Rigid Frames in Relativity*

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A new approach is presented for defining and obtaining rigid frames of reference. The results are shown to be equivalent to those of Rosen. The advantage of the present approach is that exact solutions can be obtained in certain simple cases, as well as approximate solutions in general.

1. INTRODUCTION

THE study of rigid-body motion dates back to Born's paper of 1909.[1] Rosen[2] proposed covariant conditions for rigid-body motion, which are equivalent to the Born conditions. In Sec. 2 of this paper, we propose a definition of a rigid frame of reference in the restricted case of flat two-dimensional space-time, which may be shown (Sec. 5) to be equivalent to Rosen's definition under the same restrictions. The present formulation allows us to find exact solutions for the rigid frames.

We define a rigid frame by requiring a constant spatial distance between any two points having fixed spatial coordinates. This rigid coordinate system is defined via a family of trajectories. Initially we choose an arbitrary family such that one member of the family is our original trajectory. From this we derive a differential equation whose solutions are the trajectories which define the rigid frame. Next, methods of solving this nonlinear partial differential equation are discussed. There still remains some ambiguity in the choice of this rigid frame of reference, depending upon the choice of the time coordinate. Two cases are treated; first, where the time is defined as the proper time along each trajectory, and second, time-orthogonal systems.

In Sec. 3, a specific example is considered, that of uniform acceleration. We digress briefly to discuss certain properties of the uniformly accelerated trajectories. We then illustrate the methods of Sec. 2 with this specific example.

In Sec. 4 we generalize our criteria for rigid motion to include arbitrary motion in arbitrary space-time (the criteria cannot, however, be satisfied in every case).

In Sec. 5 we prove the equivalence with Rosen's conditions. The geometric interpretation of Rosen's equations is clarified.

Finally an Appendix is devoted to a discussion of an approximate method of solving the differential equation of Sec. 2 for the transformation to the co-moving frame, for use in cases where the exact solution presents particularly difficult problems.

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[1] M. Born, Ann. Physik **30**, 1 (1909).

[2] N. Rosen, Phys. Rev. **71**, 54 (1947).

2. GENERAL THEORY OF TRANSFORMATION TO CO-MOVING REFERENCE FRAME

Consider an observer following a trajectory $x=f(t)$, and introduce the family of trajectories $x=f(t)+x_0$, where x_0 is a parameter. This can be considered as a coordinate transformation from the frame (x,t) to the frame (x_0,t). In the new coordinate system the equation of the original trajectory is $x_0=0$. The new metric obtained from $ds^2=dx^2-dt^2$ is

$$ds^2=dx_0{}^2+2\dot{f}dx_0dt-(1-\dot{f}^2)dt^2,\quad \dot{f}\equiv df/dt. \tag{2.1}$$

The general expression for spatial distance to a neighboring point is given by[3]

$$dl^2=\left(g_{rs}-\frac{g_{r4}g_{s4}}{g_{44}}\right)dx^r dx^s,\quad r,s=1,2,3. \tag{2.2}$$

In our case Eq. (2.2) takes the form

$$dl^2=\left(g_{11}-\frac{g_{14}{}^2}{g_{44}}\right)dx_0{}^2=\frac{dx_0{}^2}{1-\dot{f}^2}. \tag{2.3}$$

The value of dx_0 obtained from Eq. (2.3) by taking dl to be constant defines a trajectory infinitesimally near to the original trajectory $x_0=0$, with the property that the spatial distance between the two trajectories, as calculated from the point of view of the original observer, is constant. This trajectory, expressed in terms of the original coordinates (x,t), is

$$x=f(t)+(1-\dot{f}^2)^{\frac{1}{2}}dl. \tag{2.4}$$

Suppose we can find a family of trajectories $x=f(t,p)$ with the property that any two neighboring trajectories (that is, trajectories with infinitesimally differing values of p) are separated by constant spatial distance and such that $f(t,0)=f(t)$. If we apply the procedure which led to Eq. (2.4) to the equation $x=f(t,p)$ for a fixed but arbitrary value of p, we obtain

$$x=f(t,p)+[1-\dot{f}^2(t,p)]^{\frac{1}{2}}dl,\quad \dot{f}\equiv\partial f/\partial t. \tag{2.5}$$

Since this is assumed to be a separate member of the family, we have the result

$$\begin{aligned}f(t,p)+[1-\dot{f}^2(t,p)]^{\frac{1}{2}}dl&=f(t,\,p+dp)\\&=f(t,p)+f'(t,p)dp,\quad f'\equiv\partial f/\partial p,\end{aligned} \tag{2.6a}$$

[3] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Inc., Cambridge, Massachusetts, 1951), p. 258.

or

$$[1-\dot{f}^2]^{\frac{1}{2}}=f'dp/dl, \tag{2.6b}$$

where higher powers of dp have been neglected. If we fix the parameter p by the choice of $p=l$, we obtain the differential equation

$$\dot{f}^2+f'^2=1. \tag{2.7}$$

The solutions of this equation with the initial condition $f(t,0)=f(t)$ define a family of curves such that any one curve maintains a fixed spatial distance from any other curve.

