# Effect of Nuclear Forces on the Cross Sections of Photonuclear Reactions*†

K. OKAMOTO‡
*Department of Physics, Louisiana State University, Baton Rouge, Louisiana*
(Received May 20, 1959)

The effect of nuclear forces, or the effect of the quasi-deuteron model, is discussed for the integrated cross section and the bremsstrahlung weighted cross section. The nuclear force is assumed to be of partly Majorana exchange character. Only the central force is considered. The two-body potential is a Gaussian type without a hard core, the parameters of which are taken from the effective range theory. The calculations are performed by first-order perturbation theory and the results are that the integrated cross section is increased by about ten percent and the bremsstrahlung weighted cross section is decreased by a few percent. Therefore the independent-particle model can be regarded as a good approximation for photonuclear reactions.

## I. INTRODUCTION

As is true also in other nuclear reactions or in nuclear structure, the independent-particle model (hereafter referred to as IPM) is now known to be successful for photonuclear reactions. Especially in the sum rule calculations, in which knowledge of the wave functions of the nuclear excited states is not necessary, the results of the IPM are in fairly good agreement with experiment.[1–5] On the other hand, it is also well known that the strong correlation between nucleons due to nuclear forces plays an important role in high-energy photonuclear reactions, and if we assume a deuteron-like subunit inside the nucleus, we can explain the high-energy phenomena.[6–8] This we call the quasi-deuteron model.

These two facts seem to contradict each other, and few calculations have ever been done to explain this discrepancy. Levinger calculated the effect of the quasi-deuteron model on the bremsstrahlung-weighted cross section (hereafter denoted as $\sigma_b$) and found that $\sigma_b$ decreased by about ten percent, although the exact value depended on the parameters.[9] Brueckner calculated this for the integrated cross section (hereafter denoted as $\sigma_{\text{int}}$) and found that $\sigma_{\text{int}}$ was increased.[10]

We shall calculate this effect in detail both for $\sigma_b$ and for $\sigma_{\text{int}}$. We expand the wave function by perturbation theory taking into account the two-body interaction. The two-body potential is assumed to be of partly Majorana exchange character. Only the central force is considered, and the radial dependence is of Gaussian type without a hard core. The parameters are taken from the effective-range theory.

In Sec. II, $\sigma_{\text{int}}$ is found to increase by about ten percent. The dependence of the above results on the values of the parameters is investigated and it is found that the above result is roughly independent of the choice of the parameters, as long as they are within the reasonable range. In Sec. III, $\sigma_b$ is evaluated by two independent approximations. The results of the two approximations agree with each other that $\sigma_b$ is decreased by a few percent. This is not inconsistent with Levinger's preliminary results.[9] The effect of the Coulomb force is also estimated. For $\sigma_{\text{int}}$ this is exactly zero because only neutron-proton pairs affect the value of $\sigma_{\text{int}}$. For $\sigma_b$ this effect is not exactly zero, but is found to be negligible. The effect of the hard core is neglected in our calculation. It may not be negligible for $\sigma_{\text{int}}$, but is probably negligible for $\sigma_b$. Therefore our calculation shows that the IPM is a fairly good approximation for photonuclear reactions.

## II. CALCULATION OF THE INTEGRATED CROSS SECTION

According to the sum rule, $\sigma_{\text{int}}$ is given by[1]

$$\sigma_{\text{int}}=\int \sigma dW = 2\pi^2 e^2(\hbar^2/Mc)\textstyle\sum_n f_{0n}, \tag{1}$$

where $f_{0n}$ is the oscillator strength defined as follows[1]:

$$f_{0n}=\frac{2M(E_n-E_0)}{\hbar^2}\left|\int \psi_0{}^* \sum_k z_k \psi_n d\tau\right|^2, \tag{2}$$

where $z_k$ is the component of the displacement of the $k$th nucleon along the direction of photon polarization. The summed oscillator strength, $\sum_n f_{0n}$, is given by Levinger and Bethe[1] (hereafter abbreviated as LB).

$$\sum_n f_{0n}=\frac{NZ}{A}-\frac{M}{3\hbar^2}x\int \psi^* \sum_i \sum_j r_{ij}{}^2 V P_{ij}\psi d\tau, \tag{3}$$

* Supported by the Research Corporation.
† A portion of a dissertation submitted to Louisiana State University in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Physics. (Copies may be obtained from the Louisiana State University Department of Physics, or from University Microfilms, 313 North 5th Street, Ann Arbor, Michigan.)
‡ Now at the Department of Physics, McMaster University, Hamilton, Ontario, Canada.

1 J. S. Levinger and H. A. Bethe, Phys. Rev. **78**, 115 (1950), and **85**, 577 (1952).
2 J. S. Levinger and D. C. Kent, Phys. Rev. **95**, 418 (1954).
3 J. S. Levinger, Phys. Rev. **97**, 122 (1955).
4 Yu. K. Khokhlov, Doklady Akad. Nauk U.S.S.R. **97**, 239 (1954).
5 J. S. Levinger, Phys. Rev. **107**, 554 (1957).
6 J. S. Levinger, Phys. Rev. **84**, 43 (1951).
7 Yu. K. Khokhlov, J. Exptl. Theoret. Phys. U.S.S.R. **23**, 241 (1952).
8 S. Yoshida, Progr. Theoret. Phys. (Kyoto) **6**, 1032 (1951).
9 J. S. Levinger, Bull. Am. Phys. Soc. **1**, 37 (1956).
10 K. A. Brueckner, talk at Ann Arbor Nuclear Physics Conference, 1955 (unpublished).

where $x$ is the fraction of the Majorana exchange force, $i$ and $j$ refer to a proton and a neutron, respectively, $r_{ij}$ is the distance between them, $V$ is the two-body potential for a proton $i$ and a neutron $j$, and $P_{ij}$ is the Majorana exchange operator for them.

LB[1] took a plane wave as a wave function and found

$$\sigma_{\text{int}}=15A(1+0.8x)\text{ Mev-mb.} \tag{4}$$

Here we assume $N=Z=A/2$. The first and second terms inside the bracket correspond to the first and second terms of Eq. (3). The coefficient of $x$, which we shall call $C$, is somewhat different for different models.[3]

