Ground-State Energy of Bose-Einstein Gas with Repulsive Interaction

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The explicit evaluation of the ground-state energy of a Bose gas interacting through a two-body repulsive potential as an expansion essentially in powers of the density of the particles and the scattering length is given. It is shown that the problem can be treated by using diagrammatical analysis up to any orders. The energy was investigated up to the fourth order in the ordinary sense of a perturbation expansion in terms of the scattering length, but including terms up to infinite order for special classes of diagrams. The energy obtained contains in general a $\sum_n C_n \rho^n \ln\rho$ ($n \geqq 2$, half-integer and integer) dependence, and the coefficient C_2 is evaluated. A self-consistent treatment is given following this analysis and is shown to lead to the formulation of Beliaev.

THE purpose of this paper is to carry through the evaluation of the ground-state energy of a Bose gas, interacting with a repulsive two-body potential, in an expansion in terms of the two-body scattering length at zero energy and of the density of particles, by using an analysis in terms of diagrams. The method we use for this purpose is to expand the "strength" of the potential energy $\int v(r)d^3r$ consistently as a power series in the two-body scattering length α at zero energy, and to represent the potential as a power series in α multiplied by the "shape" $v(r)/\int v(r')d^3r'$ of the potential (Sec. 1). Then this is used for the perturbation evaluation of the energy regarding the scattering length as an expansion parameter. The idea is similar to that of the pseudopotential method[1] and the procedure we employ is that of the K-matrix method.[2] The main difference is that we keep the shape of the potential in its original form to maintain the space locality of the interaction, and makes correction term-wise because in this paper we want to evaluate everything in powers of α. In Sec. 2, in the expression of ground-state energy, we replace the operator which represents the creation and annihilation of zero momentum Bose-particles with the operator consisting of the total number of particles (c-number) and the total number of excited particles (operator). Then, analyzing the expression of energy by diagrams, we show that there appear apparently disconnected diagrams which we should take as "connected" diagrams (a kind of nonlocal interaction). The effects of these "connected" diagrams are shown to be amalgamated as the change of the one-particle energy and the change of the total occupation number which appeared in the above replacement of zero-momentum operators. The ground-state energy is evaluated in Sec. 3 up to fourth order in the ordinary sense of a perturbation in the scattering length, partially including special diagrams up to infinite order; the role of the apparently disconnected "connected" diagrams is revealed in fourth order. The energy takes the following form, a being the length which appears in the "shape" of the potential

$$E_0=N\frac{2\pi\rho\alpha}{m}\Big[1+\frac{32\sqrt{2}}{15\pi}(8\pi\rho\alpha^3)^{\frac{1}{2}}+G_1(8\pi\rho\alpha^3)^{\frac{1}{2}}(8\pi\rho\alpha a^2)^{\frac{1}{2}}$$
$$+G_2(8\pi\rho\alpha^3)+G_3(8\pi\rho\alpha^3)\ln(8\pi\rho\alpha a^2)$$
$$+\text{higher order power series in } (8\pi\rho\alpha^3)^{\frac{1}{2}},$$
$$(8\pi\rho a^2)^{\frac{1}{2}}, \ln(8\pi\rho\alpha a^2) \text{ and } (\alpha/a)\Big].$$

We have evaluated G_1 and G_3, and found G_1 to be dependent on the shape of the potential, but G_3 is independent and is $\frac{4}{3}-(3)^{\frac{1}{2}}/\pi$. These are the leading corrections to the now familiar result if $8\pi\rho\alpha^3 \ll 1$ ($\alpha \leqq a$). In Sec. 4, the self-consistent formulation, which produces amalgamation of the effect of apparently disconnected "connected" diagrams into the one-particle energy change and effective total number change, is given and found to lead to the same formulation as Beliaev's.[3]

1. EXPANSION OF POTENTIAL ENERGY INTO SERIES OF SCATTERING LENGTH

Let us consider a two-particle interaction having the form of a "soft-core" potential given by:

$$\begin{aligned} v(r)&=0 \quad \text{for } r>a \\ &=V_0 \quad \text{for } r<a, \end{aligned} \tag{1.1}$$

r being the relative coordinate. The potential $v(r)$ can be written in the following way:

$$v(r)=\int v(r')d^3r'\left[v(r)\Big/\int v(r')d^3r'\right],$$

where we call $\int v(r)d^3r$ the strength of the potential for the reason that it represents the strength of the poten-

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[1] Lee, Huang, and Yang, Phys. Rev. **106**, 1135 (1957).
[2] K. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957). The energies we evaluated in this paper are the class of corrections we stated earlier in Sec. 5 of this reference.
[3] S. T. Beliaev, J. Exptl. Theoret. Phys. U.S.S.R. 34, 289 (1958) [translation: Soviet Phys. JETP **7**, 289 (1958)].

tial in momentum space at zero momentum:

$$v(q)=\int v(r)\exp(i\mathbf{q}\cdot\mathbf{r})d^3r;$$

and call $v(r)/\int v(r')d^3r'$ the "shape" of the potential because it is normalized to unity in momentum space at zero momentum, and its high momentum tail depends strictly on the shape of the potential we use. [In this paper, we assume, besides the form (1. 1), potentials which have a general shape, mainly repulsive, in such a way that when the scattering length $\alpha\to 0+$ the potential vanishes.] For the potential (1. 1) we can write down explicitly the scattering length at zero energy:

$$\frac{\alpha}{a}=1-\frac{1}{(V_0a^2m)^{\frac{1}{2}}}\tanh[(V_0a^2m)^{\frac{1}{2}}]. \tag{1. 2}$$

Hence, by solving (1. 2) for V_0a^2m as a power series in α, we can get an expansion of the "strength" $\int v(r)d^3r=(4\pi a^3/3)V_0$. For a more general interaction it is difficult to obtain the equation corresponding to (1. 2), and we adopt the following procedure, which in fact is more suitable for practical application than is (1. 2). The scattering length at zero energy, α, is related to the scattering matrix t by the following equations[4]:

$$t_{00;00}=\frac{4\pi\alpha}{mV}=v_{00;00}+\sum_q v_{00;q-q}\frac{1}{-q^2/m}t_{q-q;00},$$
$$t_{q-q;00}=v_{q-q;00}+\sum_{q'} v_{(q,-q);q',-q'}\frac{1}{-q'^2/m}t_{q'-q';00}, \tag{1. 3}$$

where

$$v_{ij;lk}=\frac{1}{V}\delta_{i+j,k+l}\int\exp[i(\mathbf{j}-\mathbf{k})\cdot\mathbf{r}]v(r)d^3r,$$

V is the normalization volume of the system, and $v_{(ij);lk}$ means

$$v_{(ij);lk}=\tfrac{1}{2}(v_{ij;lk}+v_{ji;lk}).$$

The expansion of the "strength" in powers of the scattering length at zero energy can be performed in the following manner; first rewrite the Eq. (1. 3) by introducing the "shape" of the potential.

"Shape" of the potential:

$$f_{ij,lk}=v_{(ij),lk}\Big/\int v(r')d^3r' = v_{(ij),lk}/v_{00,00}, \tag{1. 4}$$

$$t_{00;00}=v_{00;00}\left(1+\sum_q f_{00;q-q}\frac{1}{-q^2/m}t_{q,-q;00}\right),$$
$$t_{q-q;00}=v_{00;00}\left(f_{q-q;00}+\sum_{q'} f_{q,-q;q',-q'}\frac{1}{-q'^2/m}t_{q'-q';00}\right). \tag{1. 5}$$

[4] B. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

Eliminating $v_{00;00}$ from the lower equation and solving it for $t_{q-q;00}$ as a function represented by $t_{00;00}$ and f, then putting it back into the first equation, we get $v_{00;00}$ as a series in $t_{00;00}$; then from (1. 4) we get an expansion of the potential energy into a series in $t_{00;00}$ (writing $t_{00;00}$ as t_0):

$$\begin{aligned}v_{(ij);lk}=\Bigg\{&t_0-t_0^2\left[\left(f\tfrac{1}{a}f\right)\right]-t_0^3\left[\left(f\tfrac{1}{a}f\tfrac{1}{a}f\right)-2\left(f\tfrac{1}{a}f\right)^2\right]\\&-t_0^4\left[\left(f\tfrac{1}{a}f\tfrac{1}{a}f\tfrac{1}{a}f\right)-5\left(f\tfrac{1}{a}f\right)\left(f\tfrac{1}{a}f\tfrac{1}{a}f\right)+5\left(f\tfrac{1}{a}f\right)^3\right]-t_0^5\left[\left(f\tfrac{1}{a}f\tfrac{1}{a}f\tfrac{1}{a}f\tfrac{1}{a}f\right)\right.\\&-6\left(f\tfrac{1}{a}f\right)\left(f\tfrac{1}{a}f\tfrac{1}{a}f\tfrac{1}{a}f\right)-3\left(f\tfrac{1}{a}f\tfrac{1}{a}f\right)^2\\&+21\left(f\tfrac{1}{a}f\right)^2\left(f\tfrac{1}{a}f\tfrac{1}{a}f\right)\\&\left.-14\left(f\tfrac{1}{a}f\right)^4\right]-\cdots\Bigg\}f_{ij;lk}\\=&\sum_{n=1}^{\infty}v_{(ij);lk}^{(n)},\end{aligned} \tag{1. 6}$$

where $[f(1/a)f(1/a)\cdots(1/a)f]$ means

$$\sum_{q_1\cdots q_n} f_{q_1}\frac{1}{-q_1^2/m}f_{q_1-q_1;q_2-q_2}\frac{1}{-q_2^2/m}\cdots\frac{1}{-q_n^2/m}f_{q_n}.$$

We use abbreviations $v_{00;q-q}=v_{q0;0q}\equiv v_q$, $f_{00;q-q}=f_{q-q;00}\equiv f_q$ in the following paragraphs.

