and $\Sigma_{02}^{(2)}$. With the aid of these we may determine $G^{(2)}$, and then calculate $E_0^{(3)}$ and $\mu^{(3)}$ in a fashion directly analogous to our calculation of $E_0^{(2)}$ and $\mu^{(2)}$ in the preceding section.

We find it convenient to begin by obtaining an expression for $G^{(2)}$ which differs somewhat from that, one finds on direct application of (4.12). We do this by considering the algebraic equations for G and $\tilde{G}$ which obtain if one uses $G^{(1)}(p,\epsilon)$, $\tilde{G}^{(1)}(p,\epsilon)$, and $\bar{G}^{(1)}(p,\epsilon)$ as the "bare" propagation functions instead of $G^0(p,\epsilon)$; one must likewise introduce new effective potentials, $\Sigma_{11}'=\Sigma_{11}-\Sigma_{11}^{(1)}$, and $\Sigma_{02}'=\Sigma_{02}-\Sigma_{02}^{(1)}$, in place of Σ_{11} and Σ_{02}. The new coupled integral equations may be simply obtained by analysis of the appropriate new diagrams in a way directly analogous to the procedure of Sec. 4, as shown in Fig. 7. The resulting equations are[19]

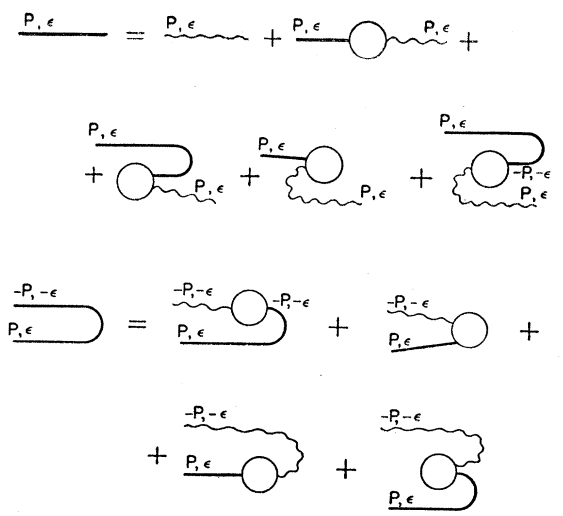

FIG. 7. The graphical representation of the equations (8.1) and (8.2). The wavy lines correspond to the Green's functions $G^{(1)}$, $\tilde{G}^{(1)}$, and $\bar{G}^{(1)}$.

$$\begin{aligned}G(p,\epsilon)=G^{(1)}(p,\epsilon)&+G(p,\epsilon)\Sigma_{11}'(p,\epsilon)G^{(1)}(p,\epsilon)\\&+\tilde{G}(p,\epsilon)\Sigma_{02}'(p,\epsilon)G^{(1)}(p,\epsilon)\\&+G(p,\epsilon)\Sigma_{20}'(p,\epsilon)\bar{G}^{(1)}(p,\epsilon)\\&+\tilde{G}(p,\epsilon)\Sigma_{11}'(p,-\epsilon)\bar{G}^{(1)}(p,\epsilon),\end{aligned}\tag{8.1}$$

and

$$\begin{aligned}\tilde{G}(p,\epsilon)=\tilde{G}(p,\epsilon)&\Sigma_{11}'(p,-\epsilon)G^{(1)}(p,-\epsilon)\\&+G(p,\epsilon)\Sigma_{20}'(p,\epsilon)G^{(1)}(p,-\epsilon)\\&+G(p,\epsilon)\Sigma_{11}'(p,\epsilon)\tilde{G}^{(1)}(p,\epsilon)\\&+\tilde{G}(p,\epsilon)\Sigma_{02}(p,\epsilon)\tilde{G}^{(1)}(p,\epsilon).\end{aligned}\tag{8.2}$$

