Stark Broadening of Hydrogen Lines in a Plasma*

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The frequency distributions of hydrogen lines broadened by the local fields of both ions and electrons in a plasma are calculated in the classical path approximation. The electron collisions are treated by an impact theory which takes into account the Stark splitting caused by the quasi-static ion fields. The ion field-strength distribution function used includes the effect of electron shielding and ion-ion correlations. The various approximations that were employed are examined for self-consistency and an accuracy of about 10% in the resulting line profiles is expected. Good agreement with experimental H_β profiles is obtained while there are deviations of factors of two with the usual Holtsmark theory. Asymptotic distributions for the line wings are given for astrophysical applications. Also here the electron effects are generally as important as the ion effects for all values of the electron density and in some cases the electron broadening is larger than the ion broadening.

INTRODUCTION

SINCE atomic hydrogen is subject to a large linear Stark effect, the principal cause of broadening of hydrogen lines in plasmas is the interaction of the emitting atoms with the local electric fields of the ions and electrons. Even at low degrees of ionization and low densities, Stark effect broadening is usually dominant, but if at high temperatures Doppler broadening becomes significant in the cores of the lines it can easily be taken into account by the usual folding procedure. But under most conditions the spectral distribution will depend mainly on the electron and ion density and only slightly on temperature or neutral density. With the aid of an adequate line broadening theory one can therefore use experimentally determined hydrogen line profiles to measure the densities of the perturbing ions and electrons in a partially ionized gas.

Detailed calculations of the broadening of Ly-α, Ly-β, H_α, H_β, H_γ, and H_δ by both ions and electrons in a plasma have been carried out using an IBM 704. The effects of electron shielding and ion-ion correlations on the statistical theory for the broadening by ions have been incorporated using Ecker's[1] distribution functions. The broadening due to electrons is calculated by means of a generalized impact theory which takes into account the Stark splitting caused by the ion fields. Large corrections to the usual Holtsmark[2] theory have been obtained which affect much of the previous work on the composition and structure of stellar atmospheres which was based on the Holtsmark distribution.

In the present calculation, the various approximations which are introduced into the numerical evaluation of the spectral distribution function have been investigated and an over-all accuracy of about 10% is indicated. This is borne out by a comparison with recent precise measurements of the H_β profile in a plasma by Bogen.[3] The theoretical and experimental profiles are in close agreement over an intensity range of two decades.

ELECTRON BROADENING

Because of the characteristically high velocities of electrons in a plasma, their influence on line broadening can be calculated with the impact approximation, as will be shown at the end of this section. A generalized impact theory is used that has been developed by the authors[4] and independently by Baranger.[5] It treats the case of overlapping lines in a multiplet and leads to new terms in the spectral distribution; i.e., asymmetric terms in addition to the usual Lorentz-dispersion terms. For the problem under consideration, the hydrogenic energy levels are split by the quasi-static ion fields. The lines which arise from transitions between these split levels are then broadened and overlap because of the electron collisions.[6]

The general problem of overlapping lines was solved explicitly when perturbations of the upper state are large compared to perturbations of the lower state.[4] The method of calculation when both states are perturbed was only indicated and is given below using a generalization of "Method II" of the earlier theory.

The spectral distribution function is found from the trace of the thermal average (denoted by $\{\ \}_T$) of the

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[5] M. Baranger, Phys. Rev. **111**, 494 (1958).
[6] A more complete discussion of the general aspects of the problem and a survey of earlier theories of hydrogen line broadening can be found in A. C. Kolb, dissertation, University of Michigan, 1947; University of Michigan Engineering Research Institute, ASTIA Document No. AD 115040 (unpublished). See also H. Margenau, Revs. Modern Phys. **31**, 569 (1959).

Fourier transform of the dipole-dipole correlation function

$$\{\mu(t)\mu(0)\}_T=\{T_a^\dagger(t,0)e^{iH_0t/\hbar}\mu(0)e^{-iH_0t/\hbar}\times T_b(t,0)\mu(0)\}_T, \quad (1)$$

where $\mu(0)$ is the dipole operator at the time $t=0$, H_0 is the unperturbed Hamiltonian of the radiating atom and $T_{a,b}(t,0)$ is the usual time development operator which satisfies the Schrödinger equation

$$i\hbar\frac{d}{dt}T_{a,b}(t,0)=e^{iH_0t/\hbar}V(t)e^{-iH_0t/\hbar}T_{a,b}(t,0), \quad (2)$$

where $V(t)$ is the perturbation. The a and b indices denote the principal quantum numbers of the initial and final states and the operator $T_a(t,0)$, say, has matrix elements between states with the same principal quantum number a. The dipole-dipole correlation function can be written in this way because we do not consider collision induced transitions between states with different principal quantum numbers and also neglect the dipole radiation arising from transitions between substates of a level with a given principal quantum number. Neglecting these transitions can be fully justified for low-energy electron collisions in a plasma and for optical line radiation.

Consider now the matrix elements

$$\begin{aligned}&\{\langle\alpha'|\mu(t)\mu(0)|\alpha'\rangle\}_T\\&=\{\langle\alpha'|T_a^\dagger(t,0)|\alpha''\rangle\exp(i\omega_{\alpha''}t)\langle\alpha''|\mu(0)|\beta'\rangle\\&\quad\times\exp(i\omega_{\beta'}t)\langle\beta'|T_\beta(t,0)|\beta''\rangle\langle\beta''|\mu(0)|\alpha'\rangle\}_T\\&=\exp(i\omega_{\alpha''\beta'}t)\langle\alpha''|\mu(0)|\beta'\rangle\langle\beta''|\mu(0)|\alpha'\rangle\\&\quad\times\langle\alpha'|\langle\beta'|\{T_a^\dagger(t,0)T_b(t,0)\}_T|\alpha''\rangle|\beta''\rangle, \quad (3)\end{aligned}$$

where we have chosen a representation in which H_0 is diagonal and $\omega_{\alpha''\beta'}\equiv\omega_{\alpha''}-\omega_{\beta'}\equiv(E_{\alpha''}-E_{\beta'})/\hbar$. The problem is so reduced to finding the thermal average of the product $T_a^\dagger(t,0)T_b(t,0)$. Consider now the change in this quantity during a time interval Δt:

$$\begin{aligned}&\Delta\{T_a^\dagger(t,0)T_b(t,0)\}_T\\&=\{T_a^\dagger(t+\Delta t,0)T_b(t+\Delta t,0)-T_a^\dagger(t,0)T_b(t,0)\}_T\\&=\{[T_a^\dagger(t+\Delta t,t)T_b(t+\Delta t,t)-1]\\&\quad\times T_a^\dagger(t,0)T_b(t,0)\}_T. \quad (4)\end{aligned}$$

If there is only one collision[7] in the time interval Δt that is not correlated with previous collisions (impact approximation), the two factors on the right side can be averaged separately and one has

$$\Delta\{T_a^\dagger(t,0)T_b(t,0)\}_T=\sum_j P_j[T_{aj}^\dagger(t_j)T_{bj}(t_j)-1]\Delta t_j\times\{T_a^\dagger(t,0)T_b(t,0)\}_T, \quad (5)$$

where we denote the change in $T(t,0)$ by a collision during the time interval Δt_j by $T(t_j)$; P_j is the probability per unit time that there is a collision of the type j. The time interval Δt_j has been chosen to be large compared to the duration of a collision but sufficiently short so that there is only a small deviation in $T_a^\dagger(t,0)T_b(t,0)$ during the time interval. This assumption will be examined later for self-consistency. In that case one obtains the differential equation

$$\frac{d}{dt_j}\{T_a^\dagger T_b\}_T=\phi_{ab}\{T_a^\dagger T_b\}_T, \quad (6)$$

where

$$\phi_{ab}\equiv\sum_j P_j[T_{aj}^\dagger T_{\beta j}-1], \quad (7)$$

which has a solution

$$\{T_a^\dagger T_b\}_T=\exp(\phi_{ab}t), \quad (8)$$

with $T_a^\dagger(0,0)T_b(0,0)=1$. This result follows because $T_{aj}(t_j)\times T_{bj}(t_j)$ due to a single collision is independent of the time t_j of the collision when Δt_j is large compared to the collision time. Then we have $T_{(a,b)j}(t_j)=T_{(a,b)j}(0)\equiv T_{(a,b)j}$, and ϕ_{ab} is independent of t_j. Substitution of (8) into the expression for $\{\mu(t)\mu(0)\}_T$ and Fourier transformation yields the spectral distribution

$$\begin{aligned}J_{ab}(\omega)&=\frac{1}{\pi}\operatorname{Re}\operatorname{Tr}\int_0^\infty dte^{-i\omega t}\rho\{\mu(t)\mu(0)\}_T\\&=\frac{e^{-E_a/kT}}{\pi Z(T)}\operatorname{Re}\sum_{\alpha'\alpha''}\sum_{\beta'\beta''}\langle\alpha'|\mu(0)|\beta'\rangle\langle\beta''|\mu(0)|\alpha'\rangle\\&\quad\times\int_0^\infty dt\langle\alpha'|\langle\beta'|\exp[(-i\Delta\omega_{\alpha''\beta'}+\phi_{ab})t]|\alpha''\rangle|\beta''\rangle, \quad (9)\end{aligned}$$

where $\omega-\omega_{\alpha''\beta'}\equiv\Delta\omega_{\alpha''\beta'}$, and $Z(T)$ is the partition function.