It can be seen quite easily that the coefficients in square brackets play a role in cancelling the energies of two interacting particles at zero energy higher than the first order in expansion of t_0,[5] in fact, the coefficients can be determined to arbitrary order by using this condition. For the case of model (1. 1), we can evaluate the coefficients immediately by comparing the expansion of V_0a^2m in terms of α/a obtained by using Eq. (1. 2) and the expansion of (1. 5):

$$t_{00;00}=\frac{4\pi\alpha}{mV}=v_{00;00}+v_{00;00}^2\left(f\tfrac{1}{a}f\right)+v_{00;00}^3\left(f\tfrac{1}{a}f\tfrac{1}{a}f\right)+\cdots, \tag{1. 7}$$

[5] The energy of the two-particle system consist of $O(1/R)+O(1/R^2)+\cdots$, where R is the radius of the normalization volume V. What is meant by "cancel" is only for the energy of order $1/R$. This part of the energy only plays an essential role in the many-body problem; see reference 1.

and we get

$$\left(\begin{matrix}1\\ f\text{-}f\\ a\end{matrix}\right)=-0.4000\frac{3Vm}{4\pi a},\quad \left(\begin{matrix}1\ 1\\ f\text{-}f\text{-}f\\ a\ a\end{matrix}\right)=0.1619\left(\frac{3Vm}{4\pi a}\right)^2,$$

$$\left(\begin{matrix}1\ 1\ 1\\ f\text{-}f\text{-}f\text{-}f\\ a\ a\ a\end{matrix}\right)=-0.0656\left(\frac{3Vm}{4\pi a}\right)^3,\quad \text{etc.}$$

and

$$v_{(ij);lk}=t_0f_{ij;lk}\left[1+0.4000\left(\frac{3Vmt_0}{4\pi a}\right)+0.1200\left(\frac{3Vmt_0}{4\pi a}\right)^2+0.0614\left(\frac{3Vmt_0}{4\pi a}\right)^3+\cdots\right].$$

We shall see in the following paragraphs that if we evaluate the energy of the system by expansion in powers of the scattering length α, the second, third, . . . order corrections to the energy, which have the same structure as the combinations of f appearing in (1. 6), are nearly cancelled and the energy of the system (this kind of energy we call energy coming from scattering) becomes a series with much smaller coefficients than the above figures. (Higher-order corrections cancel out[5] for the two-particle system with momentum zero, only the first order surviving; for the many-particle system the free-particle energy is changed by the presence of the other particles and a term of the form $(f[(1/a')-(1/a)]f)$ appears, the difference $a'-a$ being of order at least α.) The real restriction for expansion of energy of the system about scattering length should be found in the effect of many-body interaction via two-body interaction, the contribution of which to the energy has different structure in terms of f than (1. 6).

2. TRANSFORMATION TO EQUIVALENT PROBLEM

The ground-state energy of the system can be expressed explicitly by the equation given by Gell-Mann and Low,[6] which is the time-dependent version of the Brillouin-Wigner equation for the energy correction; following their notation, we have

$$\Delta E_0=\lim_{\alpha\to 0}\frac{(\phi_0, vU_\alpha^{-1}(-\infty\,;0)\phi_0)}{(\phi_0, U_\alpha^{-1}(-\infty\,;0)\phi_0)},\qquad (2.\ 1)$$

where ϕ_0 is the free ground-state wave function of N bosons, and

$$U_\alpha^{-1}(-\infty\,;0)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{0}\cdots\int_{-\infty}^{0}P(v(t_1)v(t_2)\cdots v(t_n))\times e^{\alpha(t_1+t_2+\cdots+t_n)}dt_1dt_2\cdots dt_n.\qquad (2.\ 2)$$

and P is the chronological ordering symbol,

$$v(t)=e^{iH_0t}ve^{-iH_0t},\qquad (2.\ 3)$$

$$\begin{aligned}v=&\tfrac{1}{2}v_0n_0(n_0-1)+\textstyle\sum_q n_q(v_q+v_0)n_0\\&+\tfrac{1}{2}\textstyle\sum_q \beta_q{}^*\beta_{-q}{}^*v_q\beta_0\beta_0+\text{c.c.}\\&+\textstyle\sum_q\sum_p \beta_q{}^*\beta_p{}^*\tfrac{1}{2}(v_q+v_p)\beta_{p+q}\beta_0+\text{c.c.}\\&+\tfrac{1}{2}\textstyle\sum_p\sum_{p'}\sum_q \beta_{p+q}{}^*\beta_{p'}{}^*v_q\beta_{p'+q}\beta_p,\end{aligned}$$

(sums do not include momentum 0) (2. 4)

where the β_q's are annihilation operators of bosons with momentum q and the n's are number operators.

We here note that the evaluation of (2. 1) with the wave function of the state with N free particles in the ground state is completely equivalent to the evaluation of the same quantity with replacement of v [(2. 4)] by

$$\begin{aligned}H_{\text{int}}\equiv v=&\tfrac{1}{2}v_0(N-\textstyle\sum_q n_q)(N-\sum_q n_q-1)\\&+\textstyle\sum_p n_p(v_p+v_0)(N-\sum_q n_q)\\&+\tfrac{1}{2}\textstyle\sum_q \beta_q{}^*\beta_{-q}{}^*v_q[(N-\sum_q n_q)\\&\times(N-\textstyle\sum_q n_q-1)]^{\frac{1}{2}}+\text{c.c.}\\&+\textstyle\sum_q\sum_p \beta_q{}^*\beta_p{}^*\tfrac{1}{2}(v_q+v_p)\\&\times\beta_{p+q}(N-\textstyle\sum_q n_q)^{\frac{1}{2}}+\text{c.c.}\\&+\tfrac{1}{2}\textstyle\sum_q\sum_p\sum_{p'} \beta_{p+q}{}^*\beta_{p'}{}^*v_q\beta_{p'+q}\beta_p\qquad (2.\ 5a)\\=&\tfrac{1}{2}\left\{\left[v(N)+\sum_{m=1}^{\infty}N^m\frac{d^m}{dN^m}v(N)\frac{1}{m!}\times\left(-\frac{\sum_q n_q}{N}\right)^m\right]+\text{c.c.}\right\},\qquad (2.\ 5)\end{aligned}$$

where

$$\begin{aligned}v(N)=&\tfrac{1}{2}v_0N(N-1)+\textstyle\sum_p n_p(v_p+v_0)N\\&+\textstyle\sum_q \beta_q{}^*\beta_{-q}{}^*v_q[N(N-1)]^{\frac{1}{2}}\\&+\textstyle\sum_q\sum_p \beta_q{}^*\beta_p{}^*(v_q+v_p)\beta_{p+q}(N)^{\frac{1}{2}}\\&+\tfrac{1}{2}\textstyle\sum_q\sum_p\sum_{p'} \beta_{p+q}{}^*\beta_{p'}{}^*v_q\beta_{p'+q}\beta_p\end{aligned}$$

with respect to the no-particle state,[7] where the β's now refer to particles with finite momentum.

Noticing the above fact, we can represent (2. 1) and (2. 5) by diagrams representing interactions, and we can show that they can be reduced to the connected contributions only, namely (denoting the no-particle wave function by ψ_0):

$$\Delta E_0=\left(\psi_0,\sum_{\text{connected},\,n}H_{\text{int}}\left(\frac{1}{-H_0}H_{\text{int}}\right)^n\psi_0\right),\qquad (2.\ 6)$$

by using a similar argument to that used by Goldstone[8] for the fermion case. The remarkable difference is that

[6] M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

[7] This statement holds exactly for $N\to\infty$, because for the equivalence of two quantities we need the condition $N-\sum_q n_q\geqq 0$ in every intermediate state.

[8] J. Goldstone; Proc. Roy. Soc. (London) A239, 267 (1957).

when (2. 1) and (2. 5) are represented by diagrams we have, as "connected" diagrams, diagrams of the form shown in Fig. 1 by the presence of the term which consists of a product of $N^m(d^m/dN^m)v(N)$ or $N^m(d^m/dN^m)v^\dagger(N)$ and $(-\sum_q n_q/N)^m$ in the interaction energy; for example, the first one represents

$$\tfrac{1}{2}\sum_q[N(N-1)]^{\frac{1}{2}}v_q\frac{1}{-q^2/m}\left(N\frac{d}{dN}v(N)\right)_{q\leftarrow q}$$
$$\times\frac{1}{-q^2/m}v_q[N(N-1)]^{\frac{1}{2}}\tfrac{1}{2}\sum_{q'}[N(N-1)]^{\frac{1}{2}}v_{q'}$$
$$\times\frac{1}{-(q^2+q'^2)/m}\left(-\frac{1}{N}\right)\frac{1}{(-q^2+q'^2)/m}v_{q'}[N(N-1)]^{\frac{1}{2}},$$

where $[N(d/dN)v(N)]_{q\leftarrow q}$ is the expression $\sum_p n_p \times(v_p+v_0)N$ obtained from differentiations of $\frac{1}{2}[v(N)+\text{c.c.}]$. [The interaction obtained by differentiating the first term in $v(N)$, namely $(N-\frac{1}{2})v_0(-\sum_q n_q)$, produces the change of free-particle energy.] But because these "connected" diagrams are apparently disconnected, we can write them as products of independent sums over particle momenta. This can be proved by adding

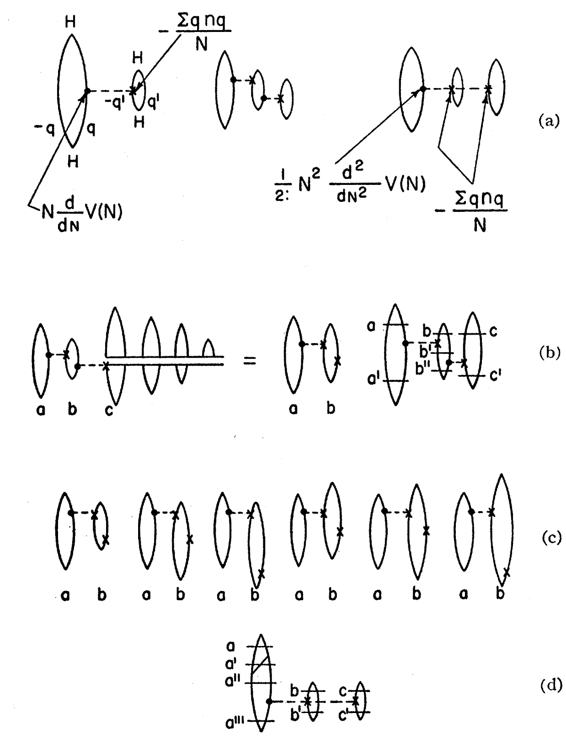

FIG. 1. First class of disconnected "connected" diagrams. (b) The energy denominator of Eq. (2. 6) for the last diagram is

$$\frac{1}{a}\frac{1}{a+c}\frac{1}{a+b+c}\frac{1}{a'+b'+c}\frac{1}{a'+b''+c'}\frac{1}{a'+c'}\frac{1}{a'}.$$

(c) Relative position of diagram b. (d) Energy denominators $b(=b')=c(=c')$. ($\times$ represents $-\sum_q n_q/N$).

