The coefficients $R(N,T)$ vary slowly with temperature and density. Values of this constant for conditions not represented in the table can therefore be easily found by interpolation or extrapolation. Corresponding calculations for higher series members of the Balmer lines are not yet available, but since the $R(N,T)$ depend smoothly on the principal quantum number of the upper state for H_β, H_γ, and H_δ, the absorption coefficients for other Balmer lines can be estimated by extrapolation.

ACKNOWLEDGMENT

We gratefully acknowledge the assistance of Dr. R. J. Arms of the National Bureau of Standards who devised a general code for the evaluation of $S(\alpha)$, Eq. (56), on the National Bureau of Standards IBM 704.

PHYSICAL REVIEW VOLUME 116, NUMBER 1 OCTOBER 1, 1959

Conductivity of a Warm Plasma*

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(Received May 15, 1959)

A theory for obtaining the conductivity of a uniform plasma as a function of frequency and temperature is presented and compared with a number of recent treatments.

INTRODUCTION

RECENTLY, several different treatments of the high-frequency properties of an anisotropic plasma have appeared.[1–4] In each case the time-dependent part of the electron distribution function is obtained and then used to determine either the conductivity tensor or the propagation constant for a plane electromagnetic wave within the plasma. The forms of the conductivity tensor reported by these authors differ and the cause of the differences is not clear. It is the purpose of this discussion to indicate the nature of the differences or similarities in the various treatments.

FORMULATION OF THE PROBLEM

We consider the plasma to consist of electrons, positive ions, and neutral particles. In the absence of any electromagnetic disturbance, the plasma has a uniform density and is electrically neutral. For simplicity, we assume that in the presence of an electromagnetic field only the motion of the electrons is affected. The procedure for determining the properties of the plasma can be applied equally well when the motion of the ions is included; the contribution of the ions can be inferred from the results for electrons by noting the change of mass and charge. Within the plasma the electrons are described by their kinetic properties. Thus, the number of electrons at time t whose position and range of velocities lie within the interval $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ and $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ is given by $f(\mathbf{r},\mathbf{v},t)d^3rd^3v$. The electron distribution function, $f(\mathbf{r},\mathbf{v},t)$, must satisfy the Boltzmann equation[5]

$$\partial f/\partial t+\mathbf{v}\cdot\nabla_r f+(q/m)(\mathbf{E}+\mu_0\mathbf{v}\times\mathbf{H})\cdot\nabla_v f =-\nu(f-f_0). \quad (1)$$

Here $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ are the electric field and magnetic intensity, respectively. The quantities $q=-|q|$ and m are the charge and mass of an electron, respectively. In MKS units, which will be used here, ϵ_0 and μ_0 are the characteristic constants of free space. The loss term, $-\nu(f-f_0)$, is included to conserve number density and momentum. For simplicity, the collision frequency, ν, is assumed to be independent of velocity. The removal of both this assumption and the limited loss term can be accomplished by following the method of Allis.[6]

We consider a plasma that is close to thermal equilibrium within which the following linearization condition holds:

$$\begin{aligned} f(\mathbf{r},\mathbf{v},t)&\sim f_0(v^2)+f_1(\mathbf{r},\mathbf{v},t),\\ \mathbf{E}(\mathbf{r},t)&\sim \mathbf{E}_1(\mathbf{r},t),\\ \mathbf{H}(\mathbf{r},t)&\sim H_0\hat{z}+\mathbf{H}_1(\mathbf{r},t),\end{aligned} \quad (2)$$

where $f_0(v^2)$, the distribution function in the absence of the electromagnetic disturbance, is chosen to be the Maxwell-Boltzmann distribution

$$f_0(v^2)=n(m/2\pi KT)^{\frac{3}{2}}\exp-(mv^2/2KT). \quad (3)$$

Here K and T are, respectively, Boltzmann's constant

* This work was supported in part by the U. S. Atomic Energy Commission.
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and the electron gas temperature, and n is the number density of electrons within the plasma. If we retain terms linear in f_1 and $\mathbf{E}_1$, then Eq. (1) can be written as

$$\partial f_1/\partial t+\mathbf{v}\cdot\nabla_r f_1+\boldsymbol{\omega}_c\times\mathbf{v}\cdot\nabla_v f_1+\nu f_1=-(q/m)\mathbf{E}_1\cdot\nabla_v f_0, \quad (4)$$