In this paper we expand the wave function by perturbation theory:

$$\psi=\psi_0+\sum_n\frac{(V_{ij})_{0n}}{E_0-E_n}\psi_n, \tag{5}$$

$$\psi_0=(1/\Omega)e^{i\mathbf{k}_i\cdot\mathbf{r}_i}e^{i\mathbf{k}_j\cdot\mathbf{r}_j}, \tag{5$'$}$$

$$\psi_n=(1/\Omega)e^{i\mathbf{k}_i'\cdot\mathbf{r}_i}e^{i\mathbf{k}_j'\cdot\mathbf{r}_j}, \tag{5$''$}$$

where $V_{ij}$ is the potential between the proton $i$ and the neutron $j$ and the relation between $V_{ij}$ and $V$ will be given later by Eq. (12). $\Omega$ is the volume of normalization defined by

$$\Omega=(4\pi/3)r_0^3A. \tag{6}$$

$\mathbf{k}_i$, $\mathbf{k}_j$, $\mathbf{k}_i'$ and $\mathbf{k}_j'$ are wave numbers of the proton $i$ and the neutron $j$. They must satisfy the following relation of momentum conservation:

$$\mathbf{k}_i+\mathbf{k}_j=\mathbf{k}_i'+\mathbf{k}_j'. \tag{7}$$

Then Eq. (3) becomes

$$\begin{aligned}\sum_n f_{0n}=&\frac{NZ}{A}-\frac{M}{3\hbar^2}x\sum_i\sum_j\int\psi_0^*r_{ij}^2VP_{ij}\psi_0d\tau\\&+\frac{M}{3\hbar^2}x\sum_i\sum_j\int r_{ij}^2V\sum_{n'}\frac{(V_{ij})_{0n'}}{E_{n'}-E_0}\\&\times(\psi_{n'}^*P_{ij}\psi_0+\psi_0^*P_{ij}\psi_{n'})d\tau+O[(V_{ij})_{0n'}^2].\end{aligned} \tag{8}$$

For convenience of later discussion we reverse the order of $E_0$ and $E_n$ in the third term. We introduce the following quantities:

$$\mathbf{q}=\mathbf{k}_i-\mathbf{k}_i'=\mathbf{k}_j'-\mathbf{k}_j, \tag{9}$$

$$\mathbf{q}'=\mathbf{k}_j'-\mathbf{k}_i=\mathbf{k}_j-\mathbf{k}_i'; \tag{9$'$}$$

then

$$\psi_{n'}^*P_{ij}\psi_0=\psi_0^*P_{ij}\psi_{n'}=\Omega^{-2}e^{i\mathbf{q}'\cdot\mathbf{r}}. \tag{10}$$

The first and second terms of Eq. (8) were calculated by LB[1] and correspond to the IPM value. We neglect the last term and take only the third term as a correction due to the quasi-deuteron effect.

$$\Delta\sum_n f_{0n}=x\frac{2M}{3\hbar^2}\frac{1}{\Omega}\int_0^\infty\sum_i\sum_j r_{ij}^2V\sum_n\frac{(V_{ij})_{0n}}{E_n-E_0}\times e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}d^3r_{ij}. \tag{11}$$

The relation between $V$ and $V_{ij}$ is given by

$$V_{ij}=V[(1-x)+xP_{ij}], \tag{12}$$

then $(V_{ij})_{0n}$ is given by

$$(V_{ij})_{0n}=\frac{1}{\Omega}[(1-x)F(\mathbf{q})+xF(\mathbf{q}')], \tag{13}$$

where $F(\mathbf{q})$ is defined by

$$F(\mathbf{q})=\int_0^\infty Ve^{i\mathbf{q}\cdot\mathbf{r}}d^3r. \tag{14}$$

We use the following approximation for the energy:

$$\begin{aligned}E&=\hbar^2k^2/2M^*\\&=(1/\rho)(\hbar^2k^2/2M),\end{aligned} \tag{15}$$

where $M^*$ is the effective mass of the nucleon and $\rho=M^*/M$, which will be determined later.

The summations with respect to $i$ and $j$ correspond to those with respect to $\mathbf{k}_i$ and $\mathbf{k}_j$ and the summation with respect to $n$ corresponds to $\mathbf{k}_i'$ and $\mathbf{k}_j'$, but can be reduced to $\mathbf{k}_i'$ only, because of Eq. (7). Then Eq. (11) becomes

$$\Delta\sum_n f_{0n}=\rho x\frac{4}{3}\frac{M^2}{\hbar^4}\frac{1}{\Omega^2}\int_0^\infty\sum_{\mathbf{k}_i}\sum_{\mathbf{k}_j}\sum_{\mathbf{k}_i'}\frac{(1-x)F(\mathbf{q})+xF(\mathbf{q}')}{\mathbf{k}_i'^2+\mathbf{k}_j'^2-\mathbf{k}_i^2-\mathbf{k}_j^2}\times e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}\mathbf{r}_{ij}^2Vd^3r_{ij}. \tag{16}$$

The term involving $F(\mathbf{q})$ is difficult to evaluate, but we expect that the integral involving $F(\mathbf{q})e^{i\mathbf{q}'\cdot\mathbf{r}}$ is smaller than that involving $F(\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{r}}$ because of the interference between $\mathbf{q}$ and $\mathbf{q}'$. Therefore, as an *upper limit*, we replace $F(\mathbf{q})$ by $F(\mathbf{q}')$.

$$\Delta\sum_n f_{0n}\leqq\rho x\frac{4}{3}\frac{M^2}{\hbar^4}\frac{1}{\Omega^2}\int_0^\infty\sum_{\mathbf{k}_i}\sum_{\mathbf{k}_j}\sum_{\mathbf{q}'}\frac{F(\mathbf{q}')e^{i\mathbf{q}'\cdot\mathbf{r}_{ij}}}{\mathbf{k}_i'^2+\mathbf{k}_j'^2-\mathbf{k}_i^2-\mathbf{k}_j^2}\times\mathbf{r}_{ij}^2Vd^3r_{ij}. \tag{17}$$

The summation with respect to $\mathbf{k}_i'$ is again replaced by $\mathbf{q}'$ using Eq. (9$'$).

The energy denominator of Eq. (17) can be written as

$$\mathbf{k}_i'^2+\mathbf{k}_j'^2-\mathbf{k}_i^2-\mathbf{k}_j^2=2\mathbf{q}'(\mathbf{q}'+\mathbf{k}_i-\mathbf{k}_j). \tag{18}$$

The summations can be transformed into integrals as follows:

$$\sum_{\mathbf{k}}=\frac{2\Omega}{(2\pi)^3}\int d^3k. \tag{19}$$

Here the factor 2 comes from spin. However, this factor is not necessary for $\mathbf{q}'$, because once we specify the spins of $i$ and $j$, they remain unchanged in the excited states. Then Eq. (17) becomes

$$\Delta\sum_n f_{0n}=\rho x\frac{8}{3}\frac{M^2}{\hbar^4}\frac{\Omega}{(2\pi)^9}\int_0^\infty r_{ij}^2Vd^3r_{ij}\int d^3k_i\times\int d^3k_j\int_0^\infty d^3q'\frac{F(\mathbf{q}')e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}}{\mathbf{q}'\cdot(\mathbf{q}+\mathbf{k}_i-\mathbf{k}_j)}. \tag{20}$$

For simplicity we omit the inequality sign. We shall also express the wave numbers in units of the Fermi wave number $k_F$, which is defined by

$$k_F=\frac{1}{r_0}\frac{3}{2}\left(\frac{\pi}{3}\right)^{\frac{1}{3}}=\frac{1.52}{r_0}. \tag{21}$$

Then the wave numbers can be written

$$\mathbf{k}_i=k_F\mathbf{p},\quad \mathbf{k}_j=k_F\mathbf{n},\quad \mathbf{q}'=k_F\mathbf{s}'. \tag{22}$$

According to the Pauli principle they are restricted by the following relations:

$$|\mathbf{p}|,\ |\mathbf{n}|<1;\quad |\mathbf{p}+\mathbf{s}'|,\ |\mathbf{n}-\mathbf{s}'|>1. \tag{23}$$

Equation (20) then becomes

$$\Delta\sum_n f_{0n}=\rho x\frac{8}{3}\frac{M^2}{\hbar^4}\frac{\Omega}{(2\pi)^9}k_F{}^7\int_0^\infty r_{ij}{}^2Vd^3r_{ij}\int d^3p \times\int d^3n\int_0^\infty d^3s\frac{F(s)e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}}{\mathbf{s}\cdot(\mathbf{s}+\mathbf{p}-\mathbf{n})}. \tag{24}$$

Of course the limit of integrations with respect to $\mathbf{p}$ and $\mathbf{n}$ is given by Eq. (23). We omit the prime and write $\mathbf{s}'$ as $\mathbf{s}$ for the sake of simplicity.