The general solution to Eq. (2.7) is obtained[4] by eliminating α between the equations

$$f(t,l)=\alpha t+(1-\alpha^2)^{\frac{1}{2}}l+\phi(\alpha) \tag{2.8}$$

and

$$\frac{\partial f}{\partial\alpha}\equiv t-\frac{\alpha l}{(1-\alpha^2)^{\frac{1}{2}}}+\frac{d\phi}{d\alpha}=0, \tag{2.9}$$

where ϕ is a function to be determined by the initial data according to the following procedure. It is easily calculated from Eqs. (2.8) and (2.9) that

$$\dot{f}(t,l)=\alpha(t,l). \tag{2.10}$$

Since $f(t,0)$ is a known function $f(t)$, $\phi(\alpha)$ can be determined by setting $l=0$ in Eqs. (2.8) and (2.10) and eliminating t between them. Specifically, we have

$$f(t)=\alpha_0 t+\phi(\alpha_0),\quad \dot{f}(t)=\alpha_0,\quad \alpha_0\equiv\alpha(t,0). \tag{2.11}$$

Solving the second of Eqs. (2.11) for $t=t(\alpha_0)$ and substituting in the first, we get

$$f[t(\alpha_0)]=\alpha_0 t(\alpha_0)+\phi(\alpha_0). \tag{2.12}$$

Since the functional form of ϕ is independent of the particular value of α, we can drop the subscript 0 and obtain

$$\phi(\alpha)=f[t(\alpha)]-\alpha t(\alpha). \tag{2.13}$$

In practice it is often difficult to obtain this exact solution; therefore in the Appendix we discuss an approximate means of solving Eq. (2.7).

The transformation to the rigid coordinate frame is not uniquely determined until we have chosen the time coordinate. The most obvious choice on physical grounds would be to use the proper time along each trajectory. We may write the proper time as

$$\tau=\int_0^t(1-\dot{f}^2)^{\frac{1}{2}}dt=\int_0^t f'dt, \tag{2.14}$$

where the last equality holds by virtue of the differential Eq. (2.7). We may solve Eq. (2.14) for t as a function of τ and l, which we may write in the form

$$t=g(\tau,l). \tag{2.15a}$$

We may now use Eq. (2.15a) to eliminate t from the equation of the trajectories $x=f(t,l)$, obtaining x as a function of τ and l, namely

$$x=f[g(\tau,l),l]\equiv h(\tau,l). \tag{2.15b}$$

Equations (2.15) now determine the transformation from the frame (x,t) to the frame (l,τ). We may easily calculate the line element in the latter frame to be

$$ds^2=(h_l^2-g_l^2)dl^2+2(h_lh_\tau-g_lg_\tau)dld\tau-(g_\tau^2-h_\tau^2)d\tau^2, \tag{2.16}$$

where the subscript on h or g refers to the partial derivative with respect to that variable. It is possible to simplify Eq. (2.16) by noting that, since τ is the proper time along a trajectory $l=\text{const}$, the coefficient of $d\tau^2$ must be -1. This may be verified directly from Eqs. (2.15) and (2.14). We have from Eq. (2.15b),

$$h_\tau=\dot{f}g_\tau, \tag{2.17}$$

and from Eqs. (2.15a) and (2.14), we have

$$g_\tau=dt/d\tau=(1-\dot{f}^2)^{-\frac{1}{2}}. \tag{2.18}$$

Thus, we see that

$$g_\tau^2-h_\tau^2=g_\tau^2(1-\dot{f}^2)=1. \tag{2.19}$$

Hence the line element (2.16) assumes the form

$$ds^2=(h_l^2-g_l^2)dl^2+2(h_lh_\tau-g_lg_\tau)dld\tau-d\tau^2. \tag{2.20}$$

Furthermore, since the coordinate system (l,τ) is rigid, the application of Eq. (2.2) to the line element (2.20) results in the identity

$$(h_l^2-g_l^2)+(h_lh_\tau-g_lg_\tau)^2\equiv 1. \tag{2.21}$$

This identity may be verified explicitly in the same manner as Eq. (2.19).