These equations may also be obtained by suitable algebraic manipulations from (4.12) and (4.13), using (7.10) and (7.11). We now remark that for the calculation of $E_0^{(3)}$ it will suffice to work directly with (8.1), substituting $G^{(1)}$, $\tilde{G}^{(1)}$, and $\bar{G}^{(1)}$ for G, $\tilde{G}$, and $\bar{G}$ on the right-hand side of the equation, and then using the resulting expression in (4.10). Thus we may write $G=G^{(1)}+G^{(2)}$, where

$$\begin{aligned}G^{(2)}(p,\epsilon)=G^{(1)}(p,\epsilon)&\Sigma_{11}^{(2)}(p,\epsilon)G^{(1)}(p,\epsilon)\\&+G^{(1)}(p,\epsilon)\Sigma_{02}^{(2)}(p,\epsilon)\bar{G}^{(1)}(p,\epsilon)\\&+\tilde{G}^{(1)}(p,\epsilon)\Sigma_{11}^{(2)}(p,-\epsilon)\bar{G}^{(1)}(p,\epsilon)\\&+\tilde{G}^{(1)}(p,\epsilon)\Sigma_{02}^{(2)}(p,\epsilon)G^{(1)}(p,\epsilon),\end{aligned}\tag{8.3}$$

and the $\Sigma^{(2)}$'s denote the appropriate expressions of lowest order in n_0 for Σ's.

[19] Actually, the $G^{(1)}(p,\epsilon)$, and $\bar{G}^{(1)}(p,\epsilon)$ of (8.1) and (8.2) differ from the expressions (7.10) and (7.11), in that one should include as well the higher terms in the expansion for μ. To the order of the calculations carried out in this section, these higher terms make no contribution, so we drop them at the outset.

There is a further very great simplification which results because $E_0^{(3)}$ turns out to depend logarithmically on $n_0 f_0^3$. As will presently become obvious, we can obtain the correct coefficient of this term by taking only the leading terms for small n_0 in (8.3). We therefore find

$$G^{(2)}(p,\epsilon)\cong\frac{\Sigma_{11}^{(2)}(p,\epsilon)}{(\epsilon-\frac{1}{2}p^2+i\delta)^2}-\frac{2n_0f_0\Sigma_{02}^{(2)}(p,\epsilon)}{(\epsilon-\frac{1}{2}p^2+i\delta)^2(\epsilon+\frac{1}{2}p^2-i\delta)}+\frac{n_0^2f_0^2\Sigma_{11}^{(2)}(p,-\epsilon)}{(\epsilon-\frac{1}{2}p^2+i\delta)^2(\epsilon+\frac{1}{2}p^2-i\delta)^2}. \quad (8.4)$$

The corresponding expression for $E_0^{(3)}$ is

$$\frac{E_0^{(3)}}{\Omega}-\tfrac{1}{2}n\mu^{(3)}=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int_C d\epsilon\frac{i}{(2\pi)}(\epsilon+\tfrac{1}{2}p^2)G^{(2)}(p,\epsilon),$$

which upon substitution of (8.4) becomes

$$\frac{E_0^{(3)}}{\Omega}-\tfrac{1}{2}n\mu^{(3)}=\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int_C\frac{d\epsilon i}{(2\pi)}\left\{\frac{[\epsilon+\frac{1}{2}p^2]\Sigma_{11}^{(2)}(p,\epsilon)}{(\epsilon-\frac{1}{2}p^2+i\delta)^2}-\frac{2n_0f_0\Sigma_{02}^{(2)}(p,\epsilon)}{(\epsilon-\frac{1}{2}p^2+i\delta)^2}+\frac{n_0^2f_0^2\Sigma_{11}^{(2)}(p,-\epsilon)}{(\epsilon-\frac{1}{2}p^2+i\delta)^2(\epsilon+\frac{1}{2}p^2-i\delta)}\right\}. \quad (8.5)$$

In carrying out the integral over ϵ in (8.5), it is important to bear in mind the fact that any singularities of $G(p,\epsilon)$ must lie slightly above the negative real ϵ axis, or slightly below the positive real ϵ axis, as follows from the definition of G, (4.2). Therefore, because the contour of integration in (8.5) is closed above the real axis, the only contributions to $E_0^{(3)}$ arise from the singularities along the negative real ϵ axis. As a result we need consider only those parts of $\Sigma_{02}^{(2)}(p,\epsilon)$ which have singularities on the negative real ϵ-axis, while for the terms involving $\Sigma_{11}^{(2)}(p,\epsilon)$ and $\Sigma_{11}^{(2)}(p,-\epsilon)$ we shall need to consider the leading term in n_0 for $\Sigma_{11}^{(2)}(p,p^2/2)$, as well as the leading terms which possess singularities above the negative real ϵ axis.