Finally we have

$$\begin{aligned}J_{ab}(\omega)&=\frac{e^{-E_a/kT}}{\pi Z(T)}\operatorname{Re}\sum_{\alpha'\alpha''}\sum_{\beta'\beta''}\langle\alpha'|\mu(0)|\beta'\rangle\\&\quad\times\langle\alpha'|\langle\beta'|\frac{1}{(i\Delta\omega_{\alpha''\beta'}-\phi_{ab})}|\alpha''\rangle|\beta''\rangle\\&\quad\times\langle\beta''|\mu(0)|\alpha'\rangle. \quad (10)\end{aligned}$$

In obtaining this formula we have taken the matrix elements of the density operator

$$\langle\alpha'|\rho|\alpha'\rangle=\langle\alpha'|\exp(-H_0/kT)|\alpha'\rangle/Z(T)\cong e^{-E_a/kT}/Z(T)$$

[7] In the case of hydrogen, most of the electron broadening is due to distant, i.e., weak collisions, which overlap in time. If the combined effect of such weak collisions in Δt is small, $T_a(t+\Delta t,t)\times T_b(t+\Delta t,t)$ will be given by the first terms of the iterated solution of the Schrödinger equation. The linear superposition is still valid in that case because terms due to the perturbations produced by several different electrons will average out since the perturbation energy is an odd function of the perturber coordinates. This is only true up to the third iteration and the validity criterium for the impact theory may therefore be reformulated as follows: There must be either only one strong collison or several weak collisions which cause together only a small deviation in the matrix elements of the $T_a^\dagger T_b$ operator, in a time interval characteristic for the correlations of the perturbations.

to be independent of the splitting. This is a good approximation for the optical region where the perturbation energies are small compared with the mean thermal energy.

In order to calculate the spectral distribution with (7) and (10) it is necessary to solve the Schrödinger equation for the time-development operator. We now make the classical path assumption and take the field produced by the perturbing electron to be uniform over the atom. The classical Hamiltonian for a single electron collision of type j is

$$V_j(t)=e^2\frac{\mathbf{r}\cdot\mathbf{r}_j}{|\mathbf{r}_j|^3}=\frac{e^2\mathbf{r}\cdot(\boldsymbol{\varrho}_j+\mathbf{v}_j t)}{(\rho_j^2+v_j^2t_j^2)^{\frac{3}{2}}}. \tag{11}$$

Here $\mathbf{r}$ is the position vector of the atomic electron; $\boldsymbol{\varrho}_j$ and $\mathbf{v}_j$ are the impact parameter and velocity, respectively, of the perturbing electron at the time of closest approach $t=0$. For rapidly varying perturbations the exponentials in (2) can be replaced by unity. This can be seen as follows: The average Stark effect splitting within a level of principal quantum number a will be $\Delta\omega\simeq e^2a^2a_0/\hbar\rho_i^2$ in angular frequency units with the Bohr radius $a_0=\hbar^2/me^2$ and a typical distance ρ_i between the radiating atom and the nearest ion.† The duration of an electron collision is of order $\Delta\tau_e=\rho_e/\bar{v}_e$, ρ_e being a typical impact parameter and $\bar{v}_e$ the average velocity of the electrons. Now ρ_i and ρ_e will both be of the order $N^{-\frac{1}{3}}$ if N is the number of electrons per unit volume. The exponents are of the order $\Delta\omega\Delta\tau_e$, i.e., $a^2N^{\frac{1}{3}}\hbar/m\bar{v}_e$. When this quantity is small compared to unity, the exponentials are not important. This limits the theory to densities that are not too high but still covers most of the range of practical interest as can be seen from Table I.

Now we can extend all time integrals from $-\infty$ to $+\infty$ and obtain the following expression for the operator T_j from the iterated solution of the Schrödinger equation (2) using (11):

$$T_{aj}=1+\frac{e^2}{i\hbar}\int_{-\infty}^{\infty}dt\frac{\mathbf{r}_a\cdot\mathbf{r}_j(t)}{|\mathbf{r}_j(t)|^3}-\left(\frac{e^2}{\hbar}\right)^2\int_{-\infty}^{\infty}dt\frac{\mathbf{r}_a\cdot\mathbf{r}_j(t)}{|\mathbf{r}_j(t)|^3}\times\int_{-\infty}^{t}dt'\frac{\mathbf{r}_a\cdot\mathbf{r}_j(t')}{|\mathbf{r}_j(t')|^3}+\cdots. \tag{12}$$

It can also be shown for many lines of interest that we need only consider ϕ_a instead of ϕ_{ab}, since the broadening arises mainly from perturbations of the upper state which is most easily polarized; i.e., we can take $T_b(t,0)=1$ if perturbations of the final state are numerically unimportant. This is rigorous for the Lyman series since the ground state is nondegenerate and is an excellent approximation for the higher members of the Balmer series as is shown later. Calculations of spectral line intensities where perturbations of both the initial and final states must be considered, using the general formulas given here, will be the subject of future publications.

TABLE I. Limiting values of $\Delta\omega\cdot\Delta\tau_e$ for $T=10\,000°$K.

	Ly-α	Ly-β	H$_\alpha$	H$_\beta$	H$_\gamma$	H$_\delta$
N(cm^{-3})	10^{19}	10^{18}	10^{18}	10^{17}	10^{17}	10^{16}
$\Delta\omega\Delta\tau_e$	0.2	0.2	0.2	0.15	0.25	0.15

† The subscripts i and e denote ions and electrons and should be distinguished from the collision index j.

For the present purposes we need only to calculate the quantity $\int d\sigma_j P_j(T_{aj}-1)$, which involves an average over the directions defined by $\boldsymbol{\varrho}_j$ and $\mathbf{v}_j$. All odd terms in (12) drop out and one obtains

$$T_{aj}-1=-\left(\frac{e^2}{\hbar}\right)^2\int_{-\infty}^{\infty}dt\frac{\mathbf{r}_a\cdot\mathbf{r}_j(t)}{|\mathbf{r}_j(t)|^3}\times\int_{-\infty}^{t}dt'\frac{\mathbf{r}_a\cdot\mathbf{r}_j(t')}{|\mathbf{r}_j(t')|^3}+\cdots. \tag{13}$$

On evaluation of the double integral all terms containing components of both $\mathbf{v}_j$ and $\boldsymbol{\varrho}_j$ will average out, and the terms of the form $v_{jx}v_{jy}$ give no contribution because they are multiplied by odd functions of t and the double integral is consequently zero. With these simplifications we have

$$T_{aj}-1=-\left(\frac{e^2}{\hbar}\right)^2(\mathbf{r}_a\cdot\boldsymbol{\varrho}_j)^2\int_{-\infty}^{\infty}\frac{dt}{(\rho_j^2+v_j^2t^2)^{\frac{3}{2}}}\times\int_{-\infty}^{t}\frac{dt'}{(\rho_j^2+v_j^2t'^2)^{\frac{3}{2}}}. \tag{14}$$

All terms of the type $\rho_{jx}\rho_{jy}$ will again average out and the quantities $(\rho_{jx})^2$, $(\rho_{jy})^2$, $(\rho_{jz})^2$ can be replaced by $\frac{1}{3}\rho_j^2$ for an isotropic distribution of electrons. The integrals then reduce to

$$T_{aj}-1=-\frac{2}{3}\frac{e^4}{\hbar^2}\frac{\mathbf{r}_a\cdot\mathbf{r}_a}{(\rho_jv_j)^2}, \tag{15}$$

and we find

$$\phi_a=N\int d\sigma_j\,P_j(T_{aj}-1)=-2\pi N\int\int d\rho dv\,f(v)\left[\frac{2}{3}\frac{e^4}{\hbar^2}\frac{\mathbf{r}_a\cdot\mathbf{r}_a}{(\rho v)^2}\rho\right], \tag{16}$$

where $f(v)$ is the velocity distribution which is assumed to be Maxwellian:

$$f(v)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{m}{kT}\right)^{\frac{3}{2}}v^2\exp\left(-\frac{mv^2}{2kT}\right). \tag{17}$$

Evidently the ρ integration leads to logarithmic divergencies for both small and large impact param-

eters. At small impact parameters higher order terms in the iterated solution of the Schrödinger equation become of importance. But it will turn out that close collisions are relatively unimportant compared to the more distant collisions. We can estimate their effects from the Lorentz-Weisskopf collision frequency $N\pi\rho_w^2 v$, where the cross section $\pi\rho_w^2$ is defined by the condition that the leading terms in (13) approach unity, i.e., by

$$\rho_w=(\tfrac{2}{3})^{\frac{1}{2}}\frac{e^2}{\hbar}\frac{\langle \mathbf{r}_a\cdot\mathbf{r}_a\rangle^{\frac{1}{2}}}{v}. \tag{18}$$