The integrations with respect to $\mathbf{p}$ and $\mathbf{n}$ is given by Eq. (23). We omit the prime and write $\mathbf{s}'$ as $\mathbf{s}$ for the sake of simplicity.

The integrations with respect to $\mathbf{p}$ and $\mathbf{n}$ were already done by Euler[11] and are given in the Appendix.

$$\int d^3p\int d^3n\frac{1}{\mathbf{s}\cdot(\mathbf{s}+\mathbf{p}-\mathbf{n})}=\frac{4\pi^2}{15}\frac{P(s)}{s}, \tag{25}$$

where $P(s)$ is the polynomial given in the Appendix. Then Eq. (24) becomes

$$\Delta\sum_n f_{0n}=\rho x\frac{16}{45}\frac{M^2}{\hbar^4}\frac{\Omega}{(2\pi)^6}k_F{}^7\int_0^\infty r_{ij}{}^2Vd^3r_{ij} \times\int_0^\infty sds\,F(s)p(s)e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}. \tag{26}$$

The first integral with respect to $\mathbf{r}_{ij}$ is carried out very easily for a Gaussian potential, with parameters taken from the effective-range theory.[12]

$$V=-s_0V_0\exp(-r_{ij}{}^2/\lambda^2), \tag{27}$$

$$V_0=(229.21/b^2)\ \mathrm{Mev}\times10^{-26}\ \mathrm{cm}^2, \tag{27$'$}$$

$$\lambda=b/(2.06)^{\frac{1}{2}}, \tag{27$''$}$$

where $b$ is the intrinsic range and $s_0$ is the well depth parameter. Their values are to be determined later.

[11] H. Euler, Z. Physik **105**, 553 (1937).
[12] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).

For simplicity of calculation we introduce the following quantity:

$$(M/\hbar^2)V=-s_0W=-s_0W_0\exp(-r_{ij}{}^2/\lambda^2), \tag{28}$$

$$W_0=(M/\hbar^2)V_0=5.53/b^2. \tag{28$'$}$$

Equation (26) then becomes

$$\Delta\sum_n f_{0n}=-\rho x\frac{16}{45}\frac{M}{\hbar^2}\frac{\Omega}{(2\pi)^6}k_F{}^7s_0W_0 \times\int_0^\infty r_{ij}{}^2\exp(-r_{ij}{}^2/\lambda^2)e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}d^3r_{ij} \times\int_0^\infty F(s)P(s)sds. \tag{29}$$

The first integral is evaluated as follows:

$$\int_0^\infty r_{ij}{}^2\exp(-r_{ij}{}^2/\lambda^2)e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}d^3r_{ij} =\tfrac{1}{4}\pi^{\frac{3}{2}}\lambda^5(6-\lambda^2k_F{}^2s^2)\exp(-\lambda^2k_F{}^2s^2/4). \tag{30}$$

Inserting Eq. (30) into Eq. (29), we get

$$\Delta\sum_n f_{0n}=-s_0x\rho\frac{4\pi^{\frac{3}{2}}}{45}\frac{M}{\hbar^2}\frac{\Omega}{(2\pi)^6}k_F{}^7W_0\lambda^5 \times\int_0^\infty(6-\lambda^2k_F{}^2s^2)F(s)P(s) \times\exp(-\lambda^2k_F{}^2s^2/4)sds. \tag{31}$$

$F(s)$ is defined by Eq. (14), and for the potential of Eq. (28) it is

$$(M/\hbar^2)F(s)=-s_0W_0\pi^{\frac{3}{2}}\lambda^3\exp(-\lambda^2k_F{}^2s^2/4). \tag{32}$$

Inserting this into Eq. (31) and putting $s=2u$, we get

$$\Delta\sum_n f_{0n}=+s_0{}^2x\rho\frac{2}{45}\frac{\Omega}{(2\pi)^3}k_F{}^7W_0{}^2\lambda^8 \times\int_0^\infty(6-4\lambda^2k_F{}^2u^2) \times\exp(-2\lambda^2k_F{}^2u^2)P(u)udu. \tag{33}$$

Let us define $K$ as follows[13]:

$$K=(2/45)[\Omega/(2\pi)^3]k_F{}^7W_0{}^2\lambda^8 =0.024A\xi^4,\quad(\xi=b/r_0). \tag{34}$$

Equation (33) then becomes $Ks_0{}^2x\rho J$, where $J$ is given by

$$J=\int_0^\infty(6-\beta u^2)\exp(-\alpha u^2)P(u)udu, \tag{35}$$

$$\alpha=2\lambda^2k_F{}^2, \tag{35$'$}$$

$$\beta=2\alpha. \tag{35$''$}$$

[13] It should be pointed out here that $r_0$ in our case is *not* the effective range, but the nuclear radius parameter which is taken in this paper as $1.2\times10^{-13}$ cm [see Eq. (6)].

We expand $P(u)$ and break it up into two parts as given in the Appendix.

$$J=J_1+J_2, \tag{36}$$

$$J_1=\int_0^1 (6-\beta u^2)[40(1-\ln 2)u^3-10u^5 + \tfrac{4}{3}u^7+0.21u^9]\exp(-\alpha u^2)du, \tag{36'}$$

$$J_2=\int_1^\infty (6-\beta u^2)\left[\frac{10}{3}+\frac{2}{3}\frac{1}{u^2}+0.16\frac{1}{u^4}\right] \times\exp(-\alpha u^2)du. \tag{36''}$$

$J_1$ can be calculated analytically.

$$\begin{aligned}J_1=&36.822\alpha^{-2}-(12.274\beta+60)\alpha^{-3}\\&+(30\beta+24)\alpha^{-4}-(16\beta-15.12)\alpha^{-5}\\&-e^{-\alpha}[(-1.804\beta+11.452)\alpha^{-1}\\&+(0.059\beta-8.658)\alpha^{-2}+(9.726\beta-28.44)\alpha^{-3}\\&+(14\beta+39.12)\alpha^{-4}-(16\beta-15.12)\alpha^{-5}].\end{aligned} \tag{37}$$