We notice that the coordinate system we have introduced is not a time-orthogonal one. We shall now show that it is always possible to introduce a time-orthogonal system. Let us make a coordinate transformation from the frame (l,τ) to the frame $(l,\bar{t})$ according to

$$\tau=m(\bar{t},l). \tag{2.22}$$

The line element in this new frame is

$$ds^2=\left[(h_l^2-g_l^2)+2\frac{\partial m}{\partial l}(h_lh_\tau-g_lg_\tau)-\left(\frac{\partial m}{\partial l}\right)^2\right]dl^2+2\left[\frac{\partial m}{\partial\bar{t}}(h_lh_\tau-g_lg_\tau)-\frac{\partial m}{\partial l}\frac{\partial m}{\partial\bar{t}}\right]dld\bar{t}-\left(\frac{\partial m}{\partial\bar{t}}\right)^2d\bar{t}^2. \tag{2.23}$$

In order to obtain a time-orthogonal system, the function m must be chosen such that the coefficient of the cross term $dld\bar{t}$ vanishes. We thus see that the function

[4] The solutions of this equation are obtained through use of Charpit's method; see, for example, Frederic H. Miller, *Partial Differential Equations* (J. Wiley and Sons, Inc., New York, 1941), Secs. 44–45.

m is to be chosen in accordance with the requirements on its partial derivatives,

$$\partial m/\partial\bar{t}\neq 0, \quad \partial m/\partial l = h_l h_\tau - g_l g_\tau. \tag{2.24}$$

With the aid of Eqs. (2.24) and (2.21), the line element (2.23) takes the simple form

$$ds^2 = dl^2 - \left(\frac{\partial m}{\partial \bar{t}}\right)^2 d\bar{t}^2. \tag{2.25}$$

Once we have made a particular choice of the metric, and hence the rigid coordinate system, it is easy to predict the results of measurements made by the moving observer. For example, two observers at rest in such a system would see a Doppler shift which they would interpret as a gravitational effect.

3. UNIFORM ACCELERATION

Before applying the above procedure to uniform acceleration we shall digress a moment and discuss some properties of uniform acceleration in special relativity. Uniform acceleration is usually defined in terms of the trajectories

$$x^2 - t^2 = a^{-2}, \tag{3.1}$$

where a is the acceleration.[5]

It is well known that the hyperbolas (3.1) are invariant under the homogeneous Lorentz transformations (taking $c=1$)

$$x' = \frac{x - vt}{(1-v^2)^{\frac{1}{2}}}, \quad t' = \frac{t - vx}{(1-v^2)^{\frac{1}{2}}}. \tag{3.2}$$

Hence two observers connected by such a transformation would agree as to the numerical value of the acceleration.[6] We may prove the stronger statement that the hyperbolas (3.1) are the only curves invariant under the Lorentz transformations (3.2). It is clear that the following two conditions are necessary for the invariance of any curve under the transformations (3.2): (a) The x intercept of the curve must be invariant; (b) the slope with which the curve crosses the x axis must be invariant. We shall show that conditions (a) and (b) lead uniquely to the family (3.1) and hence they are also sufficient conditions for the invariance. Consider two coordinate systems (x,t) and (x',t') connected by one of the Lorentz transformations (3.2) corresponding to a particular value of v. Next consider a particle following a trajectory that satisfies condition (b). Then if at $t=0$ in the original coordinate system the particle's velocity was v_0 its velocity at $t'=0$ as measured in the other coordinate system must also be v_0. The particle's velocity dx/dt in the original system at the time t which corresponds to $t'=0$ may be found by the usual relativistic law for the composition of velocities to be

$$\frac{dx}{dt} = \frac{v + v_0}{1 + vv_0}. \tag{3.3}$$

We see from the second of Eqs. (3.2) that the time $t'=0$ corresponds to the time

$$t = vx \tag{3.4}$$

in the original system. If we use Eq. (3.4) to eliminate v from Eq. (3.3), we obtain

$$\frac{dx}{dt} = \frac{t + v_0 x}{x + v_0 t}. \tag{3.5}$$

The general solution of the differential Eq. (3.5) is

$$(x+t)^{1-v_0}(x-t)^{1+v_0} = a^{-2}. \tag{3.6}$$

We must now impose condition (a). It is seen from Eq. (3.6) that if $t=0$ we have $x=\pm a^{-1}$. Thus condition (a) will only be satisfied if the curves represented by

