probability for $X_\beta=1.19$; they are:

$$w_\mu=0.396\times10^3 \text{ sec} \quad \text{(assumption 1)}$$
$$0.317\times10^3 \text{ sec} \quad \text{(assumption 2)}$$
$$0.409\times10^3 \text{ sec} \quad \text{(assumption 3)},$$

where $(ft)_\beta=845$ sec is used according to Sec. V.

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Determination of the Pion-Nucleon Coupling Constant from n-p Scattering Angular Distribution*†

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By the use of a method recently proposed by Chew, the pion-nucleon coupling constant is determined from differential cross sections for neutron-proton scattering. Data at 90 and 400 Mev were used. Details of the extrapolation procedure are discussed and the statistical methods used in interpreting the results are explained. The resulting value of the coupling constant is between 0.06 and 0.07, depending on the range and energy of the data included in the analysis. The discrepancy between this value and the usually quoted 0.08 should not be taken seriously, however, because several nonstatistical uncertainties could not be taken into account. The origin of these uncertainties is discussed.

I. INTRODUCTION

In a recent paper Chew[1,2] suggested a method of determining the pion-nucleon coupling constant from differential cross sections for nucleon-nucleon scattering. The method is based on the conjectured occurrence of poles in the nucleon-nucleon scattering amplitude at certain unphysical values of the scattering angle. If p_1 and q_1 are the initial four-momenta of the nucleons, and p_2 and q_2 the final four-momenta, the momentum transfer is defined as $t=-(p_2-p_1)^2$ and the crossed momentum transfer as $\bar{t}=-(p_2-q_1)^2$. We use a metric such that $p^2=\mathbf{p}^2-E^2$, and our units are $\hbar=c=1$. There is then a pole at $t=\mu^2$ and another at $\bar{t}=\mu^2$, where μ is the pion mass. In terms of the barycentric scattering angle θ and the barycentric three-momentum p, the first of these poles[3] corresponds to $\cos\theta=+(1+\mu^2/2p^2)$, and the second to $\cos\theta=-(1+\mu^2/2p^2)$. If in the case of neutron-proton scattering one associates p_1 and p_2 with the proton, and q_1 and q_2 with the neutron, then in terms of Feynman diagrams the first pole gives the contribution of the exchange of a single neutral pion (forward scattering) whereas the second pole gives the contribution of the exchange of a single charged pion (charge-exchange scattering). In addition to the poles, one conjectures also the existence of branch points, corresponding to higher-order processes, when t or $\bar{t}$ becomes $(2\mu)^2$, $(3\mu)^2$, etc. In terms of $\cos\theta$ these branch points occur at

$$\cos\theta=\pm(1+4\mu^2/2p^2), \quad \pm(1+9\mu^2/2p^2), \quad \cdots. \tag{1}$$

They are thus considerably farther from the ends of the physical region than are the poles. One may therefore hope that the poles will tend to dominate the physical region, especially near the ends.[1] The residues of the poles are known exactly and are proportional to g^2, the pion-nucleon coupling constant. In fact the pole terms are formally identical with the two second-order one-pion exchange diagrams in perturbation theory. It must be pointed out, however, that we are not merely doing perturbation theory. In the first place the coupling constant and masses that are involved refer to actual physical particles and not to bare particles as would be the case in perturbation theory. Secondly, once the existence and position of the poles are accepted, their contribution to the scattering amplitude can be calculated without any reference whatever to perturbation theory, for instance by the method used by Goldberger, Nambu, and Oehme.[4]

Chew's suggestion then entails determining the resi-

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† A preliminary account of this work was given in Bull. Am. Phys. Soc. Ser. II, 3, 404 (1958).

[1] Geoffrey F. Chew, Phys. Rev. **112**, 1380 (1958).

[2] Geoffrey F. Chew, *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1958), p. 96.

[3] It might be instructive to point out that this pole also occurs in the classical Born approximation which is simply the Fourier transform of $g^2r^{-1}\exp(-\mu r)$. For a massless carrier of the field, the corresponding pole is *at* the edge of the physical region, which is the reason why the differential cross section for Coulomb scattering at 0° is infinite.