Then the collisions are strong and completely disrupt the radiation process. To estimate the magnitude of ρ_w it is sufficient to replace the matrix element $\langle \mathbf{r}_a\cdot\mathbf{r}_a\rangle$ by a mean value for a given principal quantum number a, which will be of the order $a^4a_0^2$; then

$$\rho_w\simeq(\tfrac{2}{3})^{\frac{1}{2}}(h/mv)a^2. \tag{19}$$

The divergence at large impact parameters corresponds to the breakdown of all binary collision theories for the case of Coulomb interactions. Actually the distant collisions outside the Debye radius will be screened in a neutral plasma and one can cut off the integral at the Debye radius ρ_D given by

$$\rho_D=(kT/4\pi Ne^2)^{\frac{1}{2}}. \tag{20}$$

By dividing the collisions into three classes, strong collisions inside ρ_w, weak collisions between ρ_w and ρ_D and screened collisions outside ρ_D, the ρ integration can be performed and (16) becomes

$$\phi_a=-\pi N\int dv\, vf(v)\left[\rho_w^2+\frac{4}{3}\frac{e^4}{\hbar^2}\frac{\mathbf{r}_a\cdot\mathbf{r}_a}{v^2}\ln\left(\frac{\rho_D}{\rho_w}\right)\right]. \tag{21}$$

The first term in the bracket determines the relative contribution of strong collisions and the logarithmic term represents the contribution of weak collisions. Comparison with (18) shows that the coefficient of the logarithm is $2\rho_w^2$. The ratio of the weak- to strong-collision contributions is therefore given by

$$\phi_w/\phi_s=2\ln(\rho_D/\rho_w), \tag{22}$$

or if we express ρ_D by (20) and ρ_w by (19) and use the mean velocity $\bar{v}=(3kT/m)^{\frac{1}{2}}$, the ratio

$$\frac{\phi_w}{\phi_s}\simeq 2\ln\left[\frac{3kT}{2\hbar a^2}\left(\frac{m}{2\pi Ne^2}\right)^{\frac{1}{2}}\right] \tag{23}$$

is 5 or larger for the range of densities and temperatures considered in this paper. The contribution of close collisions to the ϕ matrix will therefore be less than 20%.

This result has some bearing on the validity of the classical path assumption also, because we can now see from (19) that the angular momentum of electrons giving significant contributions is $(\frac{2}{3})^{\frac{1}{2}}\hbar a^2$ or larger. The most effective impacts which take place near the Debye radius correspond to angular momenta larger by one order of magnitude. Errors due to the classical path assumption are therefore entirely negligible for hydrogen because of the large angular momentum quantum numbers.[8] This also shows that the field strength produced by the perturbing electron is practically constant over the atom because

$$\rho_w=(\tfrac{2}{3})^{\frac{1}{2}}\frac{e^2}{\hbar v}a^2a_0\gtrsim 3a^2a_0, \tag{24}$$

for the most probable electron velocities. Since the most effective impacts occur at impact parameters several times larger than ρ_w it can be concluded that the assumption of a homogeneous perturbing field, i.e., the choice of the perturbation Hamiltonian (11), is justified for the majority of effective collisions which take place at distances large compared to the Bohr radius.

Neglecting the strong-collision term, we finally obtain after a partial integration

$$\phi_a=-\frac{4}{3}\frac{e^4}{\hbar^2}\left(\frac{2\pi m}{kT}\right)^{\frac{1}{2}}\left\{-\ln\left(\frac{\rho_D}{\rho_w}\right)\exp\left(\frac{-mv^2}{2kT}\right)\Big|_{v_{\min}}^{\infty}+\int_{v_{\min}}^{\infty}\frac{dv}{v}\exp\left(\frac{-mv^2}{2kT}\right)\right\}\mathbf{r}_a\cdot\mathbf{r}_a. \tag{25}$$

For the very small velocities the limiting impact parameter for weak collisions $\rho_w(v)$ is larger than the Debye radius ρ_D. The effect of these slow electrons can so be estimated with the Lorentz-Weisskopf formula and turns out to be entirely negligible because only very few electrons are in this velocity range. We will therefore proceed by cutting off the integral at a minimum velocity $v_{\min}$ defined by the relation $\rho_w(v_{\min})=\rho_D$ or with (19) and (20):

$$v_{\min}=(8\pi Ne^2/3kT)^{\frac{1}{2}}(\hbar a^2/m). \tag{26}$$

Then the first term in (25) vanishes and we have, with the substitutions

$$y=mv^2/2kT, \tag{27}$$

and

$$y_{\min}=mv_{\min}^2/2kT=(4\pi N/3m)(e\hbar a^2/kT)^2, \tag{28}$$

the final expression for the ϕ operator which we have used in the numerical calculations

$$\phi_a=\frac{1}{3}\left(\frac{8\pi m}{kT}\right)^{\frac{1}{2}}N\frac{e^4}{\hbar^2}\mathbf{r}_a\cdot\mathbf{r}_a\int_{y_{\min}}^{\infty}dy\frac{\exp(-y)}{y}. \tag{29}$$

The values of $y_{\min}$ are mostly smaller than 0.1 so that the exponential integral may often be approximated by

$$\int_{y_{\min}}^{\infty}dy\frac{\exp(-y)}{y}\simeq-\ln y_{\min}-0.577, \tag{30}$$

[8] It was shown earlier in reference 6 that for hydrogen the quantum mechanical and classical path calculations lead to identical results for the matrix elements of the perturbation if the relative angular momenta are large and the interaction is weak.

and ϕ is only a very weak function of $y_{\min}$ which justifies the cutoff procedures. The major uncertainties in ϕ are due to the neglect of the exponentials,[9] the extension of the time integrals in (12) and the neglect of strong collisions in (25). These various approximations introduce errors of the order 10–20%. But these errors will partially compensate each other and Eq. (25) is accurate to about 10%.[10] The errors due to the approximate nature of the perturbation Hamiltonian, i.e., the assumption of classical path and homogeneous perturbing field, are of lesser importance. They essentially only cause an uncertainty in $y_{\min}$, the exact value of which is not very critical.

This does not yet mean that the line profiles calculated with these approximations for ϕ_a will be of corresponding accuracy. The validity of the impact assumption must first be determined. This requires for weak collisions an estimate of the change $\langle \Delta T_a \rangle$ of the matrix elements of the transition operator in a time interval $\Delta\tau_c$ characteristic for the correlations of the perturbations. Now the correlation time $\Delta\tau_c$ is of the order of $\rho_D/\bar{v}$ or smaller, $\langle \Delta T_a \rangle \cong \phi_a \Delta\tau_c$, and using (29) with $\langle \mathbf{r}_a \cdot \mathbf{r}_a \rangle \cong a^4 a_0^2$ and also (20), we find

$$|\langle \Delta T_a \rangle| \simeq \frac{1}{3}\left(\frac{2N}{me^2}\right)^{\frac{1}{2}} \frac{\hbar^2}{m\bar{v}} a^4 \int_{y_{\min}}^{\infty} dy \frac{\exp(-y)}{y}. \quad (31)$$

For the lines considered in this paper this quantity is 0.1 or smaller. Therefore, the impacts only cause a small change in the operator $T(t,0)$ in a time where the perturbations are correlated, and the impact assumption is accordingly justified.

Using standard perturbation theory for the linear Stark effect for a field strength F which is here produced by the ions and is taken to be in the z direction, we obtain the usual expression in terms of parabolic wave functions:

$$\Delta\omega_{\alpha''\beta} = \Delta\omega_{ab} - (eF/h)(\langle \alpha'' | z | \alpha'' \rangle - \langle \beta | z | \beta \rangle), \quad (32)$$

where $\Delta\omega_{ab} = \omega - \omega_a + \omega_b$ is the frequency measured from the frequency of the unperturbed line.

Substitution of (29) and (32) into the general expression (10) for the spectral intensity yields the final formula for the impact profile used in our numerical calculations:

$$J_{ab}(\omega,F) = \frac{1}{\pi} \sum_{\alpha'\alpha''\beta} \mathrm{Re}\langle \beta | \mu(0) | \alpha' \rangle \times \left\langle \alpha' \left| \left[i[\Delta\omega_{ab} - (eF/\hbar)(\langle \alpha'' | z | \alpha'' \rangle - \langle \beta | z | \beta \rangle)] + \frac{1}{3} N \left(\frac{8\pi m}{kT}\right)^{\frac{1}{2}} \frac{e^4}{\hbar^2} \mathbf{r}_a \cdot \mathbf{r}_a \int_{y_{\min}}^{\infty} dy \frac{\exp(-y)}{y} \right]^{-1} \right| \alpha'' \right\rangle \times \langle \alpha'' | \mu(0) | \beta \rangle, \quad (33)$$

where we drop $[\exp(E_a/kT)]/Z(T)$ for convenience. This describes the influence of the electron perturbations on the Lyman lines within a $\sim$10% numerical accuracy as discussed previously. For the Balmer lines the broadening of the $b=2$ state by electron impacts is neglected. But again this will only introduce small errors because the ϕ-matrix elements vary as a^4 and the number of elements contributing to the electron broadening of a Stark component varies with a. The relative contribution of the broadening of the $b=2$ state to the broadening of the initial state for the Balmer lines is therefore of the order $(2/a)^5$ and consequently is negligible except perhaps in the case of H_α.