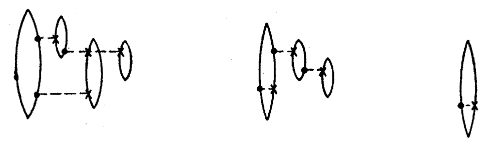

FIG. 2. Second class of disconnected "connected" diagrams.

diagrams which have all possible relative positions of interaction with respect to each other; for example, in the second graph, we first sum over all possible relative positions of the last interaction as shown in Fig. 1(b); then the energy denominator of the last connected graph is disentangled from the rest and acts at $\times$ of graph b. For each possible relative position of graph b, by summing over all possible positions of the interaction $\times$ as above, we have Fig. 1(c). Again in this sum the energy denominator is disentangled and the complete sum becomes a product of separate denominators $(1/aa')(1/bb'b'')(1/cc')$.[9]

There appear in (2. 6) also graphs of the form Fig. 2, namely diagrams in which one or more $-\sum_q n_q/N$ are acting in the diagram without introducing new isolated loops, but as we shall see in the following we can discard these graphs completely.

To estimate the contribution to the energy from diagrams, first we can show by a simple counting that the connected diagrams which consists only of $v(N)$ and $v^\dagger(N)$ give an energy proportional to the total number N, $\rightarrow O(N)+O(1)+\cdots$. The effect of the "interaction"

$$\frac{1}{2}\left[\sum_{m=1}^{\infty}N^m\frac{d^m}{dN^m}v(N)\frac{1}{m!}\left(-\frac{\sum_q n_q}{N}\right)^m+\text{c.c.}\right],$$

produces two two kinds of diagrams [apart from the one-particle energy change $(N-\frac{1}{2})v_0(-\sum_q n_q)$ as mentioned above]. The first one is of the form given in Fig. 1, the contribution of which to the energy can be represented as a product of contributions from each diagram as was shown above. In these graphs, each $-\sum_q n_q/N$ acts on the completely separate self-connected loop. In estimating the order of energy coming from these diagrams, we note that $(-\sum_q n_q/N)$ is of order $1/N$, while each $N^m(d^m/dN^m)v(N)$ or $N^m(d^m/dN^m)v^\dagger(N)$ is of order $v(N)$ or $v^\dagger(N)$. Hence in the first class of diagrams $(-\sum_q n_q/N)^m$ produces $(1/N)^m$, with $m+1$ loops which consist of the interaction of order $v(N)$ and $v^\dagger(N)$, and the latter loops give rise to energy $O(N^{m+1})$ (because a single loop with interaction $v(N)$ and $v^\dagger(N)$ gives a factor proportional to N, the total number); the total contribution is thus proportional to the total

[9] If two or more of the same diagrams are contained in apparently disconnected "connected" diagrams, the above process counts diagrams 2!, 3!$\cdots$times, hence we must divide by these figures. For instance, Fig. 1(d) for diagrams b equal to c, gives as the contribution

$$\frac{1}{2!}\frac{1}{aa'a''a'''}\frac{1}{bb'}\frac{1}{cc'}.$$

number N. In the second kind of diagram $-\sum_q n_q/N$ acts one or more times without producing new loops as in Fig. 2. These give, with given $(-\sum_q n_q/N)^m$, a number of loops less than $m+1$; hence the energy is equal to or lower than order one. To get the total energy of the ground state up to order N, we need therefore only consider the first class of "connected" diagrams.

3. GROUND-STATE ENERGY

In this section, we take into account the effect of the change of one-particle energy in zeroth order, which comes from the first and the second terms of v (2. 5a); they give

$$\tfrac{1}{2}v_0N(N-1)+\textstyle\sum_p n_p(v_pN+\tfrac{1}{2}v_0) -\tfrac{1}{2}v_0(\sum_q n_q)^2-\sum_p n_pv_p\sum_q n_q,$$

and we have for the Hamiltonian,

$$H=\textstyle\sum_p n_p[(p^2/2m)+v_pN+\tfrac{1}{2}v_0]+\tfrac{1}{2}v_0N(N-1) -\tfrac{1}{2}v_0(\sum_q n_q)^2-\sum_p n_pv_p\sum_q n_q +\tfrac{1}{2}\left\{\left[v'(N)+\sum_{m=1}^{\infty} N^m\frac{d^m}{dN^m}v'(N)\frac{1}{m!}\times\left(-\frac{\sum_q n_q}{N}\right)^m\right]+\text{c.c.}\right\}, \quad (3.1)$$

with

$$v'(N)=\textstyle\sum_q \beta_q{}^*\beta_{-q}{}^*v_q[N(N-1)]^{\frac{1}{2}} +\sum_q\sum_p \beta_q{}^*\beta_p{}^*(v_q+v_p)\beta_{p+q}N^{\frac{1}{2}} -\frac{1}{2}\sum_q\sum_p\sum_{p'}\beta_{p+q}{}^*\beta_{p'}{}^*v_q\beta_{p'+q}\beta_p.$$

In (3.1) we put for v the expansion of v in powers of t_0 (1. 6), and represent the Hamiltonian as a power series in t_0. We take as zeroth order energy of the excited particles[10]

$$\textstyle\sum_p[(p^2/2m)+t_0f_pN]n_p, \quad (3.1')$$

because, as was discussed in B.S.,[2] without inclusion of the "gap" we cannot proceed with a perturbation evaluation of energy in the expansion in powers of t_0 owing to the appearance of infrared divergences in the higher-order energies.

We evaluate the ground-state energy following the order of the ordinary perturbation calculation; namely, we call the energy which has two factors of t_0 the second-order energy . . . etc., and taking into account some higher order contributions simultaneously, we make correspondence with the actual expansion parameter, namely, as we shall see, $(8\pi\rho\alpha^3)^{\frac{1}{2}}$, $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$, and α/a.

(*a*) *Lowest order.*—The second term of (3.1) with the first term in (1. 6) is

$$\Delta E_0{}^{(1)}=\tfrac{1}{2}N(N-1)t_0=N2\pi\rho\alpha/m. \quad (3.2)$$

[10] We can drop $-\frac{1}{2}v_0N$ and $\sum_p n_p\frac{1}{2}v_0$ from (3.1), because they give only an energy of order one.

(*b*) *Second order.*—The contribution of iteration of the first order interaction, and the second term of (3.1) with $v_0{}^{(2)}$ [$v^{(2)}$ is the second term of (1.6)], is

$$\Delta E_0{}^{(2)}=\tfrac{1}{2}N^2t_0{}^2\sum_q f_q\left(\frac{1}{-(q^2/m)-2Nt_0f_q}-\frac{1}{-q^2/m}\right)f_q; \quad (3.3)$$

we actually have for the energy denominator $-2Nt_0f_q-2t_0f_0$ instead of $-2Nt_0f_q$ and for the first factor $N(N-1)$ instead of N^2, but the omission of $-2t_0f_0$ and $-N$ is justified because they contribute to the energy of order one.

Here we note that f_q is

$$f_q(\equiv f_{00;q-q})=\int\exp(i\mathbf{q}\cdot\mathbf{r})v(r)d^3r\Big/\int v(r)d^3r,$$

and is only comparable to one for momentum q smaller than $1/a$, where a is the core radius for model (1. 1), and for more general interaction it is the length appearing in f. Changing the variable of summation in (3. 3) from q to x defined by the equation

$$q=(2Nt_0m)^{\frac{1}{2}}x=(8\pi\rho\alpha)^{\frac{1}{2}}x, \quad (3.4)$$

one can write (3. 3) symbolically [using the abbreviation $b=1/(2Nt_0m)^{\frac{1}{2}}a$] as

$$N^2t_0{}^2mV(2Nt_0m)^{\frac{1}{2}}\int_0^{\sim b}\left(\frac{1}{-x^2-1}-\frac{1}{-x^2}\right)d^3x.$$

The integral over x converges even for $(2Nt_0m)^{\frac{1}{2}}a\to 0$, and the integral gives a series of the form

$$A+B(8\pi\rho\alpha a^2)^{\frac{1}{2}}+C(8\pi\rho\alpha a^2)+\cdots.$$

The actual value is

$$\Delta E_0{}^{(2)}=N\frac{2\pi\rho\alpha}{m}(8\pi\rho\alpha^3)^{\frac{1}{2}}\times\left(1+\begin{Bmatrix}-(15/8)(8\pi\rho\alpha a^2)^{\frac{1}{2}}+\cdots\\-(2/\pi)(8\pi\rho\alpha a^2)^{\frac{1}{2}}+\cdots\end{Bmatrix}\right), \quad (3.5)$$

where the first term is for the potential with Yukawa shape

$$f_q=1/[1+(qa)^2],$$

and the second is for a sharp cutoff in momentum space (an oscillatory potential with the main part repulsive

FIG. 3. Pair creation-annihilation diagram (I).

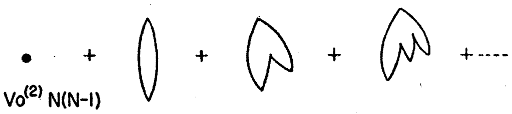

FIG. 4. Diagram which gives energy of the order $\rho\alpha\times(\rho\alpha^3)^{\frac{1}{2}}$.

in coordinate space)

$$f_q=1 \quad \text{for } q<1/a$$
$$\;=0 \quad \text{for } q>1/a.$$

We can see that the effect of the "shape" function appears as a power series in $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$, a number which is usually larger than $(8\pi\rho\alpha^3)^{\frac{1}{2}}$.