[4] Goldberger, Nambu, and Oehme, Ann. Phys. **2**, 226 (1957). See especially pp. 243 to 245.

due of a pole by multiplying the differential cross section by the pole term's denominator and extrapolating to the position of the pole.

It might be worth pointing out that the basic idea underlying the present procedure has also been used for other processes. The pion-nucleon coupling constant has been determined in this way by Taylor, Moravcsik, and Uretsky,[5] and the application to various reactions involving strange particles has been suggested by Taylor.[6] In particular, evidence for a pseudoscalar K^+ meson from photoproduction data on the basis of this procedure has been found by Moravcsik.[7] Finally the proposal by Chew and Low for measuring scattering amplitudes involving targets that do not exist in the laboratory is also akin, in spirit, to the above procedure.[8] It appears, therefore, that the idea of determining coupling constants or scattering amplitudes of one process by extrapolating the experimental differential cross sections of another process to the poles in the unphysical region is becoming a very powerful tool in elementary-particle physics.

II. SUMMARY OF THEORETICAL BACKGROUND

In this analysis we shall be concerned exclusively with the charge exchange pole of neutron-proton scattering. In case of proton-proton scattering the effects of the poles on the angular distribution appear to be masked by the Coulomb effect and by cancellations between the scalar amplitudes. In that case, therefore, a complete phase-shift analysis is required to obtain g^2. Indeed, such determination of the coupling constant has been one of the aims of the modified analysis of nucleon-nucleon scattering,[9,10] and a quite accurate value of g^2 has actually been obtained for 310-Mev p-p scattering.[11] In the case of n-p scattering, however, Coulomb effects are absent, and thus one can immediately see a rise toward each end of the physical region that may be taken as indicative of the presence of poles. We consider only the charge-exchange pole for two reasons. Firstly, there are no measurements available of n-p angular distribution in the forward direction, whereas a considerable amount of data has been taken in the backward direction. Secondly, since the coupling constant for charged pions is $(2)^{\frac{1}{2}}$ times that for neutral pions, the charge-exchange pole will be four times as strong as the other one.

An element in spin space of the p-n scattering amplitude may be written as

$$T_{r's',rs}=\frac{g^2m^2}{p^2E}\frac{\bar{u}_{r'}(p_2)\gamma_5u_s(q_1)\bar{u}_{s'}(q_2)\gamma_5u_r(p_1)}{x_0+\cos\theta}+G_{r's',rs}, \quad (2)$$

where p is the magnitude of the barycentric three-momentum, m the nucleon mass, $E^2=p^2+m^2$, and $x_0=1+(\mu^2/2p^2)$. Furthermore, g^2 is the pion-nucleon coupling constant such that $g^2=(2m/\mu)^2f^2$, and the expected value of f^2 is around 0.08. The quantity $G_{r's',rs}$ represents everything in the scattering amplitude except the charge-exchange pole. It will thus remain finite at $\cos\theta=-x_0$. The differential cross section is now given by

$$\sigma(\theta)=\tfrac{1}{4}\,\mathrm{Tr}(T^\dagger T),$$

and may be written as

$$\sigma(\theta)=\frac{g^4}{4E^2}\frac{(1+\cos\theta)^2}{(x_0+\cos\theta)^2}+\frac{A}{(x_0+\cos\theta)}+B, \quad (3)$$

where A and B are unknown functions of $\cos\theta$ and E which, however, are known to be finite at $\cos\theta=-x_0$. The analysis is made more difficult by the fact that the term in Eq. (3) in which we are interested vanishes at $\cos\theta=-1$, so that it tends to be small just in that region from which one might hope to extract the most information.