ION BROADENING

Since the ions move slowly relative to the electrons, the impact approximation cannot be used to describe the ion broadening except in a narrow frequency interval near the line center.[6] Instead it is necessary to treat the ion field in the static approximation and to obtain the ion-field distribution function.

The instantaneous shift in the position of a particular Stark level $\delta\omega_{\alpha'\beta}$ corresponding to the ion field strength F is given by time-independent perturbation theory:

$$\delta\omega_{\alpha'\beta} = (e/\hbar)(\langle \alpha' | z | \alpha' \rangle - \langle \beta | z | \beta \rangle) F \equiv C_{\alpha'\beta} F. \quad (34)$$

The spectral distribution is found from the distribution of Stark splittings, which can be expressed in terms of the field-strength distribution function $W(F)$, describing the probability of finding the ion field between F and $F+dF$ as $W(F)dF$. With this, the spectral distribution in the statistical approximation is

$$J_{ab}(\Delta\omega) = \sum_{\alpha'\beta} |\langle \alpha' | \mu(0) | \beta \rangle|^2 W(F) \frac{dF}{d\delta\omega_{\alpha'\beta}} = \sum_{\alpha'\beta} \frac{1}{C_{\alpha'\beta}} |\langle \alpha' | \mu(0) | \beta \rangle|^2 W\left(\frac{\delta\omega_{\alpha'\beta}}{C_{\alpha'\beta}}\right). \quad (35)$$

The sum is over all the Stark components contributing to a line.

Lenz,[11] Burkhardt,[12] Spitzer,[13] and Holstein[14] pointed

[9] By expanding the exponentials in (2) one can show that the errors introduced are of the order

$$\Delta\omega\Delta\tau_c \Big/ \int_{y_{\min}}^{\infty} [\exp(-y)/y] dy,$$

which is never larger than 0.1 (see Table I).

[10] If the time integrations in (12) are extended only to a finite time given by $\rho^2 + v^2\beta^2 = \rho_D^2$, one obtains instead of (21)

$$\phi_a = -\pi N \int dv v f(v) \left\{ (\rho_w/\rho_D)^2 \rho_w^2 + \frac{4}{3} \frac{e^4}{\hbar^2} \frac{\mathbf{r}_a \cdot \mathbf{r}_a}{v^2} \ln\left(\frac{\rho_D}{\rho_w}\right) \right\},$$

i.e., the errors are smaller by a factor $(\rho_w/\rho_D)^2$ than estimated for strong collisions above [see Eq. (21)].

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[14] T. Holstein, Phys. Rev. **79**, 744 (1950).

out that the statistical theory is valid only when

$$\delta\omega_{\alpha'\beta}\Delta\tau_i \gg 1. \tag{36}$$

Here $\Delta\tau_i$ is a time characteristic for the ion field variations. For an ion passing by with an impact parameter ρ_i with a velocity v_i this time will be of the order ρ_i/v_i. The corresponding field strength is e/ρ_i^2, and with (34) we can rewrite the validity criterion:

$$(eC_{\alpha'\beta}\delta\omega_{\alpha'\beta}/v_i)^{\frac{1}{2}} \gg 1. \tag{37}$$

If we introduce a mean value $\bar{C}_{\alpha'\beta}=(\hbar/me)(a^2-b^2)$ for the Stark coefficients of a line whose initial and final states have principal quantum numbers a and b, this gives finally

$$(\hbar\delta\omega_{ab}(a^2-b^2)/mv_i^2)^{\frac{1}{2}} \gg 1. \tag{38}$$

This condition is generally satisfied for the ions except near the very core of the lines.

Holtsmark[2] derived the field-strength distribution by neglecting all correlations between the ions. Then a dimensional analysis showed that distribution functions for different ion densities N have the same shape and that they scale with $N^{\frac{2}{3}}$. If one introduces the normal field strength F_0 defined by

$$F_0 = 2.61eN^{\frac{2}{3}}, \tag{39}$$

the field-strength distribution function for any density can be expressed in terms of a universal distribution function $W_H(F/F_0)$ as follows:

$$W(F) = (1/F_0)W_H(F/F_0), \tag{40}$$

and the spectral distribution can be conveniently written as

$$J_{ab}(\Delta\omega) = \sum_{\alpha'\beta} \frac{1}{C_{\alpha'\beta}F_0} |\langle\alpha'|\mu|\beta\rangle|^2 W_H\left(\frac{\delta\omega_{\alpha'\beta}}{C_{\alpha'\beta}F_0}\right), \tag{41}$$

where we put $\mu \equiv \mu(0)$ for simplicity.

It is now customary to introduce a reduced wavelength scale[15] $\alpha = \Delta\lambda/F_0$. With $\Delta\lambda = (\Delta\omega/2\pi c)\lambda^2$ ($\Delta\lambda \ll \lambda$), (41) finally reduces to

$$\begin{aligned}[S_H(\alpha)]_{ab}d\alpha &= J_{ab}(\Delta\omega)d\Delta\omega \\ &= \sum_{\alpha'\beta} K_{\alpha'\beta}^{-1} |\langle\alpha'|\mu|\beta\rangle|^2 W_H(\alpha/K_{\alpha'\beta})d\alpha,\end{aligned} \tag{42}$$

where the constants $K_{\alpha'\beta}$ are defined by

$$K_{\alpha'\beta} = \frac{\lambda^2}{2\pi c}C_{\alpha'\beta} = \frac{e\lambda^2}{2\pi c\hbar}(\langle\alpha'|z|\alpha'\rangle - \langle\beta|z|\beta\rangle). \tag{43}$$

The functions $S_H(\alpha)$ for the first four Balmer lines were calculated by Verweij[16] and simultaneously by Schmaljohann,[17] who properly took into account the average over the polarization of the radiation. Asymptotic relations for the wing distributions were also obtained by these authors. Here the corresponding field-strengths are high and are therefore caused by an ion very close to the radiating atom. Then the effect of the other more distant ions can be neglected and the distribution function will be proportional to the probability of finding the ion inside a certain shell between r and $r+dr$:

$$W(F)dF \cong 4\pi r^2 dr/V, \tag{44}$$

where $F = e/r^2$ and the volume V available for one ion is $V = N^{-1}$. It follows that

$$W(F) \cong 2\pi N e^{\frac{3}{2}} F^{-\frac{5}{2}}, \tag{45a}$$

or

$$W_H(F/F_0) \cong 1.50(F/F_0)^{-\frac{5}{2}}. \tag{45b}$$

With Eqs. (42) and (45b) one finally obtains for the line wings:

$$[S_H(\alpha)]_{ab} \cong \frac{1.50}{\alpha^{\frac{5}{2}}} \sum_{\alpha'\beta} K_{\alpha'\beta}^{\frac{3}{2}} |\langle\alpha'|\mu|\beta\rangle|^2. \tag{46}$$

The most complete tables for the $S_H(\alpha)$ were calculated by Underhill and Waddell[18] for the lines forming the different series of the hydrogen spectrum.

Using the Markoff method, Holtsmark also derived the distribution function W_H for smaller values of F/F_0, taking into account the fields of all the ions, not just the nearest neighbor. However, in calculating the probability of weak fields, the assumption that the ions are statistically independent becomes questionable because it is usually the case that the thermal energies are not very large compared with the Coulomb interaction energies. Furthermore, the ion fields will be partially shielded by the electron clouds surrounding the ions. These effects can be accounted for approximately by using Debye screened fields instead of Coulomb fields for the contribution of individual ions to the field at the radiating atom. This procedure is valid only if the number of particles within the Deybe sphere is not too small. The distribution function obtained in this way will depend on N_D, the average number of particles within the Debye sphere. From (20) we find

$$N_D = \frac{4\pi}{3}\rho_D^3 N = \frac{1}{6\sqrt{\pi}}\left(\frac{kT}{e^2}\right)^{\frac{3}{2}} N^{-\frac{1}{2}}. \tag{47}$$

Ecker[1] has calculated the field-strength distribution function $W_E(F/F_0)$ as a function of a parameter $\delta \equiv 2^{-\frac{2}{3}}N_D$ using a Coulomb field with a cutoff at the

[15] The variable α and $S(\alpha)$ are introduced in conformity with standard astrophysical notation. The quantity α is not to be confused with the quantum numbers α'.

[16] S. Verweij, Publ. Astronomical Inst. Amsterdam, No. 5 (1936); dissertation, Amsterdam, 1936.

[17] P. Schmaljohann, State examination work, 1936 (unpublished); see also A. Unsöld, *Physik der Sternatmosphären* (Springer-Verlag, Berlin, 1955), Second edition, Chap. XI.