Inspection of the terms of $\Sigma_{02}^{(2)}$ which might then contribute shows that the only diagram of importance (in the limit $n_0\to 0$) is that shown in Fig. 8(a). If we

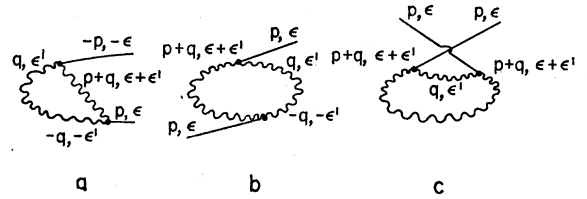

FIG. 8. The relevant diagrams for those terms in $\Sigma_{02}^{(2)}$ and $\Sigma_{11}^{(2)}$, which contribute in Eq. (8.5). The wavy lines correspond to $G^{(1)}$, $\bar{G}^{(1)}$, and $\bar{G}^{(1)}$.

now apply the rules of Sec. 4 and make use of (7.10) and (7.11) to calculate the contribution from this diagram, we find for small n_0

$$\Sigma_{02}^{(2)}\cong 4n_0f_0^2\int\frac{d^3q}{(2\pi)^3}\int d\epsilon'\frac{i}{2\pi}\times\left\{\frac{u_k^2}{\epsilon'+\epsilon-\frac{1}{2}k^2+i\delta}-\frac{v_k^2}{\epsilon'+\epsilon+\frac{1}{2}k^2-i\delta}\right\}\times\left\{-\frac{u_qv_q}{\epsilon-\frac{1}{2}q^2+i\delta}+\frac{u_qv_q}{\epsilon+\frac{1}{2}q^2-i\delta}\right\}, \quad (8.6)$$

where we have introduced $\mathbf{k}=\mathbf{q}+\mathbf{p}$, and approximated the vertices by $(n_0f_0)^{\frac{1}{2}}$. On carrying out the integral over ϵ', we find for the relevant term,

$$\Sigma_{02}^{(2)}\cong 4n_0^2f_0^3\int\frac{d^3q}{(2\pi)^3}\frac{1}{q^2[\epsilon+\frac{1}{2}q^2+\frac{1}{2}k^2-i\delta]}. \quad (8.7)$$

There are two kinds of contributions to $\Sigma_{11}^{(2)}(p,\epsilon)$ which are relevant to our purposes. Those of the first kind give rise to singularities above the negative real ϵ axis, and arise from the diagrams of Fig. 8(b) and (c). The contribution from (b) is

$$4n_0f_0^2\int\frac{d^3q}{(2\pi)^3}\int d\epsilon'\frac{i}{2\pi}\left\{\frac{u_qv_q}{\epsilon'-\frac{1}{2}q^2+i\delta}-\frac{u_qv_q}{\epsilon'+\frac{1}{2}q^2-i\delta}\right\}\times\left\{\frac{u_kv_k}{\epsilon+\epsilon'-\frac{1}{2}k^2+i\delta}-\frac{u_kv_k}{\epsilon+\epsilon'+\frac{1}{2}k^2-i\delta}\right\}, \quad (8.8)$$

while that from (c) is

$$4n_0f_0^2\int\frac{d^3q}{(2\pi)^3}\int d\epsilon'\frac{i}{2\pi}\times\left\{\frac{u_k^2}{\epsilon+\epsilon'-\frac{1}{2}k^2+i\delta}-\frac{v_k^2}{\epsilon+\epsilon'+\frac{1}{2}k^2-i\delta}\right\}\times\left\{\frac{u_q^2}{\epsilon'-\frac{1}{2}q^2+i\delta}-\frac{v_q^2}{\epsilon'+\frac{1}{2}q^2-i\delta}\right\}. \quad (8.9)$$

On carrying out the integrals over ϵ', and keeping only the relevant terms for small n_0, we find for the sum of (8.8) and (8.9)

$$-4n_0^3f_0^4\int\frac{d^3q}{(2\pi)^3}\left(\frac{1}{q^2k^2}+\frac{1}{k^4}\right)\frac{1}{\epsilon+\frac{1}{2}q^2+\frac{1}{2}k^2-i\delta}. \quad (8.10)$$