Multiplying Eq. (1) by x^2, where $x\equiv x_0+\cos\theta$, we obtain

$$y(x)\equiv(x_0+\cos\theta)^2\sigma(\theta)$$
$$=\frac{g^4}{4E^2}(1+\cos\theta)^2+Ax+Bx^2. \quad (4)$$

To the values of $y(x)$ calculated from Eq. (4), we make a least-squares fit by means of a polynomial of the form

$$f(x)=\sum_{k=0}^{n}a_kx^k. \quad (5)$$

From Eqs. (4) and (5), we see immediately that

$$a_0=f(0)=(g^4/4E^2)(1-x_0)^2=g^4\mu^4/16E^2p^4, \quad (6)$$

or

$$a_0=f^4m^4/[p^4(p^2+m^2)]. \quad (7)$$

III. STATISTICAL REMARKS

In order to discuss the errors in our results, we list here some of the relevant formulas from the theory of least squares. For a derivation of these formulas, we refer the reader to the work of Hildebrand[12] and Cziffra and Moravcsik.[13]

For each value of $i=1,2,\cdots,N$ let there be an abscissa x_i and a corresponding ordinate y_i, with an uncertainty ξ_i in the ordinate. We wish to fit these values with a polynomial of order n, of the form

$$f(x_i)=\sum_{k=0}^{n}a_kx_k{}^k. \quad (8)$$

[5] Taylor, Moravcsik, and Uretsky, Phys. Rev. **113**, 689 (1959).
[6] John G. Taylor, Nuclear Phys. **9**, 357 (1959).
[7] Michael J. Moravcsik, Phys. Rev. Letters **2**, 352 (1959).
[8] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).
[9] Michael J. Moravcsik, University of California Radiation Laboratory Report UCRL-5317-T, August, 1958 (unpublished).
[10] Cziffra, MacGregor, Moravcsik, and Stapp, Phys. Rev. **114**, 880 (1959).
[11] MacGregor, Moravcsik, and Stapp (to be published).
[12] F. B. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill Book Company, New York, 1956), Chap. 6.
[13] P. Cziffra and M. J. Moravcsik, University of California Radiation Laboratory Report UCRL-8523, October, 1958 (unpublished).

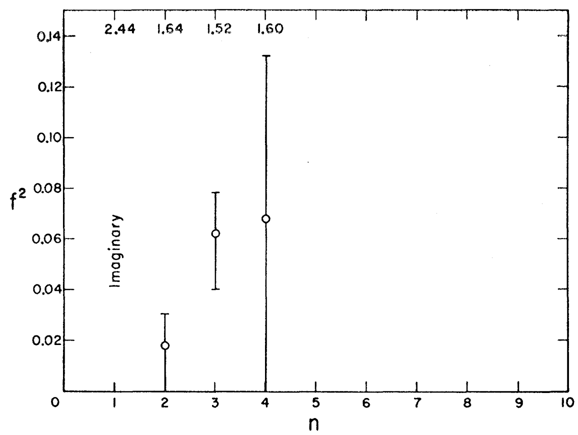

FIG. 1. Values of the pion-nucleon coupling constant f^2 *vs* degree of the polynomial (n) at 90 Mev using 24 experimental points, with scattering angles between 129° and 180°. The numbers above each point are the corresponding ρ^2 values. For $n=5$, $f^2=0.36_{-0.16}^{+0.11}$ with $\rho^2=1.52$.

To determine the a_k's, we minimize with respect to the a_k's the expression

$$Q\equiv \sum_{i=1}^{N} \xi_i^{-2}(y_i-\sum_{k=0}^{n} a_k x_k{}^k)^2, \tag{9}$$

which leads to the set of linear simultaneous equations

$$\sum_{j=0}^{n} H_{kj}a_j=v_k, \tag{10}$$

where

$$v_k=\sum_{i=1}^{n} y_i x_i{}^k \xi_i{}^{-2};\quad H_{kj}=\sum_{i=1}^{N} x_i{}^{j+k}\xi_i{}^{-2}. \tag{11}$$

Equation (10) may be formally solved as

$$a_j=\sum_{k=0}^{n} G_{jk}v_k, \tag{12}$$

where the matrix G is the inverse of the matrix H.