The considerations in B.S. show that there are also terms which contribute to the same order of the energy expansion in $\rho\alpha(\rho\alpha^3)^{\frac{1}{2}}$ in the apparently higher order graphs (in the usual sense of expansion in powers of t_0). Namely, if we add two factors Nt_0 to some converging graph, then we increase two energy denominators and introduce no further momentum summations, producing a pair-type interaction as shown in Fig. 3. The resultant contribution to the energy can be represented by multiplying the original expression of energy inside the momentum summation by the factor

$$\text{(I)} \qquad (Nt_0)^2\frac{1}{[(q^2/m)+BNt_0]^2},$$

$$\text{by changing to } x(3.4)\rightarrow\frac{1}{(x^4+\cdots)},$$

where . . . contains the number of the order one which is provided by the presence of the "gap" Nt_0 in the one-particle energy in (3. 1) (in the x variable it is of order one), and saves the infrared divergence of the integral. Namely, these interactions must be included to get the correct coefficient of $\rho\alpha(\rho\alpha^3)^{\frac{1}{2}}$ in the expansion.

This class of diagrams is shown in Fig. 4, and the sum of these diagrams is just given in B.S.,[2] with a slight change in notation

$$\Delta E_0^{(2)\prime}=\tfrac{1}{2}N^2t_0^2\sum_q f_q\left(\frac{1}{-(q^2/2m)-Nt_0f_q-\{[(q^2/2m)+Nt_0f_q]^2-(Nt_0f_q)^2\}^{\frac{1}{2}}}-\frac{1}{-q^2/m}\right)f_q$$
$$=N\frac{2\pi\rho\alpha}{m}(8\pi\rho\alpha^3)^{\frac{1}{2}}\left(\frac{32\sqrt{2}}{15\pi}+\left\{\begin{matrix}-(9/4)(8\pi\rho\alpha a^2)^{\frac{1}{2}}+\cdots\\-(2/\pi)(8\pi\rho\alpha a^2)^{\frac{1}{2}}+\cdots\end{matrix}\right\}\right), \qquad (3.6)$$

where again the first line is for the Yukawa shape and the second is for the sharp cutoff in momentum space. The formula for the coefficient of the $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$ term is

$$G_1=\frac{2}{\pi}\int_0^\infty\frac{dy}{y^2}[(f_y)^3-1], \quad \text{where } y=qa. \qquad (3.7)$$

It was assumed in obtaining (3. 7) that f_y is at least $1+O(y^2)$ for $y\rightarrow 0$.

(*c*) *Third order*.—The third order energy can be written down as follows: The first term comes from repeated scattering by $v^{(1)}$, the second and last terms are $\frac{1}{2}v_0^{(3)}N(N-1)$ with $v_0^{(3)}$ the third term of (1. 6), and the third term comes from scattering due to $v^{(2)}$ and $v^{(1)}$:

$$\Delta E_0^{(3)}=\tfrac{1}{2}N(N-1)t_0^3\left[\left(f\frac{1}{a'}f\frac{1}{a'}f\right)-\left(f\frac{1}{a}f\frac{1}{a}f\right)-2\left(f\frac{1}{a'}f\right)\left(f\frac{1}{a}f\right)+2\left(f\frac{1}{a}f\right)^2\right], \qquad (3.8)$$

where $a=-q^2/m$ as before, $a'=-q^2/m-2Nt_0f_q$, and the quantity in square brackets is the two-body scattering-matrix element $00\rightarrow 00$. Transforming to the x variable (3. 4), one can see that the integral over x converges in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}\rightarrow 0$, so the leading term does not depend on a (we can take $f=1$ to get the leading term):

$$\Delta E_0^{(3)}=N(2\pi\rho\alpha/m)(8\pi\rho\alpha^3)\,[1+\text{higher order power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}]. \qquad (3.9)$$

The pair-type diagrams for which argument (I) holds and which contribute to this order in ρ, α are given in Fig. 5. But this S_1 is the first order expectation value of energy which represents the scattering of opposite-momentum particles;

$$S_1=(\Psi_0,\ \tfrac{1}{2}\textstyle\sum_q\sum_{q'}\beta_q{}^*\beta_{-q}{}^*t_0f_{q,-q;q'-q'}\beta_{q'}\beta_{-q'}\Psi_0), \qquad (3.10)$$

with the wave function including all pair excitations with opposite momentum; namely, ground-state wave function with Hamiltonian

$$H_T{}^0=\textstyle\sum_p n_p[(p^2/2m)+t_0f_pN] +\tfrac{1}{2}(\sum_q\beta_q{}^*\beta_{-q}{}^*t_0f_qN+\text{c.c.}).$$

If we transform variables to the μ-representation which was given in B.S. (20),[2] defined by $\Psi_0=U\psi_0$, $\mu_q=U^{-1}\beta_qU$, we can take the vacuum expectation value for μ-variables; then we get for the "shape"-independent part [we take all f's equal to one because the final expression converges for $f=1$ and the effect of f appears as a

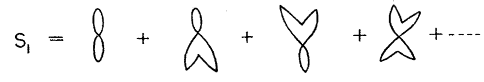

FIG. 5. Third order diagram with inclusion of effect (*I*).

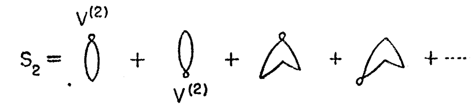

FIG. 6. Subtractive third order diagrams.

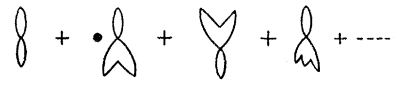

FIG. 7. Diagram which contains scattering. (Second order scattering.)

power series in $(8\pi\rho\alpha a^2)^{\frac{1}{2}}]$

$$S_1=\frac{t_0}{2}\left(\sum_q \frac{A_q}{1-A_q^2}\right)^2$$
$$=\tfrac{1}{2}N^2t_0^3\left(\sum_q \frac{1}{-2\{[(q^2/2m)+Nt_0]^2-(Nt_0)^2\}^{\frac{1}{2}}}\right)^2, \tag{3.11}$$

where A_q is defined in B.S. (23),[2] (the α in that equation is now t_0). But here, we must take into account of the second order interaction in (1. 6) which contains the second order pair creation and annihilation operator giving the diagrams of Fig. 6 where the small circle represents the second term in (1. 6), namely $v^{(2)}=-t_0^2[f(1/a)f]f$. These terms are to be taken into account together with the diagrams which represent scattering of excited particles, as shown in Fig. 7, because our second order term in the expansion (1. 6) is designed to compensate the scattering. The small-circle interaction S_2 can occur at every vertex in S_2 [the first two terms of S_2 is the third term of (3. 8)]. The sum obtained by replacing the small circle with the ordinary vertex t_0f is just the same as the term appearing in Fig. 4 multiplied by the number of vertices in each diagram (in Fig. 4); hence

$$S_2=-t_0^2\left(f\frac{1}{a}f\right)\frac{d}{d\bar{t}_0}\left(\tfrac{1}{2}N^2\bar{t}_0^2\sum_q \frac{1}{-(q^2/2m)-N\bar{t}_0-\{[(q^2/2m)+N\bar{t}_0]^2-(N\bar{t}_0)^2\}^{\frac{1}{2}}}\right). \tag{3.12}$$

where because we must remember that Nt_0 in combination with the kinetic energy is not the pair creation annihilation vertex, we put a bar above the vertex. (3. 12) gives

$$S_2=-\tfrac{1}{2}N^2t_0^3\sum_q \frac{1}{-2\{[(q^2/2m)+Nt_0]^2-(Nt_0)^2\}^{\frac{1}{2}}}\sum_q \frac{1}{-(q^2/m)}2. \tag{3.13}$$

Finally, the diagonal element of $[N(N-1)/2]v^{(3)}$ is (we take all f's equal to one)

$$S_3=\frac{N^2t_0^3}{2}\left(\sum_q \frac{1}{-(q^2/m)}\right)^2.$$

The sums of these three contributions give

$$\Delta E_0^{(3)\prime}=\frac{N^2t_0^3}{2}\left[\sum_q\left(\frac{1}{-2\{[(q^2/2m)+Nt_0]^2-(Nt_0)^2\}^{\frac{1}{2}}}-\frac{1}{-q^2/m}\right)\right]^2+\text{shape-dependent terms}$$
$$=N(2\pi\rho\alpha/m)(8\pi\rho\alpha^3)[(8/\pi^2)+\text{power series in } 8\pi\rho\alpha a^2)^{\frac{1}{2}}]. \tag{3.14}$$

From (3. 5), (3. 6), (3. 9), and (3. 14), we can see, as was remarked already in B.S. (51)–(54),[2] that the inclusion of creation and annihilation of opposite-momentum virtual pairs (I) affects the result rather slightly; namely for $\Delta E_0^{(2)\prime}$ it changes 1 to $32\sqrt{2}/15\pi=0.9603$, and for $\Delta E_0^{(3)\prime}$, 1 to $8/\pi^2=0.8106$.