The contribution to $\Sigma_{11}^{(2)}(p,\epsilon)$ to be substituted in the last term of (8.5) arises from the momentum dependence of $f_s(\frac{1}{2}\mathbf{p},\frac{1}{2}\mathbf{p})$ and the integral in (7.5). It is, to lowest order in n_0,

$$2n_0\,\mathrm{Im}f_s(\tfrac{1}{2}\mathbf{p},\tfrac{1}{2}\mathbf{p})+2n_0f_0^2\int\frac{d^3q}{(2\pi)^3}\times\left\{\frac{1}{\epsilon-\frac{1}{4}p^2-q^2+i\delta}+\frac{1}{q^2-\frac{1}{4}p^2-i\delta}\right\},\quad(8.11)$$

where, as Beliaev has shown, one may apply (7.6) to obtain

$$\mathrm{Im}f_s(\tfrac{1}{2}\mathbf{p},\tfrac{1}{2}\mathbf{p})=-\mathrm{Im}f_0^2\int\frac{d^3q}{(2\pi)^3}\frac{1}{q^2-\frac{1}{4}p^2-i\delta}=-f_0^2\pi\int\frac{d^3q}{(2\pi)^3}\delta(q^2-\tfrac{1}{4}p^2).$$

On summing (8.10) and (8.11) we find then:

$$\Sigma_{11}^{(2)}(p,\epsilon)=2n_0f_0^2\int\frac{d^3q}{(2\pi)^3}\times\left\{\frac{1}{\epsilon-\frac{1}{2}q^2-\frac{1}{2}k^2+i\delta}-\frac{2n_0^2f_0^2[k^2+q^2]}{q^2k^4(\epsilon+\frac{1}{2}q^2+\frac{1}{2}k^2-i\delta)}\right\}+2n_0f_0^2\,\mathrm{P.P.}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\frac{1}{2}q^2+\frac{1}{2}k^2-\frac{1}{2}p^2},\quad(8.12)$$

where we have shifted variables in order to write the contribution from (8.11) in a more symmetric way, and P.P. denotes the principle part. The results (8.7) and (8.12) are in agreement with the high momentum ($p\gg(n_0f_0)^{\frac{1}{2}}$) expansion of the Beliaev effective potentials.

The singularities in (8.7) and (8.12), which give rise to an imaginary part of Σ_{02} and Σ_{11}, are associated with the fact that in this order it is possible for an excitation of momentum $\mathbf{p}$ to decay into two excitations of momentum $-\mathbf{q}$ and $\mathbf{p}+\mathbf{q}$.

If we now substitute (8.7) and (8.12) into (8.5) and carry out the integration over ϵ, we find

$$\frac{E_0^{(3)}}{\Omega}-\tfrac{1}{2}n\mu^{(3)}=2n_0^3f_0^4\int\frac{d^3p}{(2\pi)^3}\times\left[\int\frac{d^3q}{(2\pi)^3}\{A(p,q)+B(p,q)+C(p,q)\}+\mathrm{Im}\int\frac{d^3q}{(2\pi)^3}\frac{1}{p^2(q^2+k^2-p^2-i\delta)}\right],\quad(8.13)$$

where

$$A(p,q)=\frac{2(p^2-q^2-k^2)(1+q^2/k^2)}{(p^2+q^2+k^2)^2k^2q^2},\quad(8.14a)$$

$$B(p,q)=\frac{8}{q^2(p^2+q^2+k^2)^2},\quad(8.14b)$$

and

$$C(p,q)=\frac{4}{(p^2+q^2+k^2)(p^2-q^2-k^2+i\delta)},\quad(8.14c)$$

are the contributions from the singularities of $\Sigma_{11}(p,\epsilon)$, $\Sigma_{02}(p,\epsilon)$, and $\Sigma_{11}(p,-\epsilon)$, respectively, and the last term in (8.13) arises from the singularity at $\epsilon=-\frac{1}{2}p^2$ in (8.5).