where $\boldsymbol{\omega}_c=-q(\mu_0/m)\mathbf{H}_0$ is the electron cyclotron frequency. Following Drummond,[4] we note the definition of the Boltzmann operator and write the solution of (4) as

$$f_1(\mathbf{r},\mathbf{v},t)=-2(q/m)(df_0/dv^2)\int_0^\infty e^{-\nu s}[\mathbf{v}\cdot\mathbf{E}_1(\mathbf{r},t)]'ds, \quad (5)$$

where the prime denotes the fact that the variables $\mathbf{r}$, $\mathbf{v}$, t in the integrand are to be replaced by $t'=t-s$, $\mathbf{v}'=\mathbf{R}\cdot\mathbf{v}$, and $\mathbf{r}'=\mathbf{r}-\int_0^s\mathbf{R}(x)\cdot\mathbf{v}dx$. Here $\mathbf{R}(s)$ is a unitary matrix with the following elements:

$$R_{ij}(s)=\begin{bmatrix}\cos\omega_c s & -\sin\omega_c s & 0\\ \sin\omega_c s & \cos\omega_c s & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (6)$$

With this form of $\mathbf{R}$ we see that

$$\mathbf{r}'=\mathbf{r}-(2/\omega_c)\sin(\omega_c s/2)\mathbf{R}(s/2)\cdot\mathbf{v}_\perp-v_z s\hat{\mathbf{z}}, \quad (7)$$

where $\mathbf{v}=\mathbf{v}_\perp+v_z\hat{\mathbf{z}}$.

The vector current density,

$$\mathbf{J}(\mathbf{r},t)=q\int\mathbf{v}f(\mathbf{r},\mathbf{v},t)d^3v, \quad (8)$$

can now be calculated by using (5). The specific components of the current are given by

$$J_{1i}(\mathbf{r},t)=-(2q^2/m)\int d^3v(df_0/dv^2)\times v_i\int_0^\infty ds\, E_{1j}(\mathbf{r}',t-s)R_{jk}(s)v_k e^{-\nu s}, \quad (9)$$

where $\mathbf{r}'$ is given by (7). The velocity dependence can be simplified by introducing the coordinates,

$$\boldsymbol{\xi}=\mathbf{R}(s/2)\cdot\mathbf{v}=\mathbf{R}(-s/2)\cdot\mathbf{v}', \quad (10)$$

and, if we note the unitary property of $\mathbf{R}$, the introduction of these coordinates into (9) leads to the expression

$$J_{1i}(\mathbf{r},t)=-(2q^2/m)\int_0^\infty ds\, e^{-\nu s}R_{ij}(-s/2)\times R_{kl}(s/2)\int d^3\xi(df_0/d\xi^2)\xi_j\xi_k E_{1k}(\mathbf{r}',t-s), \quad (11)$$

where a summation over repeated indices is implied and where $\mathbf{r}'=\mathbf{r}-(2/\omega_c)\sin(\omega_c s/2)\boldsymbol{\xi}_\perp-sf_z\hat{\mathbf{z}}$. Equation (11) is in the form derived by Drummond. If we now introduce the Fourier transforms

$$\mathbf{E}_1(\mathbf{r},t)=(1/2\pi)^2\int\mathbf{E}_1(\mathbf{k},\omega)\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)d^3k d\omega,$$
$$\mathbf{J}_1(\mathbf{r},t)=(1/2\pi)^2\int\mathbf{J}_1(\mathbf{k},\omega)\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)d^3k d\omega \quad (12)$$

into (11), then the vector current density has the simple form

$$\mathbf{J}_1(\mathbf{k},\omega)=\boldsymbol{\sigma}(\mathbf{k},\omega)\cdot\mathbf{E}_1(\mathbf{k},\omega), \quad (13)$$

where the elements of $\boldsymbol{\sigma}$ are

$$\sigma_{il}=-(2q^2/m)\int_0^\infty ds\, R_{ij}(-s/2)R_{kl}(-s/2)\times\int d^3\xi(df_0/d\xi^2)\xi_j\xi_k\exp\Phi(\xi), \quad (14)$$

$$\Phi(\xi)=-(2i\mathbf{k}_\perp\cdot\boldsymbol{\xi}/\omega_c)\sin(\omega_c s/2)-isk_zE_z-\nu s+i\omega s,$$

where $\mathbf{k}=\mathbf{k}_\perp+k_z\hat{\mathbf{z}}$. For a uniform medium σ_{il} represents the conductivity tensor associated with the characteristic electric fields $\mathbf{E}(\mathbf{r},t)$, which for a uniform medium are plane waves. The various elements of $\boldsymbol{\sigma}$ for the assumed form of $f_0(v^2)$ and for a uniform medium are listed below.