[18] A. Underhill and J. Waddell, National Bureau of Standards Circular No. 603 (U. S. Government Printing Office, Washington, D. C., 1959).

radius $\rho_D/\sqrt{2}$, instead of the exact Debye screened field. The factor $\sqrt{2}$ arises from the ion-ion correlations, i.e., an additional Boltzmann factor for the ions in the Poisson equation normally used to derive the screened Debye potential. The use of the usual Debye length as a cutoff takes into account only the screening and is appropriate as a cutoff parameter for electrons moving in a static ion field. Then the ion-ion correlations are irrelevant and the cutoff is larger by the factor $\sqrt{2}$.

The Ecker procedure can only be expected to yield satisfactory results when the ion fields are larger than the field produced by an ion at the Debye radius.

$$F/F_0 \gg F_D/F_0 = N_D^{-\frac{2}{3}}. \tag{48}$$

For smaller values of F/F_0 one obtains sufficiently accurate distribution functions by extrapolation from $F/F_0 \approx \delta^{-\frac{3}{2}}$ to $F/F_0=0$, maintaining the normalization $\int_0^\infty W(F/F_0)dF/F_0=1$. We used curves which were determined by Ecker by an extrapolation preserving the general character of the Holtsmark function, which is equivalent to drawing smooth curves from the origin to a point $F/F_0 \approx 2\delta^{-\frac{3}{2}}$. The extrapolation procedure actually employed is not critical. If one replaces Ecker's smooth distribution functions in Fig. 1 for $F/F_0 < 1.5\delta^{-\frac{3}{2}}$ by a step function determined by the normalization condition, the difference in the resulting line profiles is less than 10% under practically all circumstances even at the line center. This was shown numerically for H_β (see the following section) and is a consequence of the relative importance of the electron broadening. The lines with a central component, i.e., Ly-α, H_α and H_γ will be will be less affected and also H_δ, which has no central Stark component, is not seriously affected because its central dip is smaller than that of H_β. The only exception might be Ly-β so that measurements of this line could give direct information on the ion field-strength distribution function.

Theimer and Hoffman[19] and Ecker and Müller[20] also computed distribution functions using the complete Debye field. For large δ, these functions agree with Ecker's earlier results but for δ near unity the new distributions are still narrower. This is due to the shielding action of the electrons inside the Debye sphere. These electrons, however, give rise to impact broadening and it is more appropriate to omit them here, i.e., to use the cutoff method, so as not to use the electrons inside the Debye sphere twice in the calculation of the line profile.

Baranger and Mozer[21] have recently derived distribution functions using cluster integral expansions. For fields produced at a neutral point they essentially confirm the screened field results. Their curves are in between those obtained with ρ_D and $\rho_D/\sqrt{2}$ as screening parameter, the actual value of which is usually not critical. Therefore, this is additional evidence that the field-strength distribution functions used here are sufficiently accurate where their behavior is reflected in the resulting line profiles which include electron broadening.

ION AND ELECTRON BROADENING

The motion of the "fast" electrons and "slow" ions produces an electric field which behaves like a random function having rapid variations superimposed on a slowly varying component of comparable amplitude. The ion field will stay nearly constant in times which contain a number of electron collisions. This number is approximately the ratio of electron velocity to the ion velocity, v_e/v_i. In calculating the spectral distribution we now have to consider a sum of Fourier integrals of the following type:

$$J_K(\omega)=\sum_i \int_{t_i}^{t_{i+1}} dt \exp[-(i\Delta\omega - iCF_i(K)+\gamma)t]$$

$$=\sum_i \frac{\exp[-(i\Delta\omega - iCF_i(K)+\gamma)t_i]-\exp[-(i\Delta\omega + iCF_i(K)+\gamma)t_{i+1}]}{i(\Delta\omega - CF_i(K))+\gamma}. \tag{49}$$

Here $\Delta\omega$ stands for the frequency measured from the unperturbed line; C is a Stark-effect coefficient. $F_i(K)$ is the ion field strength at the Kth radiating atom in the time between t_i and t_{i+1} and γ is the electron damping constant. We have dropped the α',β indices for simplicity. The time interval $t_{i+1}-t_i$ is taken to be long enough to include many electron collisions but short compared to times in which the ion field changes appreciably. This permits the evaluation of the integrals. The observable profile is found from the sum over $J_K(\omega)$, averaging over $F_i(K)$. If the damping is large in times characteristic for the time variations of the ion field, only the term with $t_i=0$ contributes and the resulting profile is given by

$$J(\omega)=\mathrm{Re}\sum_K \frac{W(F_0(K))}{i[\Delta\omega - CF_0(K)]+\gamma}, \tag{50}$$

where $W(F)$ is the field-strength distribution function. We conclude that the statistical theory for ions can be applied to the entire line profile if γt_1 is large compared

[19] O. Theimer and H. Hoffman, Astrophys J. **127**, 477 (1958).
[20] G. Ecker and K. G. Müller, Z. Physik **153**, 317 (1958).
[21] M. Baranger and B. Mozer, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, Technical Report No. 2, Nonr-760(15), Office of Naval Research (1959) (to be published).

with unity. Now γ is of the order of the ϕ matrix elements multiplied by the number of interacting states, i.e., with (29), $\langle \mathbf{r}_a \cdot \mathbf{r}_a \rangle \cong a^4(\hbar^2/me^2)^2$ and the number of interacting states $\sim a$

$$\gamma \sim a\langle \phi_a \rangle \approx N\left(\frac{m}{kT}\right)^{\frac{1}{2}}\left(\frac{\hbar}{m}\right)^2 a^5 \int_{y_{\min}}^{\infty} dy \frac{\exp(-y)}{y}. \quad (51)$$

The time t_1 will be of the order $\rho/\bar{v}_i \approx N^{-\frac{1}{3}}(m_i/kT)^{\frac{1}{2}}$, the duration of an ion collision, so that we have

$$\langle \gamma t_1 \rangle_{\text{Av}} \approx N^{\frac{2}{3}} \frac{\hbar^2}{mkT}\left(\frac{m_i}{m}\right)^{\frac{1}{2}} a^5 \int_{y_{\min}}^{\infty} dy \frac{\exp(-y)}{y}. \quad (52)$$

For ion collisions for which $\gamma t_1 \approx 1$, the remaining terms in (49) cannot be neglected because of this amplitude argument. But after summation over the different atoms K they will still drop out if the phases $CF_i(K)t_1$ due to the static ion shifts are sufficiently large, because the $F_i(K)$ are random. $F_i(K)$ is of the order $eN^{\frac{2}{3}}$, C is typically $a^2\hbar/me$, so that with $t_1 \approx N^{-\frac{1}{3}}(m_i/kT)^{\frac{1}{2}}$ one finally has

$$\langle CF_i(K)t_1 \rangle_{\text{Av}} \approx \left(\frac{m_i}{kT}\right)^{\frac{1}{2}} \frac{\hbar}{m} a^2 N^{\frac{1}{3}}. \quad (53)$$

Inspection of Table II shows that $\langle \gamma t_1 \rangle_{\text{Av}}$ and $\langle CF_i(K)t_1 \rangle_{\text{Av}}$ are close to unity for the highest temperatures and lowest densities considered in this paper. The statistical theory therefore appears to be applicable for the ions in most cases even in the cores of the lines. It should be noted that this is only true if damping due to electrons is present, so that the summations over i and K can be interchanged. If this were not the case one could use the statistical theory only if the phases $\Delta\omega t_1$ are large, i.e., one would arrive at the well-known validity criterion discussed in the previous section and the statistical theory would not be valid close to the line center.[22] But in our case the effects of the ions can be taken into account by summing up the impact profiles calculated for constant ion fields. With the field-strength distribution function $W(F)$, this procedure yields the resultant line profile $J_{ab}(\omega)$:

$$J_{ab}(\omega) = \int_{-\infty}^{\infty} dF\, W(F) J_{ab}(\omega, F), \quad (54)$$

where $J_{ab}(\omega,F)$ is the electron impact profile determined by (33). It is again convenient to introduce $\alpha = \Delta\lambda/F_0$ as a measure of the distance from the line center and to write

$$S_{ab}(\alpha) = J_{ab}(\omega(\alpha))d\omega/d\alpha, \quad (55)$$

which with $\Delta\lambda = (\Delta\omega/2\pi c)\lambda^2$ and (33), (54), and (55) gives finally with Ecker's distribution $W_E(F)$

$$S_{ab}(\alpha) = \frac{1}{\pi} \sum_{m\alpha'\beta} \int_0^{\infty} df\, W_E(f)\, \mathrm{Re}\langle \beta|\mu|\alpha' m\rangle \times \left\langle \alpha' m \left| \frac{1}{i(\alpha - K_{\alpha' m\beta} f) + \gamma_a} \right| \alpha'' m \right\rangle \times \langle \alpha'' m|\mu|\beta\rangle, \quad (56)$$

with γ_a defined by

$$\gamma_a = \frac{\frac{1}{3}\lambda^2\hbar^2 N^{\frac{1}{3}}}{2.6ecm^2}\left(\frac{2m}{\pi kT}\right)^{\frac{1}{2}} \frac{\mathbf{r}_a \cdot \mathbf{r}_a}{a_0^2} \int_{y_{\min}}^{\infty} dy \frac{\exp(-y)}{y}. \quad (57)$$

The integrands are even functions of $f = F/F_0$ and the integration needs only be carried from 0 to ∞. Within our approximations the line profiles are symmetrical,[23] that is, it is sufficient to consider only positive values of α. The sum over the magnetic quantum number m is indicated, because states with different m are not mixed by the isotropic electron perturbation and can be treated separately. The quantum numbers α', α'' are now the usual parabolic quantum numbers k_1', k_2' and k_1'', k_2''. The calculations necessary, therefore, only involve matrix inversions of order a or smaller. The trace must be performed for each reduced wavelength α for a sufficient number of f values, so that the integration involving $W_E(f)$ can be performed numerically. Then the real parts are summed over the magnetic quantum numbers of the initial states and over all final states.