It is convenient to combine A and B. On doing this, and taking advantage of the symmetry between k and q, k and p, and of course p and q (when the expression appears as the integrand of [8.13]), one finds

$$A+B=\frac{2}{k^2q^2(k^2+p^2+q^2)}-\frac{2}{q^2(k^2+p^2+q^2)^2}.\quad(8.15a)$$

We also have

$$C=\frac{2}{p^2(p^2+k^2+q^2)^2}+\frac{1}{p^4}\left\{\frac{1}{k^2+p^2+q^2}-\frac{1}{q^2+k^2-p^2+i\delta}\right\}.\quad(8.15b)$$

We now note there is considerable cancellation amongst the terms arising from the different singularities. The resultant ground-state energy is real (as of course it must be) and is given by

$$\frac{E_0^{(3)}}{\Omega}-\tfrac{1}{2}n\mu^{(3)}=2n_0^3f_0^4\int\frac{d^3p}{(2\pi)^3}\left\{\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{2}{p^2q^2(p^2+q^2+k^2)}+\frac{1}{p^4}\frac{1}{(k^2+p^2+q^2)}-\frac{\mathrm{P.P.}}{p^4}\int\frac{d^3\mathbf{q}}{(2\pi)^3(k^2+q^2-p^2)}\right\}.\quad(8.16)$$

On carrying out the integration over $\mathbf{q}$ and over the solid angle of $\mathbf{p}$, we find

$$\frac{E_0^{(3)}}{\Omega}-\tfrac{1}{2}n\mu^{(3)}=\frac{n_0^3f_0^4}{16\pi^2}\left(\frac{4}{3}-\frac{\sqrt{3}}{\pi}\right)\int\frac{dp}{p}.\quad(8.17)$$

Our limiting procedure has led us to a logarithmically divergent expression. This need not concern us unduly however. We know that, had we kept the next-order in n_0 terms, the expression would be well-behaved in the low-momentum region, and possess a natural cutoff at $p_{\min}\sim(n_0f_0)^{\frac{1}{2}}$. Further, the logarithmic divergence for large momenta is a consequence of the fact that we replaced the $f(\mathbf{p},\mathbf{q})$, which properly appear in the Σ's, by f_0; had we not done this, we would have found a natural cutoff occurring, in the case of hard spheres, for $p_{\max}\sim f_0^{-1}$. We are thus led to write

$$\int\frac{dp}{p}\approx\ln\frac{p_{\max}}{p_{\min}}=\ln\frac{g}{(n_0f_0^3)^{\frac{1}{2}}},\quad(8.18)$$

where g is a constant which will influence only the next order terms, $E_0^{(4)}$ and $\mu^{(4)}$. We have therefore

$$\frac{E_0^{(3)}}{\Omega}-\tfrac{1}{2}n\mu^{(3)}=\frac{n^3f_0^4}{32\pi^2}\left(\frac{\sqrt{3}}{\pi}-\frac{4}{3}\right)\ln(nf_0^3),\quad(8.19)$$

in which we replaced n_0 by n.

To solve this equation we proceed as in Sec. 7 and write

$$E_0^{(3)}=\beta n^3 f_0^4 \ln(nf_0^3),$$

so that

$$\mu^{(3)}=3\beta n^2 f_0^4 \ln(nf_0^3),$$

in which β is determined by substituting this in (8.19). We find

$$\frac{E_0^{(3)}}{\Omega}=\frac{n^3 f_0^4}{16\pi^2}\left\{\frac{4}{3}-\frac{\sqrt{3}}{\pi}\right\}\ln(nf_0^3), \tag{8.20}$$

$$\mu^{(3)}=\frac{3n^3 f_0^4}{16\pi^2}\left\{\frac{4}{3}-\frac{\sqrt{3}}{\pi}\right\}\ln(nf_0^3). \tag{8.21}$$

Our result (8.20) is in agreement with the results (obtained by different methods) of Wu[7] and Sawada.[6]

The present calculations also permit us to determine the second-order excitation spectrum in the high-momentum region ($p\gg(n_0f_0)^{\frac{1}{2}}$). In this region, one has, from (4.12) and (8.11) a pole at

$$\epsilon(p)\cong\omega_p+\Sigma_{11}^{(2)}(p,\tfrac{1}{2}p^2)$$
$$=\tfrac{1}{2}p^2+n_0f_0\left(1-\frac{i}{4\pi}f_0p\right)\cong\tfrac{1}{2}p^2+f(\tfrac{1}{2}\mathbf{p},\tfrac{1}{2}\mathbf{p}), \tag{8.22}$$

in agreement with the results of Beliaev[4] and Lee and Yang.[20] In this momentum region, the imaginary part of $\epsilon(p)$ is seen to be simply related to the imaginary part of the forward scattering amplitude.