$$\begin{aligned}
\sigma_{11}&=(nq^2/m)\int_0^\infty ds[\cos\omega_c s-u_\perp^2(1-\cos\omega_c s)\times(\cos\omega_c s+\cos2\phi)]\exp\Phi(s),\\
\sigma_{12}&=(nq^2/m)\int_0^\infty ds[\sin\omega_c s-u_\perp^2(1-\cos\omega_c s)\times(\sin\omega_c s+\sin2\phi)]\exp\Phi(s),\\
\sigma_{13}&=-(nq^2u_\perp u_z/m)\int_0^\infty ds\,\omega_c s[(1-\cos\omega_c s)\sin\phi+\sin\omega_c s\cos\phi]\exp\Phi(s),\\
\sigma_{21}(\omega_c)&=\sigma_{12}(-\omega_c),\\
\sigma_{22}(\phi)&=\sigma_{11}(\phi+\pi/2),\\
\sigma_{23}(\phi)&=-\sigma_{13}(\phi+\pi/2),\\
\sigma_{31}(\omega_c)&=\sigma_{13}(-\omega_c),\\
\sigma_{32}(\omega_c)&=\sigma_{23}(-\omega_c),\\
\sigma_{33}&=(nq^2/m)\int_0^\infty ds(1-u_z^2\omega_c^2s^2)\exp\Phi(s),
\end{aligned} \quad (15)$$

where $\mathbf{u}=\mathbf{k}(KT/m\omega_c^2)^{\frac{1}{2}}\approx$ (electron velocity/wave velocity); $\Phi(s)=(i\omega-\nu)s-2u_\perp^2\sin^2(\omega_c s/2)-u_z^2\omega_c^2s^2/2$; and ϕ is the angle between $\mathbf{k}$ and the x axis.

DISCUSSION

In the absence of collisions and when $\phi=0$, these elements of the conductivity tensor reduce to the elements of Bernstein's[3] $\mathbf{Q}$ (when typographical errors are corrected), which in turn are equivalent to those obtained by Sitenko and Stepanov,[1] when account is taken of their different representation of the Fourier series expansion of σ. As such, the elements of $\boldsymbol{\sigma}$ in (15) represent simply a rotation of the coordinates from the case treated by Bernstein. Pradham's results are equivalent to Bernstein's specified to propagation along the dc magnetic field—except for Pradham's example, which contains an error.

The elements of (15) do not agree, however, with those obtained by Drummond. To simplify his equivalent of (11), Drummond took components of the vector current density parallel (‖) and transverse ($\perp$) to the dc magnetic field[7]:

$$\mathbf{J}=\mathbf{J}_\perp(\mathbf{E})+\mathbf{J}_{\parallel}(\mathbf{E}). \tag{16}$$

The dependence of the current on the parallel and transverse components of the electric field was explicitly taken into account by writing

$$\begin{aligned}\mathbf{J}_\perp(\mathbf{E})&=\mathbf{J}_\perp(\mathbf{E}_\perp)+\mathbf{J}_\perp(\mathbf{E}_{\parallel}),\\ \mathbf{J}_{\parallel}(\mathbf{E})&=\mathbf{J}_{\parallel}(\mathbf{E}_\perp)+\mathbf{J}_{\parallel}(\mathbf{E}_{\parallel}).\end{aligned} \tag{17}$$

For ease of calculation the contributions from $\mathbf{E}_\perp$ were calculated separately from $\mathbf{E}_{\parallel}$. Drummond noted that with this decomposition Eq. (11) may be simplified by requiring that the electric field satisfy Maxwell's equations in the form

$$\text{curl curl}\mathbf{E}-(\omega^2/c^2)\mathbf{E}=i\mu_0\omega\,\mathbf{J}. \tag{18}$$

Then, by taking the contributions to the vector current density arising from the transverse electric field, new expressions for σ_{11}, σ_{12}, σ_{21}, σ_{22}, as well as σ_{31} and σ_{32}, were obtained. From symmetry arguments, the other off-diagonal elements were found. Finally, σ_{33} was calculated from the contribution of the parallel component of the electric field.