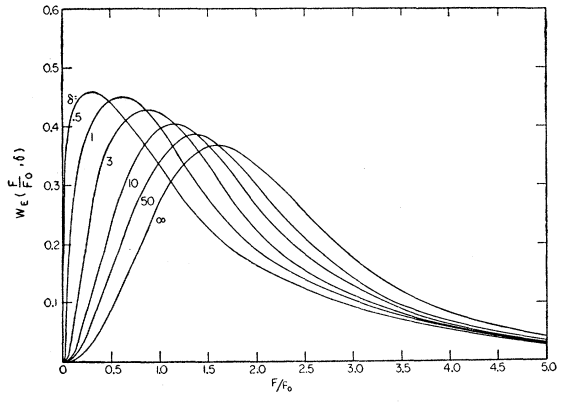

FIG. 1. Ecker's distribution function for the ion field for various values of δ which determines the average number of ions within a Debye sphere. $W(F/F_0, \infty)$ corresponds to the Holtsmark distribution.

TABLE II. Minimal values of $\langle \gamma t_1 \rangle_{\text{Av}}$ and $\langle CF_i(K)t_1 \rangle_{\text{Av}}$.

	Ly-α	Ly-β	H_α	H_β	H_γ	H_δ
$N_{\min}$[cm^{-3}]	10^{17}	10^{16}	10^{16}	10^{15}	10^{15}	10^{14}
$\langle \gamma t_1 \rangle_{\text{Av}}$	0.6	0.9	0.9	0.8	2.5	1.2
$\langle CF_i(K)t_1 \rangle_{\text{Av}}$	1.1	1.2	1.2	1.0	1.5	1.0

[22] This point is not of numerical importance, however, because the line profiles are not sensitive to the form of the ion-field distribution for small field strengths (see next section and Fig. 6).

[23] Slight asymmetries actually do occur and were discussed in H. Griem, Z. Physik **137**, 280 (1954).

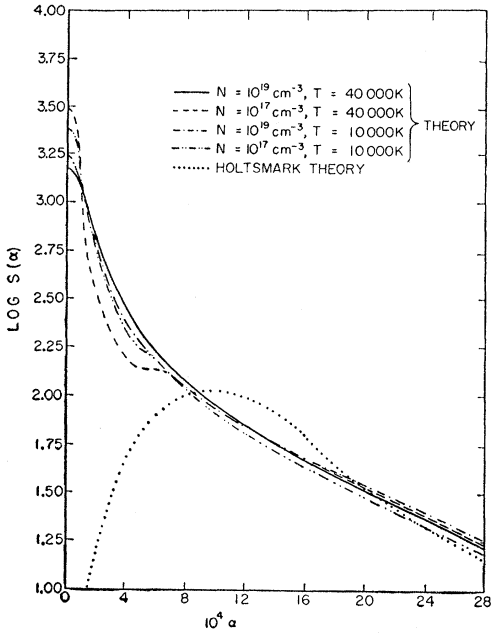

FIG. 2. Profile of the Ly-α line as a function of $\alpha=\Delta\lambda(\mathrm{A})/F_0(\mathrm{cgs})$.

Only positive m need to be considered and also only radiation polarized in the x and z direction, if proper weight factors are introduced.

The profiles of the first two Lyman lines and the first four Balmer lines were calculated from (56) for the temperatures $T=10^4$ °K, 2×10^4 °K and 4×10^4 °K and for a range of electron densities N, which are sufficiently large that the Doppler broadening is negligible and sufficiently small that the lines were still separated. The

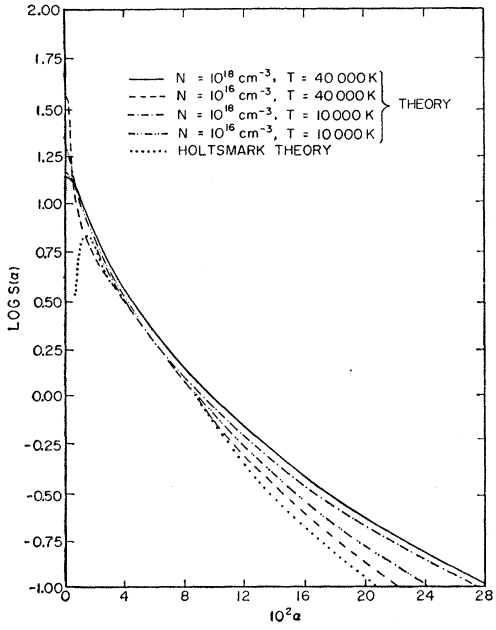

FIG. 3. Profile of H_α.

wavelength range was extended to α values for which the asymptotic formulas derived in the last section of this paper could be used without introducing errors of more than 20%.

Figures 2, 3, and 4 show some of the profiles of Ly-α, H_α and H_δ calculated in this way. The central component of H_α was corrected for the electron broadening of the $b=2$ states. The temperature dependence is quite small. The fact that the $S(\alpha)$ curves do not depend drastically on the density shows that the profiles still scale approximately with $N^{\frac{2}{3}}$, as in the Holtsmark theory. The deviations from the Holtsmark curves for H_α and H_γ are always appreciable and are even larger for Ly-α, since these lines have a central component that is not shifted in a static field.

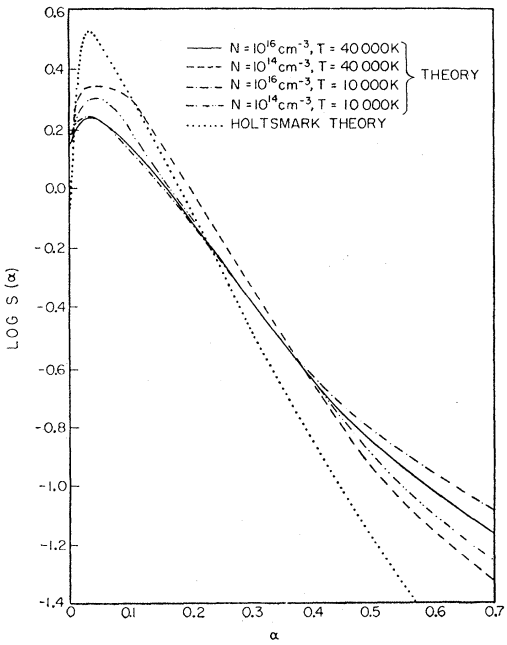

FIG. 4. Profile of H_δ.

In Fig. 5 the Holtsmark curve for H_β is compared with the statistical theory of Ecker in which the ion field strength distribution function is corrected for correlation and screening effects. For comparison, two other curves are shown which include the electron broadening; one being calculated with the Holtsmark field-strength distribution function and the other with the Ecker distribution. It is apparent that the effect of the electron broadening on the half-width is compensated to some extent by the electron shielding of the ion field and by ion-ion correlations. But otherwise the profiles are quite different. At large distances from the line center the two curves which were calculated by considering the effects of both electrons and ions agree quite well as expected, but are significantly higher than the two statistical curves which, of course, also approach one another for large values of α. It should be noted

that this comparison is only for one condition, namely $T=10\,000$ °K and $N=10^{16}$ cm^{-3}, but the general conclusions also apply to other cases because the general shape of the $\lambda S(\alpha)$ curves do not depend sensitively on either temperature or density in the range of interest.[24]

That the behavior of the field-strength distribution function for small field strengths is not critical is shown in Fig. 6, where two H_β profiles are compared which were obtained using the extreme possibilities for the extrapolated part of the field-strength distribution functions.

COMPARISON WITH EXPERIMENT

The most recent experimental data on Balmer-line profiles were obtained by Bogen,[3] who used a water-stabilized arc and made precision measurements of the H_β line profile with both photographic and photoelectric methods. The plasma densities and temperatures were calculated from measured absolute line intensities assuming thermal equilibrium. The assumption of local equilibrium is justified by the high densities and long experimental times. Figure 7 shows an experimental H_β line profile with $T=10\,400$ °K, $N=2.2\times10^{16}$ cm^{-3} and the corresponding theoretical curves. While the Holtsmark theory does not account for the spectral distribution, the complete theoretical and experimental curves agree within 10% over the whole intensity range of two orders of magnitude, i.e., within the numerical accuracy of the calculation and the experimental error.