(*d*) *Fourth order*.—In this order, we encounter the simplest type of the apparently separated "connected" diagrams. First, the scattering type of energy is always subtracting the term which comes from the interaction appearing in the expansion (1. 6) and is represented by the diagrams of Fig. 8,

$$\Delta E_0^{(4)}\ \text{scattering}=N(2\pi\rho\alpha/m)(8\pi\rho\alpha^3)^{\frac{3}{2}}\ [1+\text{higher power series in } (8\pi\rho\alpha a^2)^{\frac{1}{2}}]. \tag{3.15}$$

Because of the lack of divergence (ultraviolet) in the x integral in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}\to 0$, the first term does not depend on the shape of the potential a and is always unity even if we include higher-order scattering [see (3. 23) in (e)]. The diagram Fig. 9(a) gives

$$2(\tfrac{1}{2}N^2t_0^2)\sum_q f_q\frac{1}{-(q^2/m)-2Nt_0f_q}\times Nt_0^2\sum_{q'}\left\{f_{q0;q-q',q'}\frac{1}{-(q'^2/2m)-[(q-q')^2/2m]-(q^2/2m)-Nt_0(f_q+f_{q-q'}+f_{q'})}\right.$$
$$\left.\times 2f_{q',q-q';0q}-f_{q'}\frac{1}{-(q'^2/m)}f_{q'}\right\}\frac{1}{-(q^2/m)-2Nt_0f_q}f_q, \tag{3.16}$$

where the first factor 2 arises because there are two kinds of diagrams (namely, each of two particles can have "self-energy" diagrams), the second term in the curly bracket is due to the second order change of the one-particle energy coming from $\sum_p n_p v_p^{(2)} N$ in the second terms of the first square bracket of (3. 1), and the factor two in front of the f is due to the possibility of exchange scattering. Also from (3. 1), we find the apparently disconnected "connected" diagrams shown in Fig. 9(b) which give

$$2\frac{N(N-\frac{1}{2})t_0^2}{2}\sum_q f_q \frac{-2/N}{[-(q^2/m)-2Nt_0f_q]^2} f_q \frac{N(N-1)t_0^2}{2}\sum_{q'} f_{q'}\frac{1}{-(q'^2/m)-2Nt_0f_{q'}}f_{q'}$$

$$\doteqdot 2\frac{N^2t_0^2}{2}\sum_q f_q \frac{-2/N}{[-(q^2/m)-2Nt_0f_q]^2} f_q \frac{N^2t_0^2}{2}\sum_{q'} f_{q'}\frac{1}{-(q'^2/m)-2Nt_0f_{q'}}f_{q'}. \quad (3.17)$$

Equation (3. 16) can be interpreted as a change of energy of one particle in the energy denominator of the second order diagram by an amount

$$\Delta_q^{(1)} = Nt_0^2 \sum_{q'} \left(f_{q0;q-q',q'} \frac{1}{-(q'^2/2m)-[(q-q')^2/2m]-(q^2/2m)-Nt_0(f_{q'}+f_{q-q'}+f_q)} \times 2f_{q',q-q';0,q} - f_{q'}\frac{1}{-(q'^2/m)}f_{q'} \right), \quad (3.16')$$

because the expansion of

$$\tfrac{1}{2}N^2t_0^2\sum_q f_q \frac{1}{-(q^2/m)-2Nt_0f_q-2\Delta_q^{(1)}} f_q$$

gives (3. 16); (3. 17) can also be interpreted in a similar way with

$$\Delta^{(2)} = -Nt_0^2\sum_q f_{q'}\frac{1}{-(q'^2/m)-2Nt_0f_{q'}}f_{q'}, \quad (q\text{-independent}). \quad (3.17')$$

But for (3. 17), we must note that there is another way of interpretation, namely as a change of total number appearing in the second order energy; we note that we can write (3. 17) as follows:

$$\Delta N \frac{d}{dN}[N(N-1)]t_0^2\tfrac{1}{2}\sum_{q'} f_{q'}\frac{1}{-(q'^2/m)-2Nt_0f_{q'}}f_{q'}, \quad (3.17'')$$

where

$$\Delta N = -\tfrac{1}{2}N(N-1)t_0^2\sum_q f_q\frac{2}{[-(q^2/m)-2Nt_0f_q]^2}f_q = -\langle \textstyle\sum_k n_k\rangle_{\text{2nd order}},$$

and $\langle\ \rangle_{\text{2nd order}}$ means the second-order expectation value of excited particle numbers with respect to the ground state wave function of the system

$$H=\textstyle\sum_p(p^2/2m)n_p+\frac{1}{2}[v(N)+\text{c.c.}].$$

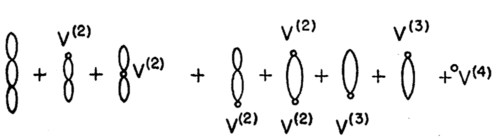

FIG. 8. Fourth order scattering contribution to the energy.

These considerations suggest that the effect of "connected" diagrams which are apparently separated is a change in the one-particle energy and a change in the total number appearing in $v(N)$ and $v^\dagger(N)$. This dual interpretation is possible only for the lowest-order diagram [Fig. 9(b)], because in this diagram we can pack the effect of either loop on the other; in higher

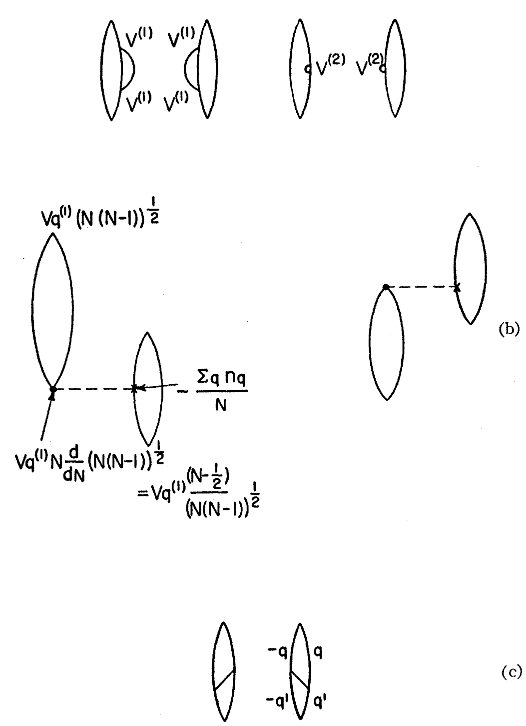

FIG. 9. (a) "Self-energy" type correction. (b) Fourth order apparently disconnedted "connected" diagram. (c) "Vertex" type correction.

orders, as is shown in Fig. 10, the interpretation which we must take becomes to some extent fixed by the diagram.

Addition of (a) and (b) gives

$$\Delta E_0{}^{(4)(a)}=\tfrac{1}{2}N^2t_0{}^2\sum_q f_q\frac{1}{[-(q^2/m)-2Nt_0f_q]^2}f_q \times 2(\Delta_q{}^{(1)}+\Delta^{(2)}). \quad (3.18)$$

Before evaluating the integrals, suppose one connects two-particle lines of some diagram, as shown in Fig. 11, it introduces two t_0 with one N and two energy denominators with one momentum summation, hence in the x variables defined in (3.4), the contribution to the energy is to multiply the original expression with (symbolically)

$$\text{(II)}\quad t_0{}^2N\sum\frac{1}{[(q^2/m)+\cdots]^2} \rightarrow t_0{}^2NVm^2\frac{1}{(2Nt_0m)^2}\frac{1}{x}\rightarrow(\rho\alpha^3)^{\frac{1}{2}}\frac{1}{x}.$$

For the graphs of Fig. 9(a) and (b) or 9(c), the original graph is given in Fig. 12(a)

$$t_0{}^2N^2\sum_q\frac{1}{(q^2/m)+\cdots}\rightarrow N\rho\alpha(\rho\alpha^3)^{\frac{1}{2}}\int dx,$$

and hence for (a) or (b) [using the abbreviation $b=1/(2Nt_0m)^{\frac{1}{2}}a=1/(8\pi\rho\alpha a^2)^{\frac{1}{2}}$]

$$N\rho\alpha(\rho\alpha^3)^{\frac{1}{2}}\int dx\,(\rho\alpha^3)^{\frac{1}{2}}\frac{1}{x}=N\rho\alpha\rho\alpha^3\int^{\sim b}\frac{dx}{x},$$

where $1/(\rho\alpha a^2)^{\frac{1}{2}}\sim b$ represents the cutoff provided by the "shape" function, and the integral converges at the lower end (generally because of the "gap" of order unity).

Thus, if we change the summation variable q into x in (3.18), then the integral over x diverges logarithmically in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}\rightarrow 0$:

$$\Delta E_0{}^{(4)(a)}=N\frac{2\pi\rho\alpha}{m}8\pi\rho\alpha^3\frac{1}{\pi}\left[\int_0^{\sim b}dx\,x^2\frac{2[(3x^2+6)^{\frac{1}{2}}-1]}{(x^2+1)^2}+\text{power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}\right],$$

where the numerator of the integrand is the sum of two "corrections" (3.16′) and (3.17′) to the one-particle energy, and the power series in $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$ is the shape-dependent part of this energy change (note that $\Delta_q{}^{(1)}+\Delta^{(2)}$ converges even if one puts $f\rightarrow 1$). We can determine the coefficient of the logarithmic term independent of the shape of the potential:

$$\Delta E_0{}^{(4)(a)}=N(2\pi\rho\alpha/m)8\pi\rho\alpha^3[-((3)^{\frac{1}{2}}/\pi)\ln(8\pi\rho\alpha a^2)+\text{constant}+\text{power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}]. \quad (3.19)$$

One should note that besides the ln dependence, the $\rho\alpha^3$ dependence of the constant term is the same as the third-order energy $\Delta E_0{}^{(3)}$.

The inclusion of pair creation and annihilation diagrams (I), which we expect to contribute to the same order in $\rho\alpha\rho\alpha^3$, brings in more energy denominators without increasing the number of momentum summations and hence there appears no divergence in the x-integral in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$ going to zero. For this reason, there appears no logarithmic term from these diagrams; of course, the constant term is affected by these diagrams [the converging x-integral in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}\rightarrow 0$ is the series in $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$ of the form $A+B(8\pi\rho\alpha a^2)^{\frac{1}{2}}+C(8\pi\rho\alpha a^2)+\cdots$]. The contribution from Fig. 9(c) can be written in the following way:

$$\Delta E_0{}^{(4)(b)}=2\frac{N^2}{2}t_0{}^2\sum_q\sum_{q'}2Nt_0{}^2f_{q'}\frac{1}{-(q'^2/m)-2Nt_0f_{q'}} \times f_{0q';q,q'-q}\frac{1}{-(q^2/2m)-[(q-q')^2/2m]-(q'^2/2m)-Nt_0(f_q+f_{q-q'}+f_{q'})}2f_{q'-q,-q';-q0}\frac{1}{-(q^2/m)-2Nt_0f_q}f_q. \quad (3.20)$$

When transformed to x-variables, this integral also diverges logarithmically in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}\rightarrow 0$, and the coefficient of the logarithm can be determined without knowing the detailed shape of the potential:

$$\Delta E_0{}^{(4)(b)}=N(2\pi\rho\alpha/m)8\pi\rho\alpha^3[\tfrac{4}{3}\ln(8\pi\rho\alpha a^2)+\text{const}+\text{higher power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}]. \quad (3.21)$$

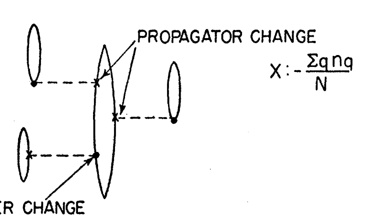

FIG. 10. Higher order corrections due to an apparently disconnected connected diagram.