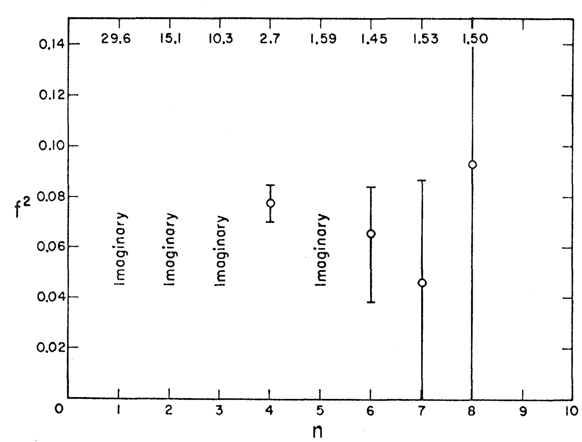

FIG. 2. Values of the pion-nucleon coupling constant f^2 *vs* degree of the polynomial (n) at 90 Mev using 48 experimental points, with scattering angles between 5.1° and 180°. The numbers above each point are the corresponding ρ^2 values.

It can now be shown that the error in the coefficient a_k due to the uncertainty in the data is given by[12,13]

$$\delta a_k=[G_{kk}\rho_n{}^2]^{\frac{1}{2}}, \tag{13}$$

where

$$\rho_n{}^2=(N-n-1)^{-1}\sum_{i=1}^{N} \xi_i{}^{-2}(y_i-\sum_{k=0}^{n} a_k x_i{}^k)^2. \tag{14}$$

For a given set of data as the degree of the polynomial is increased, $\rho_n{}^2$ at first decreases monotonically until it reaches a plateau on which it will fluctuate mildly. The value of n for which $\rho_n{}^2$ first reaches the plateau was taken to be the degree of the polynomial giving the best fit.[13]

IV. RESULTS

The calculation was performed at two different energies, 400 Mev and 90 Mev. The data were all taken from

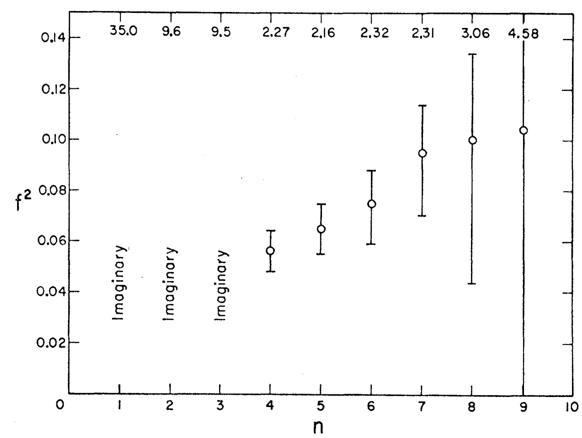

FIG. 3. Values of the pion-nucleon coupling constant f^2 *vs* degree of the polynomial (n) at 400 Mev using 12 experimental points, with scattering angles between 90° and 180°. The numbers above each point are the corresponding ρ^2 values.

the review article of Hess.[14] For each energy we made two sets of calculations, the first using the complete available range of the scattering angle, and the second using only a part of the range nearest to the pole. The results obtained are given in Figs. 1, 2, 3, and 4. The points marked "imaginary" indicate that a_0 (which is proportional to f^4) was negative for these polynomials. In Table I we list, for each of the four cases, the value of f^2 as given by the best-fitting polynomial.

In Fig. 5 the five polynomials obtained from the data at 90 Mev, with the limited range of scattering angle, are plotted. Shown also are most of the relevant experimental points.

[14] Wilmot N. Hess, Revs. Modern Phys. **30**, 368 (1958). At 90 Mev we used the data of O. Chamberlain and J. W. Easley, Phys. Rev. **94**, 208 (1954); J. Hadley *et al.*, Phys. Rev. **75**, 351 (1949); and R. H. Stahl and N. F. Ramsey, Phys. Rev. **96**, 1310 (1954). At 400 Mev we used the data of Hartzler, Siegel, and Opitz, Phys. Rev. **95**, 591 (1954).