The dip in the center of the experimental H_β curve is smaller than in the theoretical curve, which may be partly due to inhomogenieties that are present in an arc. This is borne out by the fact that the central dip in H_δ does not show up in the spectra from water-stabilized arcs, but is quite pronounced in shock tube spectra[25] where density gradients are not significant. Unfortunately, experimental profiles of comparable accuracy are not available for other than the Balmer series of hydrogen. A measurement of the first two lines of the Lyman series would be especially significant. Because the strong central Stark component is affected by electrons only, Ly-α would serve as a test case for the electron broadening theory. The Ly-β line, on the other hand, has no central component, so that the shape near the line center is only slightly influenced by electrons. It could, therefore, be used to determine the microfield distribution of the ions and to check the theory of the ion broadening.

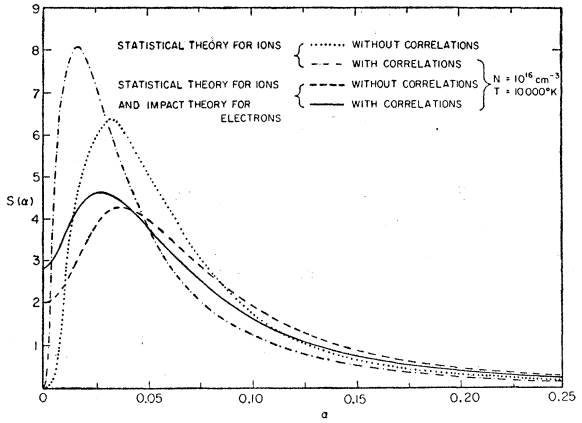

FIG. 5. Comparison of the various approximations for the H_β profile.

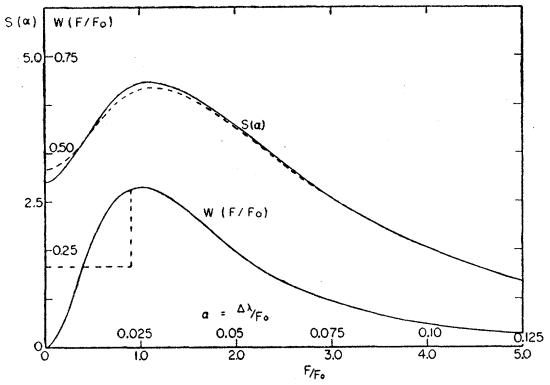

FIG. 6. Profiles $S(\alpha)$ for H_β using two extreme possibilities for the extrapolated part of the ion field strength distribution $W(F/F_0)$.

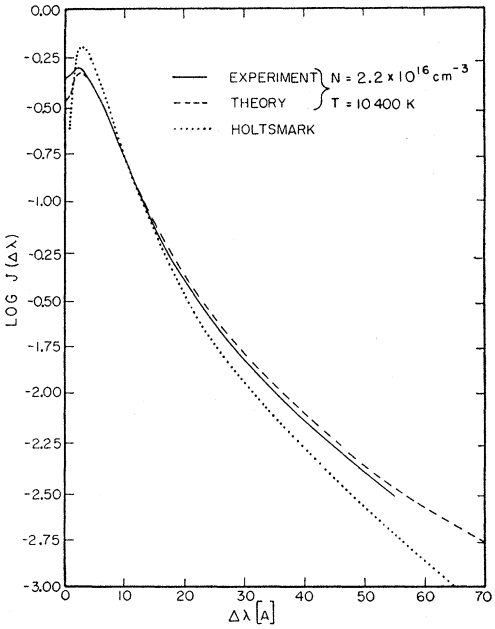

FIG. 7. Comparison between the theory for H_β and the experimental profile of Bogen.

[24] A complete set of curves for densities in the range $N=10^{14}-10^{19}$ cm^{-3} will be published as a Naval Research Laboratory report.

[25] E. B. Turner, dissertation, University of Michigan, 1956; University of Michigan Engineering Research Institute, ASTIA Document No. AD 86309 (unpublished).

The excellent agreement between the present theory and Bogen's data for H_β encourages the application of these results to deduce electron densities from measured hydrogen-line profiles. While the Holtsmark theory could be expected to yield electron densities only within a factor of two, the line-broadening theory may now be used to determine densities with relatively small errors. The accuracy now possible is quite adequate for most plasma experiments and astrophysical applications.

ASTROPHYSICAL APPLICATIONS

The Holtsmark theory has been used extensively in astrophysical applications to deduce electron densities and pressures from the profiles of the early members of the Balmer series, using temperatures obtained from relative line intensities. Since the observed radiation comes from layers with densities that vary with optical depth, a comparison between a line broadening theory and astrophysical observations is only possible after involved model atmosphere calculations. Verweij[16] was the first to employ this method, using the Holtsmark theory to deduce the structure of a stellar atmosphere. His theoretical line profiles did not agree too well in detail with the observational data, but he was able to explain qualitatively the absolute-magnitude effects in spectral classes B, A, and F. More recent calculations[26] indicate that the observed profiles can be forced to fit Holtsmark curves in the far wings, but that there are discrepancies between the theory and observations near the core of the lines and that the densities and therefore the surface gravities obtained from the line profiles are larger than those indicated by the mass and radius of the star. It was suggested[27] that the electrons might give a significant contribution to the line broadening and that the densities are accordingly smaller than those deduced using the Holtsmark theory. Recently a number of investigators[26,28–30] applied a revised line broadening theory[6] to the H_γ line emitted by B stars and found that consistent surface gravities resulted, if the contribution of the electrons to the absorption coefficient was taken into account. They estimated that the line absorption coefficient was raised by around 50% and found that one could calculate the shape function near the line core as well as in the far wing. The theory that they used can be obtained as an asymptotic limit of the present theory, but only the diagonal matrix elements of the perturbation were calculated in detail. The broadening caused by collision induced transitions was based on estimates of the magnitude of the off-diagonal matrix elements. In this section an explicit calculation, taking into account properly the effect of collision induced transitions, will be presented. The absorption coefficient for H_γ and the residual line intensity in B-type stars is found[31] to be slightly larger than that given previously by the approximate theory, but the general line shape is the same.

Because of the long optical paths encountered in stellar atmospheres the observed Balmer line profiles are only determined by the form of the wing of the theoretical absorption coefficients. The wing distribution follows from the general formula for the resultant line profile if we consider large values of α. Then only two ranges of f values contribute to the integral over f. At small f the factor $W(f)$ is large and the remaining factor in the integrand is nearly constant as a function of f. At f values given by $f=\alpha/K_{\alpha''m\beta}$ on the other hand, the second factor is large and the first factor is nearly constant. By splitting the integral and taking out the nearly constant terms, one can approximate $S_{ab}(\alpha)$ by[32]

$$S_{ab}(\alpha)=\frac{1}{\pi}\sum_{m\alpha'\alpha''\beta}\mathrm{Re}\Big[\Big(\int_0^{f_1}W(f)df\Big)\langle\beta|\mu|\alpha'm\rangle \times\langle\alpha'm|(i\alpha+\gamma)^{-1}|\alpha''m\rangle\langle\alpha''m|\mu|\beta\rangle +W\Big(\frac{\alpha}{K_{\alpha''m\beta}}\Big)\int_{f_1}^{\infty}\mathrm{Re}\langle\beta|\mu|\alpha'm\rangle \times\langle\alpha'm|[i(\alpha-K_{\alpha''m\beta}f)+\gamma]^{-1}|\alpha''m\rangle \times\langle\alpha''m|\mu|\beta\rangle df\Big]. \quad (59)$$

If now the limiting value f_1 is sufficiently large the first integral is unity because of the normalization, and if f_1 is still small compared to $f=\alpha/K_{\alpha''m\beta}$ the second integral becomes

$$(\pi/K_{\alpha''m\beta})W(\alpha/K_{\alpha''m\beta})|\langle\beta|\mu|\alpha''m\rangle|^2. \quad (60)$$

It is now consistent with these approximations to use the asymptotic expression for $W(f)$ given by (45) and to expand the electron impact contribution in powers of γ/α, retaining only the leading term. So finally we obtain

$$S_{ab}(\alpha)\cong\frac{1.5}{\alpha^{\frac{5}{2}}}\sum_{\alpha''m\beta}K_{\alpha''m\beta}^{\frac{3}{2}}|\langle\beta|\mu|\alpha''m\rangle|^2 +\frac{1}{\pi\alpha^2}\sum_{\alpha'\alpha''m\beta}\langle\beta|\mu|\alpha'm\rangle \times\langle\alpha'm|\gamma|\alpha''m\rangle\langle\alpha''m|\mu|\beta\rangle, \quad (61)$$

[26] For example, see the discussion in L. H. Aller and J. Jugaku, Astrophys. J. **128**, 616 (1958); also G. Traving, Z. Astrophys. **36**, 1 (1955); **41**, 215 (1957).
[27] G. J. Odgers, Astrophys. J. **116**, 445 (1952).
[28] Elste, Aller, and Jugaku, Publs. Astron. Soc. Pacific **68**, 23 (1956).
[29] K. Osawa, Astrophys. J. **123**, 512 (1956).
[30] H. Van Regemorter, dissertation, Paris Institute d'Astrophysique, 1958 (unpublished).
[31] L. H. Aller and J. Jugaku, Astrophys. J. **130**, 469 (1959).
[32] This result agrees with the first terms of an asymptotic series obtained earlier.[6]

TABLE III. Ratio $R(N,T)$ [$A^{-\frac{1}{2}}$] of electron to ion contribution to the absorption coefficient for the wings of the Balmer lines.