$$\left(\frac{x-vt}{(1-v^2)^{\frac{1}{2}}} + \frac{t-vx}{(1-v^2)^{\frac{1}{2}}}\right)^{1-v_0} \times \left(\frac{x-vt}{(1-v^2)^{\frac{1}{2}}} - \frac{t-vx}{(1-v^2)^{\frac{1}{2}}}\right)^{1+v_0} = a^{-2} \tag{3.7}$$

also have an x intercept of $\pm a^{-1}$. If we substitute $t=0$, $x=\pm a^{-1}$ into Eq. (3.7) we find

$$(1-v)^{1-v_0}(1+v)^{1+v_0} = 1 - v^2. \tag{3.8}$$

The only solution of Eq. (3.8) for arbitrary v is $v_0=0$. Putting this value of v_0 back into Eq. (3.6) we obtain precisely Eq. (3.1).

We shall now illustrate the methods of Sec. 2 with the example of uniform acceleration. Picking a particular trajectory from the family (3.1), corresponding to a fixed but arbitrary $a=a_0$, we shall determine a rigid frame of reference attached to it. From the second of Eqs. (2.11) we must solve for $t=t(\alpha_0)$. Since

$$f(t) = (t^2 + a_0^{-2})^{\frac{1}{2}}, \tag{3.9}$$

we have

$$\dot{f}(t) = t(t^2 + a_0^{-2})^{-\frac{1}{2}} = \alpha_0, \tag{3.10}$$

and hence

$$t = \frac{\alpha_0}{a_0(1-\alpha_0^2)^{\frac{1}{2}}}. \tag{3.11}$$

We see from Eqs. (3.9) and (3.11) that Eq. (2.13) takes the form

$$\phi(\alpha) = \left[\frac{\alpha^2}{a_0^2(1-\alpha^2)} + \frac{1}{a_0^2}\right]^{\frac{1}{2}} - \frac{\alpha^2}{a_0(1-\alpha^2)^{\frac{1}{2}}} = \frac{1}{a_0}(1-\alpha^2)^{\frac{1}{2}}. \tag{3.12}$$

[5] For a complete discussion of ways of defining uniform acceleration, see L. Marder, Proc. Cambridge Phil. Soc. **53**, 194 (1957).

[6] Under inhomogeneous Lorentz transformations the acceleration is also the same but the trajectories are displaced.

The solution (2.8), (2.9) to the differential equation becomes

$$f(t,l)=\alpha t+(1-\alpha^2)^{\frac{1}{2}}\left(l+\frac{1}{a_0}\right),$$

$$t-\frac{\alpha}{(1-\alpha^2)^{\frac{1}{2}}}\left(l+\frac{1}{a_0}\right)=0. \quad (3.13)$$

Elimination of α between the two Eqs. (3.13) gives

$$f(t,l)=[t^2+(l+a_0^{-1})^2]^{\frac{1}{2}}. \quad (3.14)$$

This is the same family of trajectories as Eq. (3.1). The parameter l is the distance, as seen by the moving observers, between the trajectories specified by $l=0$ and a given value of l. By comparison of Eq. (3.14) with Eq. (3.1), we see that the distance between any two curves is given by the difference of reciprocal accelerations.

If we choose the time coordinate in the rigid frame as the proper time τ, the transformation to the rigid frame is derived in the following manner. The relationship between τ and t, obtained from Eq. (2.14), is

$$\tau=\int_0^t\frac{(l+a_0^{-1})dt}{[t^2+(l+a_0^{-1})^2]^{\frac{1}{2}}}=(l+a_0^{-1})\sinh^{-1}\frac{t}{l+a_0^{-1}}. \quad (3.15)$$

Thus the first transformation equation (2.15a) becomes

$$t=(l+a_0^{-1})\sinh\frac{\tau}{(l+a_0^{-1})}. \quad (3.16a)$$

If we substitute this into Eq. (3.14), the other transformation equation (2.15b) becomes

$$x=(l+a_0^{-1})\cosh\frac{\tau}{(l+a_0^{-1})}. \quad (3.16b)$$

Using Eqs. (3.16), we can easily calculate that the metric is specified by

$$ds^2=\left[1-\frac{\tau^2}{(l+a_0^{-1})^2}\right]dl^2+\frac{2\tau}{l+a_0^{-1}}dld\tau-d\tau^2. \quad (3.17)$$