The uncertainties as given in Table I and the figures are misleadingly small. Only the errors in the experimental cross sections as given by the experimentors have been taken into account. Two important sources of errors have therefore been omitted, since there is no way known to the authors to take them into account. The first is the fairly substantial energy spread of the neutron beam in the scattering experiment. This affects the position of the pole, the values of the differential cross sections, and finally the relationship between a_0 and f^2 [see Eq. (7)]. The second source of error is the considerable uncertainty in the degree of the polynomial that should be chosen. The $\rho_n{}^2$ test mentioned in Sec. III is certainly not conclusive. No trustworthy test can exist when one is trying to represent an essentially infinite series by means of a polynomial. The quoted errors were calculated merely by means of Eq. (13), with the ξ_i being the uncertainties quoted by Hess.[14]

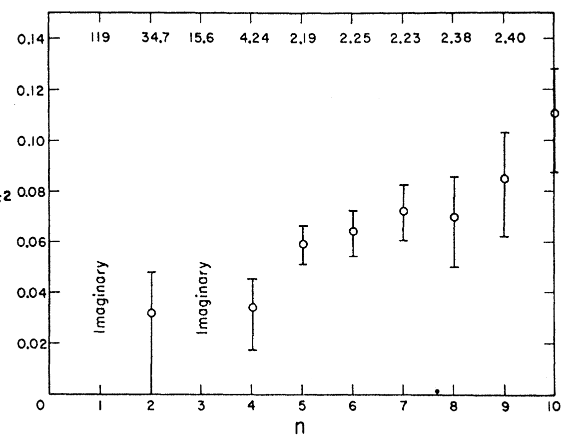

FIG. 4. Values of the pion-nucleon coupling constant f^2 *vs* degree of the polynomial (n) at 400 Mev using 23 experimental points, with scattering angles between 12.7° and 180°. The numbers above each point are the corresponding ρ^2 values.

Despite these difficulties the results as given in Figs. 1, 2, 3, and 4 are remarkably consistent. In practically all cases, once the goodness-of-fit parameter (ρ^2) dropped to a value indicating that a reasonably good fit has been achieved, the values of f^2 obtained from the different polynomials overlap. It appears that the method gives consistent results.

It will be noticed that the errors in f^2 for the 400-Mev data are much smaller than those for the 90-Mev data. The distance of extrapolation from the end of the physical region to the pole is $\mu^2/2p^2$, which is 0.052 at 400 Mev and 0.23 at 90 Mev. We see, therefore, that the former requires a much shorter extrapolation and would thus be expected to give the more accurate result.

Considering the inaccuracies inherent in the method, the values of the coupling constant obtained here are not too much at variance with the presently accepted values of $f^2=0.08$. To be sure, our values appear to be somewhat lower than 0.08, and these lower values agree well with the equally lower values obtained from the modified analysis of p-p scattering at 310 Mev.[11] Nevertheless we feel that more evidence is needed before we can say with any assurance that the pion-nucleon coupling constant as obtained from nucleon-nucleon scattering is in disagreement with the value obtained from pion-nucleon scattering and pion photoproduction.

It was mentioned in Sec. II that the coupling constant can also be obtained from the modified analysis of nucleon-nucleon scattering.[9–10] It might be illuminating to compare briefly these two methods.

Apart from its preferable theoretical features, the practical advantage of the modified analysis is that it uses, in addition to the unitarity of the S matrix, all experimental data, including results of triple scattering and correlation experiments. Thus, if such data are available, greater accuracy can be obtained in the determination. Similarly, it is also easier to use the statistical criteria to decide which order polynomial to take for the extrapolation, or, in the language of the modified analysis, how many angular momentum states to express in terms of phase shifts. Thus, for instance at

TABLE I. Determination of the pion-nucleon coupling constant f^2 from the angular distribution of n-p scattering at 90 and 400 Mev by the use of the polynomials giving the best fit.