The fallacy in the argument arises from the assumption that the parallel and transverse components of (18) have the form[7]

$$\begin{aligned}(\text{curl curl}\mathbf{E}_\perp)_\perp-(\omega/c)^2\mathbf{E}_\perp&=i\mu_0\omega\,\mathbf{J}_\perp(\mathbf{E}_\perp),\\ \hat{z}\partial(\nabla\cdot\mathbf{E}_\perp)/\partial z&=i\mu_0\omega\,\mathbf{J}_{\parallel}(\mathbf{E}_\perp),\\ (\text{curl curl}\mathbf{E}_{\parallel})_\perp&=i\mu_0\omega\,\mathbf{J}_\perp(\mathbf{E}_{\parallel}),\\ (\text{curl curl}\mathbf{E}_{\parallel})_{\parallel}-(\omega/c)^2\mathbf{E}_{\parallel}&=i\mu_0\omega\,\mathbf{J}_{\parallel}(\mathbf{E}_{\parallel}).\end{aligned} \tag{19}$$

The decomposition of (18) into the above form tacitly assumes that the parallel and transverse components of the electric field are mutually independent. That is, the independent modes of propagation within a plasma are assumed to possess components either parallel to or transverse to the dc magnetic field, but not both. However, in the case of propagation in a plasma in a direction at an arbitrary angle to the dc magnetic field, even in the absence of thermal effects the electric field of the independent modes possesses components both along and transverse to the direction of the dc magnetic field.[8]

Only in the case of propagation in a plasma in a direction along or transverse to the direction of the dc magnetic field is the form of (19) and (20) correct. For the general case of propagation at an angle to the direction of the dc magnetic field the elimination process based on (19) is not valid, and the conductivity tensor obtained by Drummond is incorrect. For the specific case of propagation parallel to or transverse to the direction of the dc magnetic field, his form of the conductivity tensor is equivalent to that obtained by Bernstein and to that obtained by Sitenko and Stepanov.

APPENDIX

If we expand (15) in powers of u^2 and retain first-order temperature effects only, and if $|u|\ll 1$ (i.e., we ignore Landau-type damping), then the elements σ_{ij} may be approximated by

$$\begin{aligned}\sigma_{11}=&(nq^2/m)\{(\nu-i\omega)/[(\nu-i\omega)^2+\omega_c^2]\}\\ &\times\{1-\omega_c^2u_z^2[(\nu-i\omega)^2-3\omega_c^2]/[(\nu-i\omega)^2+\omega_c^2]^2\\ &-3\omega_c^2u_\perp^2/[(\nu-i\omega)^2+4\omega_c^2]\\ &+2\omega_c^2u_\perp^2(\sin^2\phi)/(\nu-i\omega)^2\},\end{aligned}$$

$$\begin{aligned}\sigma_{12}=&(nq^2/m)\{\omega_c/[(\nu-i\omega)^2+\omega_c^2]\}\\ &\times\{1-6\omega_c^2u_\perp^2/[(\nu-i\omega)^2+4\omega_c^2]\\ &-\omega_c^2u_z^2[3(\nu-i\omega)^2-\omega_c^2]/[(\nu-i\omega)^2+\omega_c^2]^2\\ &-\omega_cu_\perp^2\sin2\phi/(\nu-i\omega)\},\end{aligned}$$

$$\begin{aligned}\sigma_{13}=&-(nq^2/m)\{\omega_c^2u_\perp u_z/[(\nu-i\omega)^2+\omega_c^2]^2\}\\ &\times\{2(\nu-i\omega)\cos\phi+\omega_c\sin\phi[\omega_c^2+3(\nu-i\omega)^2]/(\nu-i\omega)^2\},\end{aligned}$$

$$\sigma_{23}(\phi)=-\sigma_{13}(\phi+\pi/2),$$

$$\sigma_{21}(\omega_c)=\sigma_{12}(-\omega_c),$$

$$\sigma_{22}(\phi)=\sigma_{11}(\phi+\pi/2),$$

$$\sigma_{31}(\omega_c)=\sigma_{13}(-\omega_c),$$

$$\sigma_{32}(\omega_c)=\sigma_{23}(-\omega_c),$$

$$\begin{aligned}\sigma_{33}=&[nq^2/m(\nu-i\omega)]\{1-3\omega_c^2u_z^2/(\nu-i\omega)^2\\ &-\omega_c^2u_\perp^2/[(\nu-i\omega)^2+\omega_c^2]\}.\end{aligned}$$

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