Upon using Eqs. (21), (27″), and (35′), $\alpha$ is given by

$$\alpha=2.25\xi^2, \tag{38}$$

where[13] $\xi=b/r_0$ as given after Eq. (34)

Using Eqs. (38) and (35″), Eq. (37) becomes

$$\begin{aligned}J_1=&2.42\xi^{-4}(1-0.128\xi^{-4}+0.108\xi^{-6})\\&+\exp(-2.25\xi^2)(3.61-5.03\xi^{-2}-2.13\xi^{-4}\\&+0.0385\xi^{-6}-0.277\xi^{-8}-0.261\xi^{-10}),\end{aligned} \tag{39}$$

$J_2$ is evaluated numerically:

$$J_2=\int_1^\infty [15\xi^2u^2+1.50\xi^2-20+(0.72\xi^2-2)u^{-2}] \times\exp(-2.25\xi^2u^2)du. \tag{40}$$

As we shall see later, for reasonable values of $\xi$ the first term of Eq. (39) is predominant and we can approximate

$$J=J_1+J_2\cong J_1\cong 2.42\xi^{-4}. \tag{41}$$

Combining Eqs. (34) and (41), we obtain

$$\Delta\sum_n f_{0n}=0.058A\rho s_0^2x\cong 0.232(NZ/A)\rho s_0^2x. \tag{42}$$

Here we use the relation $N\cong Z\cong A/2$. Next we determine the values of $\rho$, and $s_0$. $\rho$ is defined by Eq. (15) and we assume[14]

$$M^*=2M/3. \tag{43}$$

This value of $M^*$ seems to be somewhat larger than is usually taken ($M^*\sim M/2$), but in our case a larger value is better, because our purpose is to find an upper limit of the quasi-deuteron effect and to show the validity of the IPM and the magnitude of the first order perturbation is proportional[15] to $M^*$.

[14] e.g., K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1955).

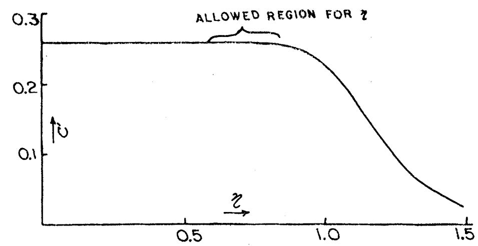


FIG. 1. The effect of the dynamical correlation on the integrated cross-section as a function of $\eta=r_0/b$, where $r_0$ is the nuclear shape parameter and $b$ is the intrinsic range of nuclear forces. $C'$ is the coefficient which appears in Eqs. (45)–(48) and represents the effect of the dynamical correlation. The "allowed region" shown by the bracket is the region of $\eta$ from our present knowledge about nuclei.

Next we determine the value of $s_0$. Strictly speaking, there are singlet and triplet states, with well depth parameters $s_1$ and $s_3$, respectively,[16,17] so $s_0$ is a weighted average of the two.

$$s_0=\tfrac{1}{4}s_1+\tfrac{3}{4}s_3\cong 1.3. \tag{44}$$

Then the coefficient fo $NZ/A$, here denoted as $C'$, is about 0.26.

$$C'\cong 0.26. \tag{45}$$

Next we shall investigate the dependence of $C'$ on the value of $\xi=b/r_0$ to test the validity of the approximation of Eq. (41).

1. $\xi\to 0$; Eqs. (35″), (38) show $\alpha$, $\beta\to 0$. Inserting this into Eqs. (36′), (36″) and combining with Eq. (34), we get

$$C'\to 0 \quad \text{for } \xi\to 0. \tag{46}$$

2. $\xi\to\infty$; we combine the results of Eqs. (34), (39), (40) and get

$$C'\to 0.26 \quad \text{for } \xi\to\infty, \tag{47}$$

assuming the above-mentioned values of $s_0$ and $M^*$.

3. $0<\xi<\infty$; $C'$ must be evaluated numerically. The calculations are performed for $1\leqq\xi\leqq 2$ and the results are shown in Fig. 1 as a function of $\eta=\xi^{-1}=r_0/b$. From our present knowledge of nuclei, $\eta$ lies in the region indicated in Fig. 1 and for these values of $\eta$, $C'$ is roughly constant. In other words, the approximation of Eq. (41) is justified.

Combining Eqs. (1), (3), (4), and (45), we get as a final result

$$\begin{aligned}\sigma_{\text{int}}&=\frac{2\pi^2e^2\hbar}{Mc}\frac{NZ}{A}\{1+(C+C')x\}\\&=15A(1+1.06x)\text{ Mev-mb.}\end{aligned} \tag{48}$$

[15] For a physical explanation of this fact, see for instance, W. J. Swiatecki, Phys. Rev. **101**, 1321 (1956). For a mathematical discussion, see for instance, E. Feenberg, Ann. Phys. (N. Y.) **3**, 292 (1958), and the references of this paper.

[16] L. Hulthén and M. Sugawara, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, pp. 52, 55.

[17] Strictly speaking, the strength of the two-body interaction inside the finite nucleus may be different from that in free space.

The coefficient of $x$ is increased by about 30% as compared with Eq. (4). However, if we put $x=\frac{1}{2}$, the whole cross section is increased by only 10%. It should be pointed out here again, that this value of $C'$ is an *upper limit* as shown by Eqs. (17), (43). Therefore we can conclude that the effect of the dynamical correlation on $\sigma_{\rm int}$ is to increase it by probably less than 10%.

### III. CALCULATION OF THE BREMSSTRAHLUNG-WEIGHTED CROSS SECTION

#### (a) Damping-Factor Method

According to the sum rule, $\sigma_b$ is given by

$$\sigma_b=\int\frac{\sigma}{W}dW$$
$$=4\pi^2\frac{e^2}{\hbar c}\sum_n\left|\left(\frac{N}{A}\sum_i z_i-\frac{Z}{A}\sum_j z_j\right)_{0n}\right|^2. \quad (49)$$

Here $i$ and $j$ refer to a proton and a neutron respectively as before. $\sigma_b$ then becomes, for $N=Z$,

$$\sigma_b=\pi^2\frac{e^2}{\hbar c}\langle(\textstyle\sum_i z_i-\sum_j z_j)^2\rangle_{00}$$
$$=\frac{\pi^2}{137}A\left[\frac{1}{5}R_0^2+\frac{2}{A}S_t\sum_{i\neq i'}\langle z_iz_{i'}\rangle_{00}\right.$$
$$\left.-\frac{2}{A}S_t\sum_i\sum_j\langle z_iz_j\rangle_{00}\right], \quad (50)$$

where $S_t$ is the statistical factor for the singlet and triplet states.