Beliaev has calculated the second-order excitation spectrum in the low-momentum region. For this calculation it is necessary to include many more terms in the calculation of $\Sigma_{11}^{(2)}$ and $\Sigma_{02}^{(2)}$, that is terms which are of the same order at low momentum as those we have considered. Beliaev finds[21]

$$\epsilon(p)\cong p(n_0f_0)^{\frac{1}{2}}[1+(7/6\pi^2)(n_0f_0^3)^{\frac{1}{2}}]$$
$$-i(3/640\pi)p^5/n_0. \tag{8.23}$$

Thus in second-order, for low momentum, there is a correction to the real part of the sound wave frequency, and an imaginary part, corresponding to the fact that a phonon of momentum $\mathbf{p}$ can decay into two phonons of momenta $\mathbf{q}$ and $\mathbf{p}-\mathbf{q}$. That this latter process goes as p^5 is a consequence of great cancellation amongst the coherence factors appearing in the $\Sigma^{(2)}$'s and in the low-momentum expansion of the denominator in (4.12).

It is interesting to note that the microscopic calculation of the sound velocity, $(n_0f_0)^{\frac{1}{2}}[1+(7/6\pi^2)(n_0f_0^3)^{\frac{1}{2}}]$, is in agreement with the macroscopic calculation. The latter derives from the fact that the velocity of sound waves of infinite wavelength is related to the compressibility, which may in turn be obtained from the ground-state energy. We calculate the first three terms in the expansion of the macroscopic sound velocity, s, with the aid of the following equations:

$$s=(dp/dn)^{\frac{1}{2}}, \tag{8.24}$$

$$p=n^2\frac{d}{dn}\left(\frac{E_0}{N}\right). \tag{8.25}$$

We find, from (7.9), (7.22), (7.24), and (8.20),

$$s=(nf_0)^{\frac{1}{2}}[1+(nf_0^3)^{\frac{1}{2}}/\pi^2$$
$$+(3/16\pi^2)nf_0^3\{\tfrac{4}{3}-\sqrt{3}/\pi\}\ln(nf_0^3)+\cdots. \tag{8.26}$$

9. DISCUSSION

In the preceding section we have seen that both E_0 and μ possess a series expansion of the form

$$E_0=\tfrac{1}{2}n^2f_0[1+a(nf_0^3)^{\frac{1}{2}}+bnf_0^3\ln(nf_0^3)+\cdots], \tag{9.1}$$

$$\mu=nf_0[1+a'(nf_0^3)^{\frac{1}{2}}+b'nf_0^3\ln(nf_0^3)+\cdots]. \tag{9.2}$$

It is not difficult to see that $n'=n-n_0$ possesses a similar series expansion

$$n-n_0=n'=\frac{1}{3\pi^2}(nf_0)^{\frac{3}{2}}[1+a''(nf_0^3)^{\frac{1}{2}}+\cdots]. \tag{9.3}$$

Depletion effects associated with the difference between n_0 and n only make their appearance in the higher order terms of the expansion (9.1) for the ground-state energy. To see this, recall Eq. (7.19), which states that

$$E_0^{(2)}-\tfrac{1}{2}n\mu^{(2)}\sim(n_0f_0)^{\frac{5}{2}}.$$

If one then substitutes the series expansion (9.3) for n_0 into (7.19), one finds contributions to higher order terms in the expansion for E_0, which have the form:

$$a_1n^3f_0^4+a_2n^{7/2}f_0^{11/2}+\cdots.$$

If one further remembers that $E_0^{(3)}-\frac{1}{2}n\mu^{(3)}\sim n_0^3f_0^4\times\ln(n_0f_0^3)$, one finds further contributions from the depletion effect which take the form

$$n^{7/2}f_0^{11/2}\ln(nf_0^3)+\cdots.$$

It thus appears likely that the series expansion for E_0 takes the form

$$E_0=\tfrac{1}{2}n^2f_0[1+a(nf_0^3)^{\frac{1}{2}}+bnf_0^3\ln(nf_0^3)+cnf_0^3$$
$$+dn^{\frac{3}{2}}f_0^{9/2}\ln(nf_0^3)+cn^{\frac{3}{2}}f_0^{9/2}+\cdots]. \tag{9.4}$$

Of these coefficients only a and b are known at present.