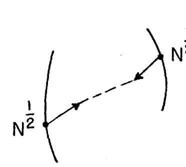

FIG. 11. Connection of two-particle lines (II).

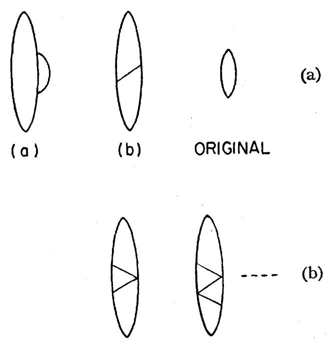

FIG. 12. (a). Original graph and induced graph for agument (II). (b) Three-particle correlation diagram.

Summing the contributions from the scattering diagrams and (a), (b), and (c), we get

$$\Delta E_0^{(4)}=N(2\pi\rho\alpha/m)\{8\pi\rho\alpha^3[G_3\ln(8\pi\rho\alpha a^2)+\text{constant}+\text{power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}]+(8\pi\rho\alpha^3)^{\frac{3}{2}}[1+\text{power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}]\},\quad(3.22)$$

with[11]

$$G_3=\tfrac{4}{3}-(3)^{\frac{1}{2}}/\pi.$$

(*e*) Finally, let us discuss higher order effects briefly. The scattering type of diagrams have always their counter terms which subtract the divergence in the x-integrals in the limit of $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$ being zero, and the resulting x-integrals converge in this limit at the upper end; hence, the contribution starts from the term which is independent of the shape of the potential, namely, only a function of α. This term can be evaluated easily, and is

$$E_{0\,\text{scattering}}{}^{(n)}=\tfrac{1}{2}N^2t_0{}^n\left[\sum_q\left(\frac{1}{-(q^2/m)-2Nt_0}-\frac{1}{-q^2/m}\right)\right]^{n-1}+\text{shape dependent term as a power series}$$
$$=N(2\pi\rho\alpha/m)(8\pi\rho\alpha^3)^{(n-1)/2}\times[1+\text{power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}].\quad(3.23)$$

The factor 1 in the first term, of course, changes when the diagrams of pair creation and annihilation (I) are taken into account.

There are also diagrams of the form of Fig. 12(b). These introduce one t_0 with one summation over momentum and one energy denominator; namely, in the x-variable (symbolically)

$$\text{(III)}\quad t_0\sum_q\frac{1}{[(q^2/m)+\cdots]}\longrightarrow t_0Vm(2Nt_0m)^{\frac{1}{2}}x\longrightarrow(\rho\alpha^3)^{\frac{1}{2}}x$$

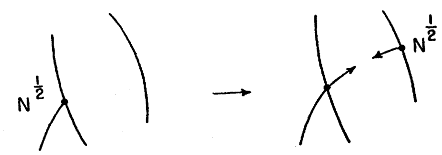

FIG. 13. Formation of three-particle correlation (III).

(see Fig. 13), to the integrand of (3.20) which has been diverging logarithmically in the limit of small $(8\pi\rho\alpha a^2)^{\frac{1}{2}}$. Whence the first diagram of Fig. 12(b) has a linear divergence and the second a quadratic one in this limit. The leading terms are (using the same abbreviation for the upper limit of the integral as before)

$$N\frac{\rho\alpha}{m}\rho\alpha^3\int^{\sim b}\frac{dx}{x}[(\rho\alpha^3)^{\frac{1}{2}}x,\,(\rho\alpha^3)x^2,\cdots]=N(\rho\alpha/m)\rho\alpha^3[\alpha/a,(\alpha/a)^2,\cdots].$$

Take for example the second diagram:

$$m^5N^3t_0{}^6V^4(q^{12}/q^{10})=NNt_0Nt_0m(Vt_0m)^4(Nt_0m)\int^b xdx$$
$$=N\frac{\rho\alpha}{m}(\rho\alpha^3)^2\left(\frac{A}{(\rho\alpha a^2)}+\frac{B}{(\rho\alpha a^2)^{\frac{1}{2}}}+c\ln(\rho\alpha a^2)+\text{constant}+\text{power series in }(\rho\alpha a^2)^{\frac{1}{2}}\right)$$
$$=N\frac{\rho\alpha}{m}\rho\alpha^3\left(\frac{\alpha}{a}\right)^2\{A+B(\rho\alpha a^2)^{\frac{1}{2}}+[\text{constant}+C\ln(\rho\alpha a^2)](\rho\alpha a^2)+\text{higher power series in }(\rho\alpha a^2)^{\frac{1}{2}}\},\quad(3.24)$$

where we have picked out the most seriously diverging expression in the second line [in the limit of $(\rho\alpha a^2)^{\frac{1}{2}}$ going to zero]; q^{12} shows four summations, and q^{10} shows five energy denominators. One concludes that this type of diagram gives rise to a power series in α/a without increasing the $\rho\alpha^3$-dependence and also they may contain a logarithmic term multiplied by a higher power of $(\rho\alpha a^2)^{\frac{1}{2}}$. Consequently, the logarithmic term in (3.19), (3.21), and (3.22) is not affected by these higher orders (even in the case of the hard core $\alpha=a$). The effect of opposite-momentum pair creation and annihilation processes (I) brings down the degree of divergence in the above considerations and so the integral over x starts from higher power in $(\rho\alpha a^2)^{\frac{1}{2}}$ than the original graph. [In (3.24), $A/\rho\alpha a^2+B/(\rho\alpha a^2)^{\frac{1}{2}}+C+\cdots$ is changed into $A+B(\rho\alpha a^2)^{\frac{1}{2}}+\cdots$ because the addition of process (I) makes the x-integral converge

[11] The same coefficient was found by T. T. Wu by the pseudopotential method and also by D. Pines and N. Hugenholtz (private communications).

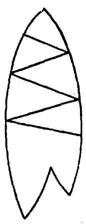

FIG. 14. Effect of pair creation-annihilation diagram (I) on three-particle correlation diagram.

at the upper end.] Take for another example, Fig. 14:

$$V^6m^9t_0{}^8N^3\frac{q^{18}}{q^{14}}(Nt_0)^2\frac{1}{q^4} = V^6m^9t_0{}^8N^3(Nt_0)^2\int^{\sim b}\frac{dx}{x}=N\frac{\rho\alpha}{m}(\rho\alpha^3)^3\times$$

$[\ln(\rho\alpha a^2)+\text{constant}+\text{power series in }(8\pi\rho\alpha a^2)^{\frac{1}{2}}]$. Hence the effect of (I) on the diagrams in Fig. 12(b) goes into higher order [Fig. 14 without process (I) is $N(\rho\alpha/m)\rho\alpha^3(\alpha/a)^4$], although they may contain logarithmic dependence. The diagrams shown in Fig. 12(b) are proper three-body interactions and make the evaluation of the constant term in $\rho\alpha\rho\alpha^3$ very difficult in the case of a hard core ($\alpha=a$), but for the soft-core case one may obtain the corresponding corrections in power series in α/a. There are also diagrams of the form Fig. 15 with forward scattering between two loops; and the contribution goes like

$$\begin{aligned}V^2m^4N^4t_0{}^5&\frac{q^6}{(q^2+\cdots)^4}\\ &=V^2m^4N^4t_0{}^5\frac{1}{Nt_0m}\int^{\sim b}\frac{dx}{x^3+\cdots}\\ &=N(\rho\alpha/m)\rho\alpha^3[A+B(\rho\alpha a^2)^{\frac{1}{2}}+\cdots],\end{aligned}\quad(3.\,25)$$

and contributes to $\rho\alpha\rho\alpha^3$.

From these arguments, it becomes clear that our ln term in (3. 22) is not affected by the higher-order corrections but to get the constant term in the order of $\rho\alpha\rho\alpha^3$, we must sum up $\Delta E_0{}^{(3)\prime}$ (3. 14) and the constant terms in (3. 22) and (3. 25), and in the latter two we must include the effect of pair creation and annihilation processes (I). For the hard-core case ($\alpha=a$), we must add to this the contribution from Fig. 12(b), for which we need not take into account the process (I), the effect of which occurs in higher order in $\rho\alpha$ as was discussed above; moreover, because there are no infrared divergences even if we omit the energy "gap" of one particle energy, we can use the completely free-particle energy in the energy denominator, and so the contribution coming from Fig. 12(b) is the free three-particle correlation energy.

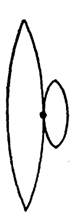

FIG. 15. Diagram showing forword scattering between excited particles contained in different loop.

4. SELF-CONSISTENT FORMULATION

As we have seen in the fourth order treatment in the above section, the "connected" diagrams which are apparently disconnected contribute to the one-particle energy change [which is momentum-independent, because the change induced comes from apparently disconnected graphs, $\Delta^{(2)}$ (3. 17′)] and the change of N which appears upon replacing the zero-momentum creation and annihilation operators by functions of the total particle number and the total excited particle number operators. Expecting this result from the beginning, we would write our Hamiltonian (2. 5) in the following form:

$$H_{\text{total}}=H_0+H_I,$$

$$H_0=\sum_q n_q[(q^2/2m)-\mu]+\tfrac{1}{2}[v(\bar{N})+\text{c.c.}],$$

$$H_I=\tfrac{1}{2}\left[\sum_{m=1}^{\infty}\bar{N}^m\frac{d^m}{d\bar{N}^m}v(\bar{N})\frac{1}{m!}\left(\frac{N-\bar{N}-\sum_k n_k}{\bar{N}}\right)^m+\text{c.c.}\right]+\sum_q n_q\mu,\quad(4.\,1)$$

which is obtained from (2. 5a) by expanding functions of $N-\sum_q n_q$ around the point $\bar{N}$; in (4. 1)

$$\begin{aligned}v(\bar{N})=&\tfrac{1}{2}v_0\bar{N}(\bar{N}-1)+\textstyle\sum_p n_p(v_p+v_0)\bar{N}\\&+\textstyle\sum_q\beta_q{}^*\beta_{-q}{}^*v_q[\bar{N}(\bar{N}-1)]^{\frac{1}{2}}\\&+\textstyle\sum_q\sum_p\beta_q{}^*\beta_p{}^*(v_q+v_p)\beta_{p+q}(\bar{N})^{\frac{1}{2}}\\&+\tfrac{1}{2}\textstyle\sum_q\sum_p\sum_{p'}\beta_{p+q}{}^*\beta_{p'}{}^*v_q\beta_{p'+q}\beta_p,\end{aligned}\quad(4.\,2)$$

μ is the one-particle energy correction which we expect from consideration of (3. 17′), and $\bar{N}$ is the "corrected" value of N corresponding to $N+\Delta N$ (3. 17″). We assume from the result of these

$$N-\bar{N}\approx O(\bar{N})\approx O(N),\quad \langle\textstyle\sum_k n_k\rangle_{\bar{N}}\approx O(\bar{N})\approx O(N),\quad \mu\approx O(1),\quad(4.\,3)$$

where

$$\langle F\rangle_{\bar{N}}=(\Psi_{\bar{N}},F\Psi_{\bar{N}}),$$

with

$$(E_{\bar{N}}-H_0)\Psi_{\bar{N}}=0.$$

Regarding H_I as perturbation, we want to determine μ and $\bar{N}$ to make the energy correction due to this perturbation one order less than the zero-order energy about the numbers $\bar{N}$ or N.