The first term was given by LB.[1] The second term corresponds to the so-called Pauli principle correlation arising from the antisymmetrization of the wave function and was calculated by Levinger and Kent.[2] The third term corresponds to the dynamical correlation. Strictly speaking, the second term also includes the dynamical correlation, but if we assume charge independence of nuclear forces, the dynamical correlation between $n-n$ and $p-p$ pairs in the second term is exactly cancelled by that between singlet $n-p$ pairs in the third term. Therefore we take only triplet $n-p$ pairs and define the effect of the dynamical correlation as follows:

$$\Delta\sigma_b=-\frac{2\pi^2}{137}\times\tfrac{3}{4}\sum_i\sum_j\langle z_iz_j\rangle_{00}. \quad (51)$$

We introduce the following variables:

$$\bar{z}=(z_i+z_j)/2, \quad (52)$$

$$z_{ij}=z_j-z_i. \quad (53)$$

Therefore

$$z_iz_j=\bar{z}^2-z_{ij}^2/4. \quad (54)$$

Then for the wave function of Eq. (5)

$$\langle z_iz_j\rangle_{00}=\langle\psi_0|z_iz_j|\psi_0\rangle$$
$$+2\left\langle\sum_n\frac{(V_{ij})_{0n}}{E_0-E_n}\psi_n|z_iz_j|\psi_0\right\rangle$$
$$+O[(V_{ij})_{0n}^2]. \quad (55)$$

We neglect the third term. The first term vanishes because it is the integral of an odd function. Using Eq. (54), the second term becomes

$$2\left\langle\sum_n\frac{(V_{ij})_{0n}}{E_0-E_n}\psi_n|z_iz_j|\psi_0\right\rangle$$
$$=2\int\int\sum_n\frac{(V_{ij})_{0n}}{E_0-E_n}\psi_n\bar{z}^2\psi_0d\tau_id\tau_j$$
$$-\tfrac{1}{2}\int\int\sum_n\frac{(V_{ij})_{0n}}{E_0-E_n}\psi_nz_{ij}^2\psi_0d\tau_id\tau_j. \quad (56)$$

The first term vanishes because of the orthogonality of the wave function. Finally, the quantity to be calculated, $I$, is

$$I\equiv\textstyle\sum_i\sum_j\langle z_iz_j\rangle_{00}$$
$$=+\tfrac{1}{6}\sum_i\sum_j\int\int\sum_n\frac{(V_{ij})_{0n}}{E_n-E_0}\psi_nr_{ij}^2\psi_0d\tau_id\tau_j. \quad (57)$$

Inserting the explicit forms of the functions defined by Eqs. (5), (12), and (13), we get the equation corresponding to Eq. (16) of $\sigma_{\rm int}$:

$$I=\frac{\rho}{3}\frac{M}{\hbar^2}\frac{1}{\Omega^2}\int\sum_{\mathbf{k}_i}\sum_{\mathbf{k}_j}\sum_{\mathbf{k}_i'}\frac{(1-x)F(\mathbf{q})+F(\mathbf{q}')}{\mathbf{k}_i'^2+\mathbf{k}_j'^2-\mathbf{k}_i^2-\mathbf{k}_j^2}$$
$$\times e^{i\mathbf{q}'\cdot\mathbf{r}_{ij}}r_{ij}^2d^3r_{ij}. \quad (58)$$

Then again neglecting the interference between $\mathbf{q}$ and $\mathbf{q}'$ as we did after Eq. (16), and assuming $\rho=\frac{2}{3}$ from Eqs. (15), (43), we get as an *upper limit*

$$I\leq\frac{2}{9}\frac{M}{\hbar^2}\frac{1}{\Omega^2}\int\sum_{\mathbf{k}_i}\sum_{\mathbf{k}_j}\sum_{\mathbf{k}_i'}\frac{F(q)e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}}{\mathbf{k}_i'^2+\mathbf{k}_j'^2-\mathbf{k}_i^2-\mathbf{k}_j^2}r_{ij}^2d^3r_{ij}. \quad (59)$$

After that, the calculation proceeds in a very similar way to that of $\sigma_{\rm int}$. (We omit the inequality sign for simplicity.)

$$I=\frac{2}{9}\frac{M}{\hbar^2}\frac{1}{\Omega^2}4\left[\frac{\Omega}{(2\pi)^3}\right]^3\int_0^\infty r_{ij}^2d^3r_{ij}\int d^3k_i\int d^3k_j$$
$$\times\int d^3k_i'\frac{F(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}}{\mathbf{k}_i'^2+\mathbf{k}_j'^2-\mathbf{k}_i^2-\mathbf{k}_j^2} \quad (60)$$

$$=\frac{4}{9}\frac{M}{\hbar^2}\frac{\Omega}{(2\pi)^9}k_F^7\int_0^\infty r_{ij}^2d^3r_{ij}\int d^3p\int d^3n$$
$$\times\int_0^\infty d^3s\frac{F(\mathbf{q})e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}}{\mathbf{s}\cdot(\mathbf{s}+\mathbf{p}-\mathbf{n})}, \quad (61)$$

where $k_F\mathbf{s}=\mathbf{q}$. The limit of the integrations with respect to $\mathbf{p}$ and $\mathbf{n}$ are given by Eq. (23) and these integrals are given in Eq. (25) and in the Appendix. Inserting this into Eq. (61), we obtain

$$I=\frac{8}{135}\frac{M}{\hbar^2}\frac{\Omega}{(2\pi)^6}k_F^7\int_0^\infty r_{ij}^2e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}d^3r_{ij}\times\int_0^\infty sdsF(s)P(s). \quad (62)$$

However, unlike the similar expression for $\sigma_{\text{int}}$ the first integral will diverge. Therefore, we introduce a damping factor $\exp[-(r_{ij}^2/l^2)]$, where $l$ is the Gaussian mean radius of the nucleus defined as follows:

$$l=(2/5)^{\frac{1}{2}}R_0. \quad (63)$$

Physically speaking, this introduction of a damping factor corresponds to taking into account a damping of the wave function at the nuclear surface. Then the first integral can be calculated as follows:

$$\int_0^\infty \exp(-r_{ij}^2/l^2)r_{ij}^2e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}d^3r_{ij}=(\pi^{\frac{3}{2}}/4)l^5(6-l^2k_F^2s^2)\exp(-l^2k_F^2s^2/4). \quad (64)$$

After that the calculation again becomes exactly the same as that of $\sigma_{\text{int}}$, except that $\lambda$ is replaced by $l$:

$$I=-s_0'\frac{1}{135}\frac{1}{(2\pi)^3}\frac{36}{27}\left(\frac{\pi}{3}\right)^{7/3}\left(\frac{2}{5}\right)^{5/2}\pi r_0A^{8/3}W_0\lambda^3\times\int_0^\infty(6-k_F^2l^2s^2)\exp[-(l^2+\lambda^2)k_F^2s^2/4]\times P(s)sds. \quad (65)$$

Here the well depth parameter $s_0'$ is somewhat different from the value used for $\sigma_{\text{int}}$ given by Eq. (44). For $\Delta\sigma_{\text{int}}$ calculated in Sec. II, strictly speaking we should have taken both central and tensor forces. However, we took only central force, which is of course an approximation. Recent numerical calculation[18] shows that in our case of Gaussian potential, $s_0$ for the central force is 0.6–1.0 and $s_0$ for the tensor force is about 0.5–0.9. Therefore, the somewhat larger value of $s_0=1.3$ is supposed to include the effect of tensor force as well, although we did not calculate it. However, in the case of $\Delta\sigma_b$ the potential appears only once and the other factors are spherically symmetric. Therefore, the contribution of the tensor force is zero after integrating over angles. Hence we shall take $s_0'=1$.