Laboratory energy (Mev)	Number of experimental points on differential cross section	Range of scattering angle (deg)	Degree of polynomial giving best fit	f^2
90	24	180 to 129	3	$0.062_{-0.022}^{+0.016}$
90	48	180 to 5.1	6	$0.065_{-0.027}^{+0.019}$
400	12	180 to 90	5	$0.066_{-0.011}^{+0.009}$
400	23	180 to 12.7	5	$0.059_{-0.007}^{+0.007}$

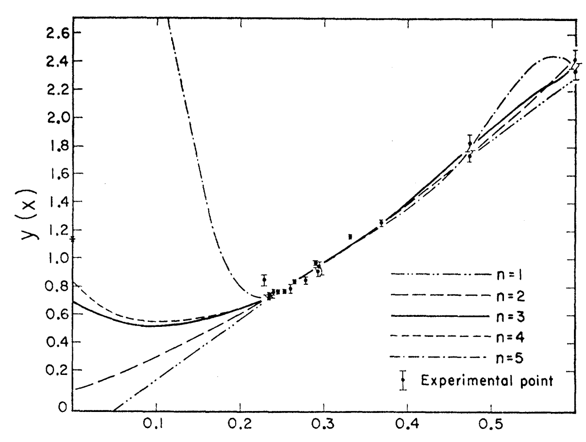

FIG. 5. Plot of $y(x)=(1+\mu^2/2p^2+\cos\theta)^2\sigma(\theta)$ *vs* $x=1+\mu^2/2p^2+\cos\theta$, for polynomials of degrees $n=1$, 2, 3, 4, and 5, using the data at 90 Mev with the limited range of the scattering angle. The experimental points are also shown except that some points near the end of the physical region are omitted. The end of the physical region occurs at $x=0.229$, and the asterisk indicates the residue corresponding to $f^2=0.08$.

310 Mev where fairly extensive data are available, a reliable determination of the coupling constant has been obtained with an error of 15% or so. Also, as has been mentioned, certain difficulties would be encountered in the use of the present scheme in the case of p-p scattering on account of the Coulomb effect which, however, can be easily taken care of in the modified analysis.

On the other hand, the modified analysis can be carried out only if sufficient number and kind of data are available. Thus, if only n-p scattering differential cross sections at 400 Mev are used, it would be completely out of the question to attempt the modified analysis. It is in such cases that the present method is useful. In addition, the modified analysis becomes complicated if not impossible at those energies at which inelastic processes play an important role. In these cases the present method, with all its uncertainties and limitations, will in fact be the only useful one. It has the advantage that nothing has to be known about the phase shifts, and that the presence of inelastic processes have no effect on it. Furthermore, its computational difficulties are far less serious than those of the modified analysis.

V. ACKNOWLEDGMENTS

We are grateful to Professor Geoffrey F. Chew for interesting discussions. The help of Dr. Richard von Holdt and Mr. Roy Clay of the Computation Division at Livermore in carrying out the numerical computations on an IBM-704 computer was also appreciated.

Note added in proof.—At the 1959 Annual International Conference on High-Energy Physics, N. S. Amaglobeli and Yu. M. Kazarimov of the U.S.S.R. reported result of n-p scattering experiments at 630 Mev. Using Chew's method of extrapolation they obtained $f^2 \approx 0.06$. We re-analyzed their data and obtained the results given in Table II. According to our criterion the best fit would occur for $\rho^2 = 1.12$ giving $f^2 = 0.040 \pm 0.015$. We would like to thank Mr. Rudolf Larsen for his assistance in analyzing these data.

TABLE II. Results of an extrapolation of the 630 n-p Russian data. Twenty experimental points extending from 11° to 180° in the barycentric system. "Imag." means that a_0 is negative and consequently f^2 would be imaginary.

Degree of Polynomial	1	2	3	4	5	6
f^2	Imag.	$0.024_{-0.024}^{+0.033}$	Imag.	Imag.	Imag.	0.039 ± 0.014
ρ^2	113	47	16.6	3.1	3.3	2.5
Degree of Polynomial	7	8	9	10	11	
f^2	Imag.	0.040 ± 0.015	0.032 ± 0.01	$0.035_{-0.035}^{+0.036}$	0.059 ± 0.025	
ρ^2	1.6	1.12	1.2	1.3	1.2	