In order to pass to a time-orthogonal system we must make a transformation of the form $\tau=m(\bar{t},l)$, subject to the requirements (2.14). In the present example, these requirements are

$$\frac{\partial m}{\partial \bar{t}}\neq 0, \quad \frac{\partial m}{\partial l}=\frac{m}{l+a_0^{-1}}. \quad (3.18)$$

Equations (3.18) may be integrated to give

$$\tau=m(\bar{t},l)=(l+a_0^{-1})n(\bar{t}), \quad (3.19)$$

where $n(\bar{t})$ is an arbitrary function. The line element in terms of l and $\bar{t}$ is easily found to be

$$ds^2=dl^2-\left[\left(l+\frac{1}{a_0}\right)\frac{dn}{d\bar{t}}\right]^2d\bar{t}^2. \quad (3.20)$$

4. GENERALIZATION

In order to generalize the definition of rigid motion given in Sec. 2, we return to the Landau and Lifshitz expression for spatial distance:

$$dl^2=\left(g_{rs}-\frac{g_{r4}g_{s4}}{g_{44}}\right)dx^rdx^s. \quad (4.1)$$

This equation, which expresses the spatial distance from an observer at rest (i.e., with fixed x^r) to a neighboring point, is valid in an arbitrary Riemannian space. The term "rigid frame" implies that the world lines of two particles with fixed spatial coordinates maintain a constant spatial separation; in other words, in a rigid frame if $dx^r/dt=0$ then $dl/dt=0$. Thus time differentiation of Eq. (4.1) gives us

$$\left(g_{rs}-\frac{g_{r4}g_{s4}}{g_{44}}\right)_{,4}dx^rdx^s=0, \quad (4.2)$$

where the subscript ",4" denotes differentiation with respect to x^4. Since the dx^r are arbitrary time-independent quantities, we obtain

$$\left(g_{rs}-\frac{g_{r4}g_{s4}}{g_{44}}\right)_{,4}=0. \quad (4.3)$$

A rigid frame is thus one whose metric tensor satisfies Eqs. (4.3).

The following question arises: Starting in an arbitrary frame, with metric $\bar{g}_{\mu\nu}$, how would one find a coordinate transformation to a rigid frame, whose metric $g_{\mu\nu}$ satisfies Eqs. (4.3)? Clearly, the transformations would have to be solutions of the differential equations obtained by substituting

$$g_{\mu\nu}=\frac{\partial\bar{x}^\rho}{\partial x^\mu}\frac{\partial\bar{x}^\sigma}{\partial x^\nu}\bar{g}_{\rho\sigma}, \quad (4.4)$$

into Eqs. (4.3), where $\bar{x}^\mu=\bar{x}^\mu(x^1,\cdots x^4)$ is the desired transformation law. It is easily verified that the transformations of Sec. 2 are solutions of these equations.

5. ROSEN'S CONDITIONS

Rosen's point of departure for relativistic rigidity is the classical definition of a rigid body as one with vanishing strain tensor. By generalizing the classical expression for the strain, he obtains the equation

$$u_{\mu;\nu}+u_{\nu;\mu}-u_{\mu;\alpha}u^\alpha u_\nu-u_{\nu;\alpha}u^\alpha u_\mu=0 \quad (5.1)$$

as his rigidity condition, where u^μ is the unit timelike velocity vector, and the semicolon denotes covariant differentiation.

We can understand the geometrical significance of Eq. (5.1) more clearly by rewriting it in the form

$$(u_{\alpha;\beta}+u_{\beta;\alpha})(\delta_\mu{}^\alpha-u^\alpha u_\mu)(\delta_\nu{}^\beta-u^\beta u_\nu)=0, \qquad (5.2)$$

which we can recognize as Killing's equation projected into the subspace orthogonal to u^μ.

The connection between Eq. (5.2) and rigid motion can be better understood if we first recall a variational-principle derivation of Killing's equation. Consider a family of trajectories with the field of unit tangent vectors u^μ, the individual members of the family being specified by a parameter λ. The four-dimensional distance along an arbitrary curve between any two members of this family is given by

$$\int_{\lambda_1}^{\lambda_2} ds=\int_{\lambda_1}^{\lambda_2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{\frac{1}{2}}d\lambda. \qquad (5.3)$$

By requiring this distance to be stationary under uniform translation of the arbitrary curve in the direction of u^μ, we obtain the variational principle

$$\delta\int_{\lambda_1}^{\lambda_2}\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{\frac{1}{2}}d\lambda =\int_{\lambda_1}^{\lambda_2}\left[\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{\frac{1}{2}}\right]_{,\alpha}\delta x^\alpha d\lambda=0, \qquad (5.4)$$

where

$$\delta x^\alpha=\epsilon u^\alpha.$$

Upon carrying out the indicated operations and simplifying, we obtain Killing's equation,

$$u_{\mu;\nu}+u_{\nu;\mu}=0, \qquad (5.5)$$

as necessary and sufficient conditions for the four-dimensional distance between the curves to be constant.