After putting $s=2u$ as before and assuming $s_0'=1$, Eq. (65) becomes

$$I=-4.61\times10^{-4}br_0A^{8/3}J'=-9.40\times A^{8/3}J'10^{-30}\text{ cm}^2. \quad (66)$$

[18] Biedenharn, Blatt, and Kalos, Nuclear Phys. **1**, 233 (1956); **6**, 359 (1958).

TABLE I. Value of $J'$.[a]

| $A$ | $\xi=1$ | $\xi=1.5$ | $\xi=2.0$ |
|---|---|---|---|
| 64 | −0.031 | −0.009 | +0.002 |
| 125 | −0.013 | −0.008 | −0.004 |
| 215 | −0.008 | −0.006 | −0.004 |

[a] $J'$ is the integral defined by Eqs. (70), (70′) and (70″), but in this table it is calculated by using the approximate formula of Eq. (71). $\xi$ is defined by $b/r_0$, where $b$ is the intrinsic range of the effective-range theory (reference 12) and $r_0$ is the nuclear radius parameter, which is taken as $1.2\times10^{-13}$ cm in this paper.

Here we assume[16] $b=r_{0t}$ (triplet effective range) $=1.7\times10^{-13}$ cm and $r_0=1.2\times10^{-13}$ cm. $J'$ is defined by

$$J'=\int_0^\infty(6-\beta'u^2)\exp(-\alpha'u^2)P(u)udu, \quad (67)$$

$$\alpha'=(l^2+\lambda^2)k_F^2, \quad (67')$$

$$\beta'=4l^2k_F^2. \quad (67'')$$

Using Eqs. (21), (27″), and (63), we find

$$\alpha'=0.93A^{2/3}+1.13\xi^2, \quad (68)$$

$$\beta'=3.71A^{2/3}. \quad (69)$$

The calculation is performed in a very similar way to that of $J$ in Sec. II.

$$J'=J_1'+J_2', \quad (70)$$

$$J_1'=\int_0^1(6-\beta'u^2)[40(1-\ln2)u^3-10u^5+\tfrac{4}{3}u^7+0.21u^9]\exp(-\alpha'u^2)du, \quad (70')$$

$$J_2'=\int_0^\infty(6-\beta'u^2)\left[\frac{10}{3}+\frac{1}{3}\frac{1}{u^2}+0.16\frac{1}{u^4}\right]\times\exp(-\alpha'u^2)du. \quad (70'')$$

In the case of $\sigma_{\text{int}}$, $\alpha=2\lambda^2k_F^2=2.25\xi^2$ and is fairly small if $\xi$ is small, so that it might be dangerous to neglect terms which include $e^{-\alpha}$ in Eq. (37). However, in this case, $\alpha'$, is fairly large except for light nuclei. Therefore we can safely neglect terms including $e^{-\alpha'}$. Similarly we can neglect $J_2'$. In other words, $J'$ is given by

$$J'\cong J_1'\cong36.822\alpha'^{-2}-(12.274\beta'+60)\alpha'^{-3}+(30\beta'+24)\alpha'^{-4}-(16\beta'-15.12)\alpha'^{-5}. \quad (71)$$

Numerical results for Eq. (71) are given in Table I.

As seen in Table I, for $A=64$ the sign of $J'$ changes for $\xi=2.0$; i.e., the results become *qualitatively* opposite. This is unreasonable. However, for $r_0=1.2\times10^{-13}$ cm, $\xi=2.0$ means $b=2.4\times10^{-13}$ cm, which is somewhat too large and should be excluded. Furthermore, our model should be regarded as a model for heavy nuclei, so this discrepancy need not be taken too seriously. In fact, for heavy nuclei the results do not

TABLE II. The results for $\Delta\sigma_b$ and $\sigma_b$ in mb. $\sigma_b$ is the bremsstrahlung cross section defined by Eq. (49). $\Delta\sigma_b$ is the change of $\sigma_b$ due to the quasi-deuteron effect and is given by Eq. (51). Calculated values of $\sigma_b$ are for the IPM. Both calculated and experimental values of of $\sigma_b$ are for $Cu^{63,65}$, $I^{127}$, $Bi^{209}$, respectively. All these values are taken from Table II of reference 5. Calculated values of $\Delta\sigma_b$ and $\Delta\sigma_b$(Coul.) are for $r_0=1.2\times10^{-13}$ cm, $b=1.7\times10^{-13}$ cm, $s_0'=1$, where $b$ and $r_0$ are the same as in Table I and $s_0'$ is the well depth parameter introduced by Eq. (65).

| $A$ | 64 | 125 | 216 |
|---|---|---|---|
| $\Delta\sigma_b$ | $-0.6$ | $-3.2$ | $-10.2$ |
| $\Delta\sigma$(Coul.) | 0.03 | 0.13 | 0.4 |
| $\sigma_b$(calc.) | 77, 74 | 188 | 354 |
| $\sigma_b$(exp.) | 65, 80 | 194 | 416 |
| $\|\Delta\sigma_b/\sigma_b\|$ (%) | $\sim$1.0 | $\lesssim$2.0 | 2.5–3.0 |

change so rapidly for different values of $\xi$, as seen in Table I.

Combining these results with Eq. (51), we obtain

$$\Delta\sigma_b=-1.02\times A^{8/3}\times|J'|\mu b. \tag{72}$$

Since we assume $b=1.7\times10^{-13}$ cm and $r_0=1.2\times10^{-13}$ cm, the value of $\xi$ is approximately 1.5. Therefore, we take the results of $J'$ for $\xi=1.5$. The value of $\Delta\sigma_b$ are listed in Table II for $A=64$, 125, 216 together with the results of previous calculations of the IPM[2,4,5] for nuclei with approximately the same mass numbers, and also with experimental values. The effect of the Coulomb force is also listed, which we shall discuss briefly later.

From Table II we see that the quasi-deuteron effect on $\sigma_b$ is to decrease it by a few percent, and that this percentage increases with increasing $A$. Therefore, we shall investigate this effect for an infinite nucleus. If we let $A\to\infty$ in Eqs. (67′), (67″) we find

$$\beta'=4\alpha' \quad \text{for } A=\infty. \tag{73}$$

The calculation of $J'$ becomes very simple and quite similar to that of $J$ in Sec. II.

$$\begin{gathered} J'\cong J_1'\to 12.274\alpha^{-2}\to 14.191A^{-4/3}, \\ \lim_{A\to\infty}\Delta\sigma_b=-0.014\ \mathrm{A}^{4/3}\ \mathrm{mb}, \end{gathered} \tag{74}$$

whereas in the IPM $\sigma_b$ is given by[5]

$$\sigma_{b,\mathrm{IPM}}=0.30\ \mathrm{A}^{4/3}\ \mathrm{mb}; \tag{75}$$

therefore

$$\lim_{A\to\infty}\frac{\Delta\sigma_b}{\sigma_b}\cong-5\%. \tag{76}$$

In other words, for an infinite nucleus the quasi-deuteron effect on $\sigma_b$ is about 5% decrease of the IPM value. This result is not inconsistent with that of Levinger.[9]

### (b) Box Normalization Method

In order to test the validity of introducing the damping factor of Eq. (64), we shall calculate $\sigma_b$ using a somewhat different method.