N[cm^{-3}]\T[°K]	$\frac{1}{2}\times10^4$	10^4	2×10^4	4×10^4	$\frac{1}{2}\times10^4$	10^4	2×10^4	4×10^4
		H_α				H_β		
10^{10}	1.50	1.05	0.79	0.60	1.39	1.05	0.80	0.60
10^{11}	1.34	0.93	0.71	0.54	1.21	0.93	0.71	0.54
10^{12}	1.17	0.82	0.63	0.48	1.04	0.81	0.62	0.48
10^{13}	1.01	0.70	0.54	0.42	0.86	0.68	0.54	0.42
10^{14}	0.85	0.59	0.46	0.36	0.69	0.56	0.45	0.35
10^{15}	0.68	0.47	0.38	0.30	0.51	0.44	0.36	0.29
10^{16}	0.52	0.35	0.30	0.25	0.34	0.31	0.27	0.23
10^{17}	0.35	0.24	0.22	0.19	0.17	0.19	0.19	0.17
10^{18}	···	0.12	0.14	0.13	···	0.07	0.10	0.11
		H_γ				H_δ		
10^{10}	1.79	1.37	1.04	0.79	2.17	1.66	1.27	0.96
10^{11}	1.56	1.20	0.92	0.70	1.87	1.45	1.12	0.85
10^{12}	1.32	1.03	0.80	0.62	1.57	1.24	0.97	0.75
10^{13}	1.08	0.87	0.68	0.53	1.27	1.03	0.82	0.64
10^{14}	0.84	0.70	0.57	0.45	0.97	0.81	0.67	0.54
10^{15}	0.61	0.53	0.45	0.37	0.67	0.60	0.52	0.43
10^{16}	0.38	0.36	0.33	0.28	0.37	0.39	0.37	0.32
10^{17}	···	0.20	0.21	0.20	···	···	···	···
10^{18}	···	···	···	···	···	···	···	···

or in terms of $\Delta\lambda=F_0\alpha$,

$$J_{ab}(\Delta\lambda)=\frac{1.5}{(\Delta\lambda)^{\frac{5}{2}}}\sum_{\alpha''m\beta}(F_0K_{\alpha''m\beta})^{\frac{3}{2}}|\langle\beta|\mu|\alpha''m\rangle|^2 + \frac{1}{\pi(\Delta\lambda)^2}\sum_{\alpha'\alpha''m\beta}\langle\beta|\mu|\alpha'm\rangle\langle\alpha'm|\gamma F_0|\alpha''m\rangle \times\langle\alpha''m|\mu|\beta\rangle\equiv J_i+J_e. \quad (62)$$

The first term J_i corresponds to the ion contribution and the second term J_e is due to the electron impacts. By factoring out the ion contribution, we can write the absorption coefficient in the form

$$a(\Delta\lambda)=a_i(\Delta\lambda)[1+R(N,T)(\Delta\lambda)^{\frac{1}{2}}]. \quad (63)$$

The coefficient $R(N,T)$, which is the ratio of the electron to ion contribution at a distance of one wavelength unit from the line center, is defined by

$$R(N,T)\equiv\frac{1}{(\Delta\lambda)^{\frac{1}{2}}}\frac{a_e(\Delta\lambda)}{a_i(\Delta\lambda)}=\frac{2(\hbar\lambda)^2(m/2\pi kT)^{\frac{1}{2}}}{3(2.6e)^{\frac{3}{2}}m^2c}\int_{y_{\min}}^{\infty}dy\frac{\exp(-y)}{y} \times\frac{\frac{1}{\pi}\sum_{\alpha'\alpha''m\beta}\langle\beta|\mu|\alpha'm\rangle\left\langle\alpha'm\left|\frac{\mathbf{r}_a\cdot\mathbf{r}_a}{a_0^2}\right|\alpha''m\right\rangle\langle\alpha''m|\mu|\beta\rangle}{1.5\sum_{\alpha''m\beta}(K_{\alpha''m\beta})^{\frac{3}{2}}|\langle\beta|\mu|\alpha''m\rangle|^2}. \quad (64)$$

Values of this coefficient calculated for the first four Balmer lines and for Ly-α and Ly-β are listed in Tables III and IV. An inspection of the tables shows that for values of astrophysical interest the term $R(N,T)(\Delta\lambda)^{\frac{1}{2}}$ is practically always of order unity and the absorption coefficients in the line wings are accordingly larger by a factor around two for a given electron density compared with the predictions of the Holtsmark theory. As mentioned earlier, this result removes an existing difficulty in the theory of stellar atmospheres.

TABLE IV. Ratio $R(N,T)$ [$A^{-\frac{1}{2}}$] of electron to ion contribution to the absorption coefficient for the wings of the Lyman lines.

N[cm^{-3}]\T[°K]	$\frac{1}{2}\times10^4$	10^4	Ly-α 2×10^4	4×10^4	$\frac{1}{2}\times10^4$	10^4	Ly-β 2×10^4	4×10^4
10^{10}	2.11	1.93	1.45	1.09	4.30	3.29	2.47	1.86
10^{12}	2.01	1.54	1.17	0.89	3.31	2.56	1.96	1.50
10^{14}	1.45	1.14	0.89	0.69	2.29	1.83	1.45	1.14
10^{16}	0.88	0.74	0.61	0.49	1.26	1.11	0.94	0.77
10^{17}	0.60	0.55	0.47	0.39	0.74	0.74	0.68	0.59
10^{18}	0.32	0.35	0.33	0.29	···	0.38	0.42	0.41

The coefficients $R(N,T)$ vary slowly with temperature and density. Values of this constant for conditions not represented in the table can therefore be easily found by interpolation or extrapolation. Corresponding calculations for higher series members of the Balmer lines are not yet available, but since the $R(N,T)$ depend smoothly on the principal quantum number of the upper state for H_β, H_γ, and H_δ, the absorption coefficients for other Balmer lines can be estimated by extrapolation.

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Conductivity of a Warm Plasma*

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A theory for obtaining the conductivity of a uniform plasma as a function of frequency and temperature is presented and compared with a number of recent treatments.

INTRODUCTION

RECENTLY, several different treatments of the high-frequency properties of an anisotropic plasma have appeared.[1–4] In each case the time-dependent part of the electron distribution function is obtained and then used to determine either the conductivity tensor or the propagation constant for a plane electromagnetic wave within the plasma. The forms of the conductivity tensor reported by these authors differ and the cause of the differences is not clear. It is the purpose of this discussion to indicate the nature of the differences or similarities in the various treatments.

FORMULATION OF THE PROBLEM

We consider the plasma to consist of electrons, positive ions, and neutral particles. In the absence of any electromagnetic disturbance, the plasma has a uniform density and is electrically neutral. For simplicity, we assume that in the presence of an electromagnetic field only the motion of the electrons is affected. The procedure for determining the properties of the plasma can be applied equally well when the motion of the ions is included; the contribution of the ions can be inferred from the results for electrons by noting the change of mass and charge. Within the plasma the electrons are described by their kinetic properties. Thus, the number of electrons at time t whose position and range of velocities lie within the interval $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ and $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ is given by $f(\mathbf{r},\mathbf{v},t)d^3rd^3v$. The electron distribution function, $f(\mathbf{r},\mathbf{v},t)$, must satisfy the Boltzmann equation[5]

$$\partial f/\partial t+\mathbf{v}\cdot\nabla_r f+(q/m)(\mathbf{E}+\mu_0\mathbf{v}\times\mathbf{H})\cdot\nabla_v f = -\nu(f-f_0). \quad (1)$$

Here $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are the electric field and magnetic intensity, respectively. The quantities $q=-|q|$ and m are the charge and mass of an electron, respectively. In MKS units, which will be used here, ϵ_0 and μ_0 are the characteristic constants of free space. The loss term, $-\nu(f-f_0)$, is included to conserve number density and momentum. For simplicity, the collision frequency, ν, is assumed to be independent of velocity. The removal of both this assumption and the limited loss term can be accomplished by following the method of Allis.[6]

We consider a plasma that is close to thermal equilibrium within which the following linearization condition holds:

$$\begin{aligned} f(\mathbf{r},\mathbf{v},t)&\sim f_0(v^2)+f_1(\mathbf{r},\mathbf{v},t),\\ \mathbf{E}(\mathbf{r},t)&\sim\mathbf{E}_1(\mathbf{r},t),\\ \mathbf{H}(\mathbf{r},t)&\sim H_0\hat{z}+\mathbf{H}_1(\mathbf{r},t), \end{aligned} \quad (2)$$

where $f_0(v^2)$, the distribution function in the absence of the electromagnetic disturbance, is chosen to be the Maxwell-Boltzmann distribution

$$f_0(v^2)=n(m/2\pi KT)^{\frac{3}{2}}\exp-(mv^2/2KT). \quad (3)$$

Here K and T are, respectively, Boltzmann's constant

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