It should further be remarked that while the coefficients a and b are independent of the particular "shape" of the interaction potential, the coefficients of the higher-order terms will depend on the specific law of force. Thus, no matter what the interaction which gives rise to the scattering length f_0, the coefficients a and b remain the same, while c is a shape-dependent parameter [as may be seen directly from (8.18), for instance].

The coefficient c appears to be quite difficult to calculate. Consider the calculation of $\Sigma_{02}^{(3)}$, for instance. It

[20] T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419 (1958).

[21] An independent calculation of (8.23) has been carried out by Lee and Yang.[20]

may readily be seen that to calculate this quantity, one must include, amongst other terms, the entire sequence of diagrams shown in Fig. 9. As Beliaev has remarked, the summation of these diagrams cannot be expressed in terms of two-body scattering amplitudes, but requires that one obtain a solution to the three-body problem in closed form.

Actually the incentive does not appear great for performing a calculation which yields c. The reason is that if one studies the relative size of the logarithmic term and the term immediately preceding it in the series expansion for E_0, one finds that when one gets to densities and scattering lengths which might characterize the behavior of liquid helium, the logarithmic term is much arger than its predecessor. For a model of hard spheres of diameter 2.2 A, with a density equal to that of liquid helium, $(nf_0^3)^{\frac{1}{2}}\cong 21.4$, and $E_0^{(3)}\sim 6E_0^{(2)}$.

The one hope, then, of carrying out a microscopic calculation of the properties of liquid helium would seem to lie in summing a selected class of higher order terms, as has been done by Brueckner and Sawada.[8] Their procedure is equivalent in the present formulation to the following approach. One takes, as in the extreme low-density limit, the effective potentials to be

$$\begin{aligned}\Sigma_{11}^{(a)}&=n[t^{(a)}(0\mathbf{q}0\mathbf{q})+t^{(a)}(0\mathbf{q}\mathbf{q}0)],\\ \Sigma_{02}^{(a)}&=nt^{(a)}(00\mathbf{q}-\mathbf{q}),\\ \mu^{(a)}&=nt^{(a)}(0000).\end{aligned}\tag{9.5}$$

We have introduced the notation $t^{(a)}$ to reflect the fact that the $t^{(a)}$'s are now determined by a nonlinear integral equation, which is similar in form to (7.3), but in which the propagators are altered to include the effect of forward scattering via the effective potential, $\Sigma_{11}^{(a)}$. Thus (7.3) is replaced by

$$t_{12;34}=v_{12;34}+v_{12;56}G_5^{(a)}G_6^{(a)}t_{56;34}^{(a)},\tag{9.6}$$

where

$$G^{(a)}(p,\epsilon)=1/(\epsilon-\tfrac{1}{2}p^2-\Sigma_{11}^{(a)}+\mu^{(a)}+i\delta).\tag{9.7}$$

In this fashion one is able to sum a selected class of higher-order diagrams.

It is always difficult to justify including some higher order terms and not others. For instance, certain terms which are of importance in the calculation of the logarithmic term, $E_0^{(3)}$, in the low-density expansion, are neglected in the Bruecker-Sawada approximation. Whether this neglect is justifiable remains to be seen. We believe the methods described in this paper offer a useful way to carry out such an investigation.

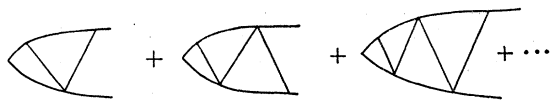

FIG. 9. An infinite series of terms which contribute to the $n_0^3f_0^4$ term in E_0/Ω.