We shall start from Eq. (62) but replace the integral with respect to $\mathbf{s}$ by a sum with respect to $\mathbf{q}$ using Eq. (19) but omitting factor 2. [See the discussion after Eq. (19).] In other words, instead of transforming all sums which appear in Eq. (59) into integrals we transform the first two of them and leave the last one as it is.

$$\begin{aligned} I&=\frac{4}{135}\frac{M}{\hbar^2}\frac{\Omega}{(2\pi)^7}k_F{}^7\int_0^\infty r_{ij}{}^2e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}d^3r_{ij}\int_0^\infty d^3s\frac{F(s)P(s)}{s} \\ &=\frac{4}{135}\frac{M}{\hbar^2}\frac{\Omega}{(2\pi)^7}k_F{}^5\int_0^\infty r_{ij}{}^2e^{ik_F\mathbf{s}\cdot\mathbf{r}_{ij}}d^3r_{ij}\int_0^\infty d^3q\frac{F(s)P(s)}{q} \\ &=\frac{4}{135}\frac{M}{\hbar^2}\frac{k_F{}^5}{(2\pi)^4}\sum_{\mathbf{q}}\int_0^\infty r_{ij}{}^2e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}d^3r_{ij}\frac{F(q)P(q)}{q}. \end{aligned} \tag{77}$$

Then we assume that the whole system is in a large cubic box of dimension $L$ and decompose $\mathbf{q}$ into its three components.

$$q_x=2\pi n_x/L,\quad q_y=2\pi n_y/L,\quad q_z=2\pi n_z/L, \tag{78}$$

where $n_x$, $n_y$, $n_z$ $=0, 1, 2\cdots$. The integral in Eq. (77) becomes

$$\begin{aligned} &\int_0^\infty r_{ij}{}^2e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}d^3r_{ij} \\ &\quad=3\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2}x^2e^{i(n_xx+n_yy+n_zz)2\pi/L}dxdydz. \end{aligned} \tag{79}$$

The integrals with respect to $y$ and $z$ are straightforward and give a delta function.

$$\begin{gathered} \int_{-L/2}^{L/2}e^{in_yy}dy=L\delta_{n_y}, \\ \int_{-L/2}^{L/2}e^{in_zz}dz=L\delta_{n_z}. \end{gathered} \tag{80}$$

Therefore, when we evaluate the integral we should put $n_y=n_z=0$. If we further assume $n_x=0$, $\mathbf{q}=0$. However, since $P(q)$ in Eq. (77) is proportional to $\mathbf{q}^2$ for small $\mathbf{q}$ (see Appendix) and $F(\mathbf{q})$ has no singularity at $\mathbf{q}=0$, Eq. (77) becomes zero if $\mathbf{q}=0$. In other words, we must assume $n_x\neq0$. Then the integral with respect to $x$ is given by

$$\int_{-L/2}^{L/2}x^2e^{in_xx}dx=\frac{(-1)^{n_x}L^3}{2\pi^2n_x{}^2}. \tag{81}$$

Therefore, from Eq. (79),

$$\int_0^\infty r_{ij}{}^2e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}d^3r_{ij}=\frac{6L^5(-1)^{n_x}}{(2\pi)^2n_x{}^2}. \tag{82}$$

Substituting this into Eq. (77), we find

$$I=\frac{8}{45}\frac{M}{\hbar^2}\frac{k_F^5}{(2\pi)^6}L^5\sum_{n_x=1}^{\infty}\frac{(-1)^{n_x}}{n_x^2}\frac{F(q)P(q)}{q}. \tag{83}$$

On the other hand, if $L$ is very large, $\mathbf{q}$ is very small for fixed $n_x$. Then from the Appendix

$$\lim_{q\to 0}\frac{F(q)P(q)}{q}=F(0)\lim_{q\to 0}\frac{P(q)}{q}$$
$$=10(1-\ln 2)\frac{F(0)}{k_F^2}\frac{2\pi}{L}n_x. \tag{84}$$

Substituting this into Eq. (83), we get

$$I=\frac{16}{9}(1-\ln 2)\frac{M}{\hbar^2}F(0)\frac{k_F^3}{(2\pi)^5}L^4\sum_{n_x=1}^{\infty}\frac{(-1)^{n_x}}{n_x}$$
$$=\frac{16}{9}(1-\ln 2)W_0\pi^{\frac{3}{2}}\lambda^3\frac{k_F^3}{(2\pi)^5}L^4\ln 2, \tag{85}$$

where $\lambda$ and $W_0$ are given by Eqs. (27″), (28′). The value of $L$ is given by

$$L=(4\pi/3)^{\frac{1}{3}}r_0A^{\frac{1}{3}}, \tag{86}$$

then $I$ becomes

$$I=\frac{16}{9}(1-\ln 2)\ln 2W_0\pi^{\frac{3}{2}}\lambda^3\frac{k_F^3}{(2\pi)^5}\left(\frac{4\pi}{3}\right)^{4/3}r_0^4A^{4/3}. \tag{87}$$

Substituting necessary values and combining with Eq. (51), we obtain

$$\Delta\sigma_b=-0.018\ A^{\frac{4}{3}}\ \text{mb}. \tag{88}$$

Comparing Eq. (75), we get

$$\Delta\sigma_b/\sigma_b\cong -6\%. \tag{89}$$

This agrees very well with Eq. (76). Therefore, we see that the damping-factor method can be regarded as a reasonable approximation.

### (c) Effect of Coulomb Force

This is exactly zero for $\sigma_{\text{int}}$ in this order of approximation, because the value of $\sigma_{\text{int}}$ is affected only by $n-p$ pairs as seen in Eq. (3). However, in the case of $\sigma_b$, this effect appears in $p-p$ pairs. [See Eq. (50) and the discussion after that.] We shall evaluate this by replacing a nuclear potential in Eq. (57) by the Coulomb potential. [For an infinite nucleus this effect will diverge, but in our case it will converge because we used a damping factor in Eq. (64).]

$$V=+e^2/r_{ij}. \tag{90}$$

The calculations are exactly the same as in the case of nuclear forces, so it is not necessary to repeat them here. The results are also listed in Table II. As seen in the table, this effect is negligible.