The zero-order energy is the energy coming from the Hamiltonian

$$H_0=K+V(\bar{N}),\quad K=\textstyle\sum_q n_q[(q^2/2m)-\mu],\quad V(\bar{N})=\tfrac{1}{2}[v(\bar{N})+\text{c.c.}],\quad(4.\,4)$$

and can be evaluated using connected diagrams only with $V(\bar{N})$, and by a simple counting we can show that the energy is proportional to $\bar{N}$; we denote this as $E_{\bar{N}}$.

To first order in the perturbation, we have the sums of diagrams Fig. 16(a),[12] where for a large circle we take an arbitrary connected diagram with $V(\bar{N})$, | | represents some internal line (two lines in Fig. 16 do not mean particle number 2), a small circle represents $\bar{N}^m(d^m/d\bar{N}^m)V(\bar{N})[(N-\bar{N})/\bar{N}]^{m-n}$ with any m, n ($n\leqq m$),[12] and each cross is for $-\sum_k n_k/\bar{N}$. See also Fig. 17. The sum becomes equivalent to a single action of $\Delta+\mu\sum_q n_q$ as shown at the end of Fig. 16(a), where[12]

$$\Delta=\sum_{m=1}^{\infty}\bar{N}^m\frac{d^m}{d\bar{N}^m}V(\bar{N})\left[\frac{1}{m!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^m+\frac{1}{(m-1)!1!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-1}\left\langle\frac{-\sum_q n_q}{\bar{N}}\right\rangle_{\bar{N}}\right.$$
$$\left.+\frac{1}{(m-2)!2!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-2}\left(\frac{-\sum_q n_q}{\bar{N}}\right)^2_{\bar{N}}+\cdots+\frac{1}{m!}\left(\frac{-\sum_q n_q}{\bar{N}}\right)^m\right]$$
$$=\sum_{m=1}^{\infty}\frac{1}{m!}\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)^m\bar{N}^m\frac{d^m}{d\bar{N}^m}V(\bar{N}),\tag{4. 5}$$

where we have written the contribution of the form Fig. 18 summed over diagrams obtained by taking all possible connected diagrams for a large circle as $\langle-\sum_q n_q/\bar{N}\rangle_{\bar{N}}$, because the sum can be written as follows: (ψ_0: no particle wave function)[13]

$$\left(\psi_0,\ \sum_{\text{connected}}\ \sum_{n,m=0}^{\infty}\left[V(\bar{N})\frac{1}{-H_0}\right]^m\frac{-\sum_q n_q}{\bar{N}}\left[\frac{1}{-H_0}V(\bar{N})\right]^n\psi_0\right)$$
$$=\left(\psi_0,\ U_\alpha^{-1\dagger}(-\infty,0)\frac{-\sum_q n_q}{\bar{N}}U_\alpha^{-1}(-\infty,0)\psi_0\right)_{\text{connected}}$$
$$=\left(\psi_0,\ U_\alpha^{-1\dagger}(-\infty,0)\frac{-\sum_q n_q}{\bar{N}}U_\alpha^{-1}(-\infty,0)\psi_0\right)\Big/(\psi_0,U^{\alpha-1}(-\infty,0)U_\alpha^{-1}(-\infty,0)\psi_0)=\left(\Psi_{\bar{N}},\frac{-\sum_q n_q}{\bar{N}}\Psi_{\bar{N}}\right),$$

by using the definition of the wave function in Gell-Mann and Low[6,14] Eq. (10). Note that each graph in Fig. 16 is of order $\bar{N}$,[15] and is of the same order as the zero-order energy.

The next higher order diagrams are obtained by mapping[16] each lower large circle of Fig. 16 by the sum represented in Fig. 19; the factors 2, 3, $\cdots$ in the second line of the figure come from the fact that there are m possibilities to choose one $-\sum_q n_q/\bar{N}$ for the shaded portion from the product of m operators,

[12] The interaction energy H_I in (4. 1) contains one special kind of interaction arising from the derivative of the 1st term of $v(N)$ and $v^\dagger(N)$ in (4. 2) [namely, $\frac{1}{2}v_0\bar{N}(\bar{N}-1)\approx\frac{1}{2}v_0\bar{N}^2=E_{\bar{N}}$ diagonal]. Because the differentiation of this term about $\bar{N}$ gives interaction consisting of the C-number multiplied by $(-\sum_q n_q/\bar{N})$, the diagrams corresponding to Fig. 16(a), 19(a), and 21(a) are different from these figures and are given in Fig. 16(b), 19(b), and 21(b). The weight in Fig. 19(b) and 21(b) (the factor 2) comes from the same reason as the weight in Fig. 19(a) and 21(a). Figure 16(b) gives

$$v_0\bar{N}^2\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)+\frac{1}{2!}v_0\bar{N}^2\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)^2,$$

[for $\langle\sum_q n_q\rangle_{\bar{N}}$, see under Eq. (4. 5)] and so Eq. (4. 5) holds including Fig. 16(b).

For Fig. 19(b) [or Fig. 21(b) which is the inverted Fig. 19(b)], we have

black square
$$=v_0\bar{N}^2\left(-\frac{\sum_q n_q}{\bar{N}}\right)+\frac{2}{2!}v_0\bar{N}^2\frac{-\sum_q n_q}{\bar{N}}\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right),$$

and if we include this contribution to Eq. (4. 6), Eq. (4. 6) holds if we omit the C-number [namely, omit from the first term in the last equality $\bar{N}(d/d\bar{N})(\frac{1}{2}v_0\bar{N}^2)$ and $\bar{N}^2(d^2/d\bar{N}^2)(\frac{1}{2}v_0\bar{N}^2)$]. The same holds for (4. 9).

[13] The meaning of the formulas is the same as given in reference 8.

[14] The proof can be seen by constructing the expectation value of the quantity F following reference 6:

$$(\Psi_0,F\Psi_0)=\lim_{\alpha\to0}\frac{(\psi_0,\ U_\alpha^{-1\dagger}(-\infty,0)FU_\alpha^{-1}(-\infty,0)\psi_0)}{(\psi_0,\ U_\alpha^{-1\dagger}(-\infty,0)U_\alpha^{-1}(-\infty,0)\psi_0)}$$

where U_α^{-1} is the expression (2. 2) for the system H_0; and putting expressions (2. 2) or Eq. (A9) of reference 6 for $U_\alpha^{-1}(-\infty,0)$, we recognize that $(\Psi_0,\ F\Psi_0)$ can be represented by connected diagrams only.

[15] The argument is the same as given in paragraph 2; each $-\sum_q n_q/\bar{N}$ is of order $1/\bar{N}$; $\bar{N}^m(d^m/d\bar{N}^m)V(\bar{N})$ is of order $V(\bar{N})$; one loop gives a contribution of order $\bar{N}$ to the energy; for each term of Fig. 16(a) one has $m+1$ loops with $(-\sum_q n_q/\bar{N})^m$; hence $(\bar{N})^{m+1}(1/\bar{N})^m\to\bar{N}$.

[16] The word "mapping" used hereafter means to take the same connected diagram for one of the lower large circle in one of the diagrams of Fig. 16 and the shaded part of one of the diagrams of Fig. 19, and form a new diagram having the shaded area as a common part.

$(-\sum_q n_q/\bar{N})^m$; the shaded portion is to be understood in Fig. 19 as a portion to map below each of Fig. 16. The sum is equivalent to taking for the black triangle in Fig. 19(a)[12]:

$$\begin{aligned}
\text{Black triangle} &= \mu \textstyle\sum_q n_q + \sum_{m=1}^{\infty} \bar{N}^m \frac{d^m}{d\bar{N}^m} V(\bar{N}) \left[\frac{1}{m!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^m + \frac{1}{(m-1)!1!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-1} \left\langle \frac{-\sum_q n_q}{\bar{N}} \right\rangle_{\bar{N}} + \cdots \right] \\
&\quad + \sum_{m=1}^{\infty} \frac{-\sum_q n_q}{\bar{N}} \left\langle \bar{N}^m \frac{d^m}{d\bar{N}^m} V(\bar{N}) \right\rangle_{\bar{N}} \left[\frac{1}{(m-1)!1!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-1} + \frac{2}{(m-2)!2!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-2} \right. \\
&\quad \left. \times \left\langle \frac{-\sum_q n_q}{\bar{N}} \right\rangle_{\bar{N}} + \frac{3}{(m-3)!3!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-3} \left\langle \frac{-\sum_q n_q}{\bar{N}} \right\rangle_{\bar{N}}^2 + \cdots \right] \\
&= \sum_{m=1}^{\infty} \frac{1}{m!}\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)^m \bar{N}^m \frac{d^m}{d\bar{N}^m} V(\bar{N}) + \mu \textstyle\sum_q n_q - \sum_q n_q \left\langle \frac{d}{d\bar{N}} V(\bar{N}) \right\rangle_{\bar{N}} \\
&\quad - \frac{\sum_q n_q}{\bar{N}} \sum_{m'=1}^{\infty} \frac{1}{m'!}\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)^{m'} \left\langle \bar{N}^{m'+1} \frac{d^{m'+1}}{d\bar{N}^{m'+1}} V(\bar{N}) \right\rangle_{\bar{N}}, \qquad (4.6)
\end{aligned}$$

where $\langle \bar{N}^m (d^m/d\bar{N}^m) V(\bar{N})\rangle_{\bar{N}}$ represents the sum of Fig. 20 over all possible connected diagrams for the large circle (the proof is the same as that for $\langle\sum_q n_q\rangle_{\bar{N}}$). If we take for the equations of $\bar{N}$ and μ

$$N-\bar{N}-\langle\textstyle\sum_q n_q\rangle_{\bar{N}} = O(1), \qquad (4.7)$$

$$\mu - \left\langle \frac{d}{d\bar{N}} V(\bar{N}) \right\rangle_{\bar{N}} = \mu - \frac{\partial}{\partial \bar{N}} E_{\bar{N}} = O\left(\frac{1}{\bar{N}}\right), \qquad (4.8)$$

then (4. 6) becomes an operator composed of

$$\text{Black triangle} = \frac{O(1)}{\bar{N}} V(\bar{N}) + O(1) \frac{\sum_q n_q}{\bar{N}}, \qquad (4.6')$$

because $\bar{N}^m(d^m/d\bar{N}^m)V(\bar{N})$ is of order $V(\bar{N})$,

$$\langle \bar{N}^m (d^m/d\bar{N}^m) V(\bar{N})\rangle_{\bar{N}} \approx \langle V(\bar{N})\rangle_{\bar{N}} \approx O(\bar{N})$$