We may further remark that Brueckner and Sawada have neglected the depletion effect. This omission would appear to lead to nonnegligible corrections for a system of hard spheres of diameter 2.2 A at the density of liquid helium (this being the Brueckner-Sawada model for liquid helium). It is not difficult to see how to apply that correction in the present formulation. In the expressions, (9.5), one should properly have n_0 appearing in place of n, where n_0 is determined, in this order by

$$n-n_0=\int\frac{d^3p}{(2\pi)^3}\int d\epsilon\,\frac{i}{2\pi}G'(p,\epsilon),\tag{9.8}$$

where $G'(p,\epsilon)$ differs from $G^{(1)}(p,\epsilon)$ only in that the modified effective potentials (with n_0) given by (9.5) are to be substituted in place of $\Sigma_{11}^{(1)}$ and $\Sigma_{02}^{(1)}$.

It is also not difficult, using the Green's function method, to formulate procedures which go well beyond Brueckner and Sawada in the number of higher-order terms which are summed. One approach, which has been proposed by one of us and Nozières,[22] consists in taking for the effective potentials

$$\begin{aligned}\Sigma_{11}^{(b)}&=n_0[t^{(b)}(0\mathbf{q}0\mathbf{q})+t^{(b)}(0\mathbf{q}\mathbf{q}0)],\\ \Sigma_{02}^{(b)}&=n_0t^{(b)}(00\mathbf{q}-\mathbf{q}),\\ \mu&=n_0t^{(b)}(0000),\end{aligned}\tag{9.9}$$

where the $t^{(b)}$'s are determined by a nonlinear integral equation with the more accurate propagators, $G^{(b)}(p,\epsilon)$, defined in terms of the effective potentials (9.9), according to (4.12). Thus

$$t_{12;34}^{(b)}=v_{12;34}+v_{12;56}G_5^{(b)}G_6^{(b)}t_{56;34}^{(b)},\tag{9.10}$$

where

$$G^{(b)}(p,\epsilon)=\frac{\epsilon+\frac{1}{2}p^2+\Sigma_{11}^{(b)}(p,-\epsilon)-\mu^{(b)}}{\{\epsilon-\frac{1}{2}[\Sigma_{11}^{(b)}(p,\epsilon)-\Sigma_{11}^{(b)}(p,-\epsilon)]\}^2-\{\frac{1}{2}p^2+\frac{1}{2}[\Sigma_{11}^{(b)}(p,\epsilon)+\Sigma_{11}^{(b)}(p,-\epsilon)]-\mu^{(b)}\}^2+[\Sigma_{02}^{(b)}(p,\epsilon)]^2},$$

and the relation between n_0 and n is to be determined according to (4.9) using the above value of $G^{(b)}(p,\epsilon)$. How successful such a procedure may be (it will certainly require extensive machine computations) remains to be seen.

10. CONCLUSION

We have shown that a simple treatment of the depletion of the ground state as a consequence of the interaction between particles permits one to apply the powerful methods of field theory to the many-boson problem. It is then possible to present a consistent divergence-free formulation of the problem to any order. This enables one to obtain certain general relationships between the quantities of interest in the theory. We were thus able to prove that the low-lying excitations of

[22] P. Nozières and D. Pines (private communication).

the system would not possess an energy gap. Another general relation, which perhaps can be proved by the methods of the present paper, is the equality which appears to obtain between the macroscopic and microscopic sound velocities.

The present approach also affords a straightforward way to calculate the series expansion of the properties of the dilute boson gas. We have calculated the next term beyond the results of Beliaev and Lee, Huang, and Yang, and find that the expansion is not a power series, but involves as well the logarithms of the expansion parameter $(nf_0^3)^{\frac{1}{2}}$. We have likewise seen that for the terms up to and including the $n^3f_0^4\ln(nf_0^3)$ term in the ground state energy, the character of the forces between the particles in a dilute boson gas is irrelevant; only the zero energy scattering amplitude enters. It is obvious that the series expansion has no meaning in the case of forces which are attractive, so that the scattering amplitude, f_0, is negative. In this case one would expect a complete breakdown of the perturbation-theoretic expansion in the low-density region (no condensed state, etc.) along with the appearance of two- or more-particle bound states.

11. ACKNOWLEDGMENTS

We should like to thank Professor C. N. Yang and Dr. K. Sawada for interesting discussions. We wish to thank Professor Robert Oppenheimer for the warm hospitality afforded us by the Institute for Advanced Study, and to acknowledge support from the Institute, from a National Science Foundation Grant (D.P.), and from a Fulbright Travel Grant (N.H.).