Let us consider the expression

$$dl^2=\gamma_{\mu\nu}dx^\mu dx^\nu, \quad \text{where} \quad \gamma_{\mu\nu}=g_{\mu\nu}-u_\mu u_\nu. \qquad (5.6)$$

Since $\gamma_{\mu\nu}$ is a projection operator into the space orthogonal to the timelike vector u^μ, dl is usually referred to as spatial distance. If we now change the foregoing derivation of Killing's equation by requiring that $\int_{\lambda_1}^{\lambda_2} dl$ be stationary, we obtain the conditions (5.2).[7] We thus see that Rosen's condition for rigid motion is just the requirement of constant spatial distance in the above sense.

It seems clear on physical grounds that our approach in Sec. 4 is equivalent to that of Rosen's; the only difference being that the $g_{\mu\nu}$ occurring in Eq. (5.6) is the metric in an arbitrary frame, whereas Eq. (4.1) is valid only in a co-moving frame. In order to prove the equivalence explicitly, we substitute u^μ, expressed in a co-moving reference frame, into Rosen's condition (5.1). Writing $u^\mu=a\delta_4{}^\mu$, where a is evaluated through the requirement $u^\mu u_\mu=1$, we obtain

$$u^\mu=g_{44}{}^{-\frac{1}{2}}\delta_4{}^\mu, \quad u_\mu=g_{44}{}^{-\frac{1}{2}}g_{\mu4}. \qquad (5.7)$$

Substitution of (5.7) into (5.1) yields, after simplification,

$$\left(g_{\mu\nu}-\frac{g_{\mu4}g_{\nu4}}{g_{44}}\right)_{,4}=0. \qquad (5.8)$$

If either μ or ν has the value 4 in Eq. (5.8), the left-hand side is identically zero; hence (5.8) reduces to our previous condition (4.3).

[7] This approach to the projected Killing's equation is due to P. G. Bergmann (private communication).

6. CONCLUSION

Since solutions to the projected Killing's equation do not, in general, exist in an arbitrary space, rigid frames of reference do not usually exist. Even in spaces that permit solutions, a rigid frame cannot be attached to an observer following an arbitrary trajectory. One advantage of the approach presented in this paper is that approximately rigid frames in the immediate vicinity of an arbitrary observer may be found by a straightforward generalization of the procedure given in the Appendix. One result of this somewhat lengthy calculation is that first order solutions for rigid motion can always be found; at higher orders, however, the strict rigidity conditions cannot be maintained.

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APPENDIX

The approximation procedure outlined here will allow one to find the rigid frame only in the immediate vicinity of the original trajectory $x=f(t)$. We begin by expanding $f(t,l)$ in a power series in l, thus obtaining

$$f(t,l)=\sum_{n=0}^{\infty}\frac{l^n}{n!}f_l{}^{(n)}, \quad f_l{}^{(n)}\equiv\left.\frac{\partial^n f}{\partial l^n}\right|_{l=0}. \qquad (A.1)$$

The problem is to determine $f_l{}^{(n)}$ in terms of the initial condition $f(t,0)=f(t)$ and its higher time derivatives. Let us substitute the expansion (A.1) into the differential Eq. (2.7) and equate like powers of l. The resulting set of equations is

$$\dot f^2(t,0)+f'^2(t,0)=1,$$

$$\sum_{n=0}^{j}\frac{1}{n!(j-n)!}(f_l{}^{(n+1)}f_l{}^{(j-n+1)}+\dot f_l{}^{(n)}\dot f_l{}^{(j-n)})=0, \quad j>0. \qquad (A.2)$$

This set of algebraic equations determines the $f_l{}^{(n)}$ in terms of $f(t)$ and its higher derivatives. The power expansion up to terms in l^2, obtained through the use of Eq. (A.2), is

$$f(t,l)=f(t)+l[1-\dot f^2(t)]^{\frac{1}{2}}+\tfrac{1}{2}l^2\ddot f^2(t)[1-\dot f^2(t)]^{-1}+\cdots. \qquad (A.3)$$