## IV. CONCLUSION

The effect of the quasi-deuteron model, i.e., the effect of the dynamical correlation between nucleons, has been treated by first order perturbation theory. The results are that $\sigma_{\text{int}}$ is increased by about ten percent and that $\sigma_b$ is decreased by a few percent. The effect of the hard core has not been treated in our calculation. This may affect the result for $\sigma_{\text{int}}$, but it will probably not be so large as to change the main conclusion. For instance, recent calculation[19] shows that for an attractive potential with a hard core the wave function outside the core is similar to that of the IPM. This seems to support the above prediction. For $\sigma_b$ this effect is clearly small, because the main contribution to $\sigma_b$ comes from the low-energy part.

Finally we conclude that the IPM is a fairly good approximation for photonuclear reactions.

## ACKNOWLEDGMENTS

The author wishes to thank Professor J. S. Levinger for his valuable advice and generous encouragement during the progress of this work. He is grateful to Professor R. E. Peierls for his kind hospitality in the Department of Mathematical Physics of the University of Birmingham, and to Professor M. A. Preston of McMaster University for his useful discussion. He also wishes to express his gratitude to the members of the University of Birmingham for their helpful discussions and advice, especially to Mr. de Providencia for his many suggestions. The help of Mr. H. Keech and Mr. D. McPherson of McMaster University in checking calculations is also appreciated.

Finally, he gratefully acknowledges receipt of financial support from the Research Corporation and the Fulbright Commission, and of a fellowship from McMaster University where this work was completed.

## APPENDIX

The integrations with respect to $\mathbf{p}$ and $\mathbf{n}$ for the energy denominator of Eqs. (24) and (61) were carried out by Euler.[11] We merely quote his results here.

$$D\equiv\int d^3s\int_{\substack{|\mathbf{p}|<1\\|\mathbf{p}+\mathbf{s}|>1}}d^3p\int_{\substack{|\mathbf{n}|<1\\|\mathbf{n}-\mathbf{s}|>1}}d^3n\frac{1}{\mathbf{s}\cdot(\mathbf{s}+\mathbf{p}-\mathbf{n})}$$
$$=\int d^3s\int_0^{\infty}\exp(-\alpha s^2)d\alpha\int_{\substack{|\mathbf{p}|<1\\|\mathbf{p}+\mathbf{s}|>1}}e^{-\alpha(\mathbf{s}\cdot\mathbf{p})}d^3p$$
$$\times\int_{\substack{|\mathbf{n}|<1\\|\mathbf{n}-\mathbf{s}|>1}}e^{+\alpha(\mathbf{s}\cdot\mathbf{n})}d^3n, \tag{A1}$$

[19] Gomes, Walecka, and Weisskopf, Ann. Phys. 3, 241 (1958).

where $\alpha$ is an auxiliary variable introduced for convenience of calculation. Putting $\alpha s=y$, we get

$$D=\int_0^2 d^3s\,\frac{4\pi^2}{s}\int_0^\infty e^{-sy}\left[\frac{1}{y^3}\{sye^{(-s/2)y}+e^{-y}(y+1)(e^{-sy}-1)\}\right]^2 dy$$
$$+\int_0^\infty d^3s\,\frac{4\pi^2}{s}\int_0^\infty e^{-sy}\left[\frac{1}{y^3}\{e^{y}(y-1)+e^{-y}(y+1)\}\right]^2 dy. \quad \text{(A2)}$$

The integration with respect to $y$ is elementary but tedious. After integration we put $s=2u$ and obtain the following results:

$$D=\frac{64}{15}\pi^3\int_0^\infty P(u)du. \quad \text{(A3)}$$

For $u<1$,

$$\begin{aligned}P(u)&=P_1(u)\\&=[4+(15/2)u-5u^3+\tfrac{3}{2}u^5]\ln(1+u)+29u^2+3u^4\\&\quad+[4-(15/2)u+5u^3-\tfrac{3}{2}u^5]\\&\quad\times\ln(1-u)-40u^2\ln 2\\&\cong 40(1-\ln 2)u^2-10u^4+\tfrac{4}{3}u^6+0.21u^8.\end{aligned} \quad \text{(A4)}$$

For $u>1$,

$$\begin{aligned}P(u)&=P_2(u)\\&=[4-20u^2-20u^3+4u^5]\ln(u+1)+4u^3+22u\\&\quad+[-4+20u^2-20u^3+4u^5]\ln(u-1)\\&\quad+(40u^3-8u^5)\ln u\cong\frac{10}{3}\frac{1}{u}+\frac{1}{3}\frac{1}{u^3}+0.16\frac{1}{u^5}.\end{aligned} \quad \text{(A5)}$$

The error of the above power-series expansions is less than 0.1% for $P_1(u)$ and less than 0.5% for $P_2(u)$.

---

PHYSICAL REVIEW VOLUME 116, NUMBER 2 OCTOBER 15, 1959

# Heavy Nuclei in the Primary Cosmic Radiation at Prince Albert, Canada. I. Carbon, Nitrogen, and Oxygen*

H. Aizu,† Y. Fujimoto, S. Hasegawa, M. Koshiba, I. Mito,‡ J. Nishimura, K. Yokoi,‡
*Institute for Nuclear Study, University of Tanashi, Tokyo, Japan*

AND

Marcel Schein, *Enrico Fermi Institute for Nuclear Studies, Department of Physics, University of Chicago, Chicago, Illinois*
(Received May 22, 1959)

A stack of G-5 emulsion, exposed at 120 000 feet for 8 hours at 61°N has been used to study the charge and energy spectrum of heavy nuclei at the low-energy end. Energy measurements have been made on C, N, and O nuclei up to 1 Bev/nucleon. The spectrum shows a broad maximum at 550 Mev/nucleon, extrapolated to the top of the atmosphere. Various possibilities to explain this spectrum are discussed. However, it seems desirable to determine the energy spectrum of the other heavy-nuclei components in this energy region in order to gain a more complete understanding of the whole problem. Measurements of this kind are in progress and will be reported.

## 1. INTRODUCTION

In order to gain information concerning the acceleration mechanism of the primary radiation, the study of the heavy-nuclei component in the cosmic radiation has certain advantages over studies on primary protons and the various secondary components. This is due to the fact that heavier nuclei cannot be created from lighter elements once they are ejected from the source into interstellar space. Any conceivable process the primary cosmic radiation might undergo in interstellar space takes place in the direction from heavier to lighter elements.

The very existence of the heavy-nuclei component[1] poses stringent restrictions on the possible types of acceleration mechanisms. Furthermore, the determination of the fluxes of Li, Be, and B [2] has so far been the only method of estimating the average age of the primary cosmic radiation.

The study of the heavier $Z$ components can yield valuable information regarding the relative abundances

* This work has been supported in part by the U. S. National Committee of IGY, the National Science Foundation, and the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

† Department of Physics, Rikkyo University, Tokyo, Japan.
‡ College of General Education, University of Tokyo, Tokyo, Japan.

[1] Freier, Ney, and Oppenheimer, Phys. Rev. **75**, 991 (1949). H. L. Bradt and B. Peters, Phys. Rev. **77**, 54 (1950); **80**, 943 (1950).

[2] Koshiba, Schultz, and Schein, Nuovo cimento **9**, 1 (1958).