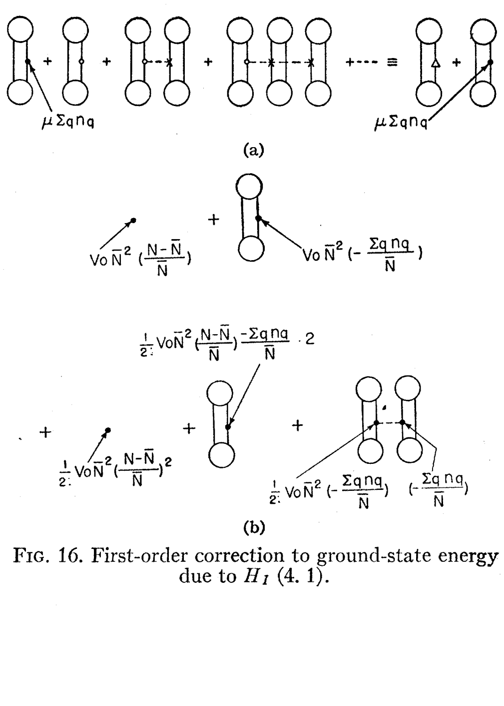

FIG. 16. First-order correction to ground-state energy due to H_I (4. 1).

[note that the quantity we denote by $O(1)$ is not an infinite sum of quantities of $O(1)$ but a few terms of $O(1)$]. Each diagram obtained by mapping (4. 6) below each one member of Fig. 16 contributes to the energy of order one, since, as we remarked before, each one of the diagrams in Fig. 16 is composed of $m+1$ apparently disconnected loops with $(-\sum_q n_q/\bar{N})^m$ (where m is an integer greater than or equal to zero) and each loop consists of interaction of order $V(\bar{N})$, hence mapping to lowest large circle with the last diagram of Fig. 19 produces diagrams consisting of either $m+1$ loops with $(-\sum_q n_q/\bar{N})^m$ and $1/\bar{N}$ [coming from first factor of (4. 6′)] or with $(-\sum_q n_q/\bar{N})^{m+1}$, so the contribution to the energy is $(\bar{N})^{m+1}(1/\bar{N})^{m+1}$, namely, order one.

The higher-order diagrams are obtained by mapping (4. 6) again to lower large circle of each of lower order diagrams. Each diagram before mapping with the last sum of Fig. 19 consists of $m+1$ loops each consisting of an interaction of order $V(\bar{N})$ with $(-\sum_q n_q/\bar{N})^m$ (where m is again an integer greater or equal to zero); hence the above consideration persists and the contributions to

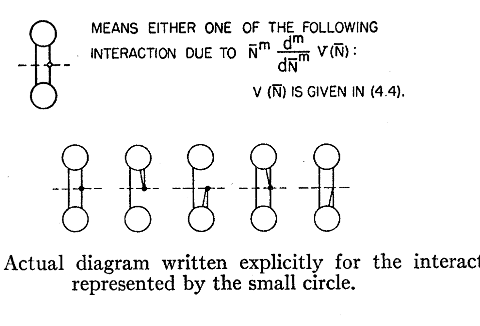

FIG. 17. Actual diagram written explicitly for the interaction represented by the small circle.

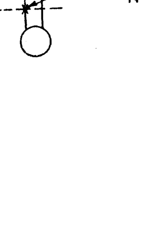

FIG. 18. Diagram which gives expectation value of excited particle operator (divided by $\bar{N}$).

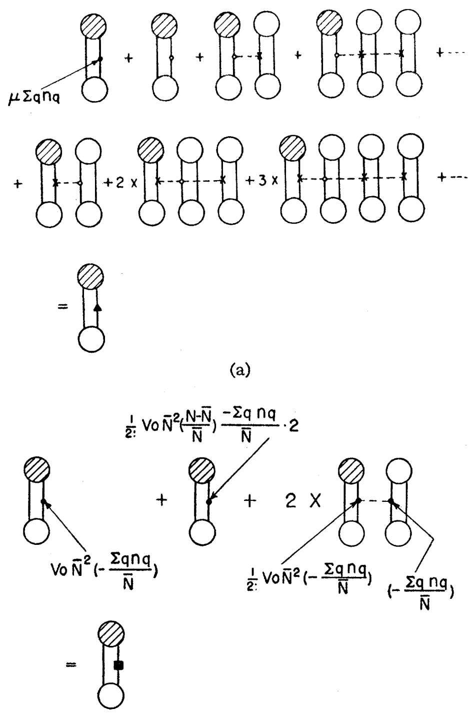

FIG. 19. Diagrams we should "map" to each of the lower large circles of Fig. 16, to get second order diagrams (the shaded area is the "mapping" area).

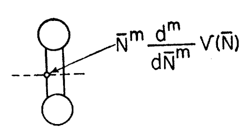

FIG. 20. Diagram which gives expectation value of $\bar{N}^m(d^m/d\bar{N}^m)V(\bar{N})$.

the energy from higher order diagrams are of order one.

Actually, the quantities of order one in the second, third, $\cdots$ order energies consist of sums of diagrams we have mapped with sum (4. 6), and the condition

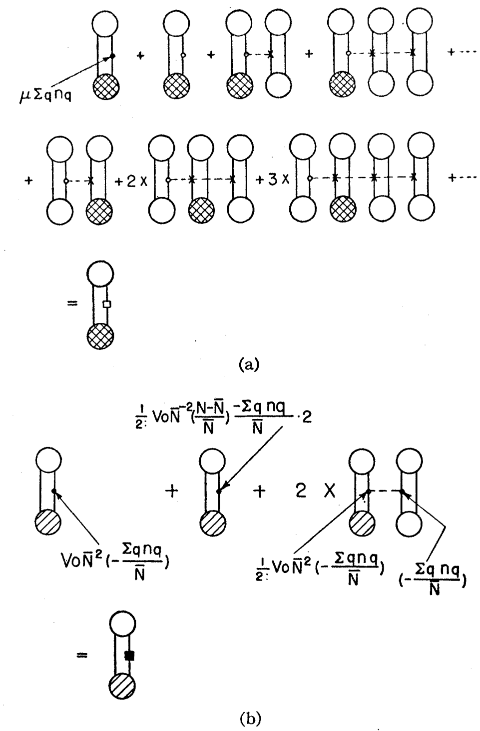

FIG. 21. Diagram showing possible position and weight of Fig. 16 to get second order diagram by mapping with Fig. 19.

(4. 7) and (4. 8) can be reapplied and the sum of order-one quantities goes into a much lower order quantity, e.g. order $1/\bar{N}$, order $(1/\bar{N})^2$, etc. For instance let us consider the second-order diagram Fig. 21(a), where the shaded area represents the portion of diagrams in Fig. 16(a) which we are going to map with the sum in Fig. 19(a) and 2, 3$\cdots$ appears for the same reason as in Fig. 19(a); namely, there are m possibilities to choose one $(-\sum_q n_q/\bar{N})$ out of $(-\sum_q n_q/\bar{N})^m$ for the shaded area. The sum Fig. 21(a) is equivalent to taking the square (of the last figure in Fig. 21(a)) as[12]

$$\begin{aligned}\text{Square}=&\mu\sum_q n_q+\sum_{m=1}^{\infty}\left\langle\bar{N}^m\frac{d^m}{d\bar{N}^m}V(\bar{N})\right\rangle_{\bar{N}}\left[\frac{1}{(m-1)!1!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-1}+\frac{2}{(m-2)!2!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-2}\left\langle\frac{-\sum_q n_q}{\bar{N}}\right\rangle_{\bar{N}}\right.\\&\left.+\frac{3}{(m-3)!3!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-3}\left\langle\frac{-\sum_q n_q}{\bar{N}}\right\rangle_{\bar{N}}^2+\cdots\right]\frac{-\sum_q n_q}{\bar{N}}\\&+\sum_{m=1}^{\infty}\bar{N}^m\frac{d^m}{d\bar{N}^m}V(\bar{N})\left[\frac{1}{m!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^m+\frac{1}{(m-1)!1!}\left(\frac{N-\bar{N}}{\bar{N}}\right)^{m-1}\left\langle\frac{-\sum_q n_q}{\bar{N}}\right\rangle_{\bar{N}}+\cdots\right]\\=&\left(\mu-\left\langle\frac{d}{d\bar{N}}V(\bar{N})\right\rangle_{\bar{N}}\right)\sum_q n_q+\sum_{m'=1}^{\infty}\left\langle\bar{N}^{m'+1}\frac{d^{m'+1}}{d\bar{N}^{m'+1}}V(\bar{N})\right\rangle_{\bar{N}}\frac{1}{m'!}\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)^{m'}\frac{-\sum_q n_q}{\bar{N}}\\&+\sum_{m=1}^{\infty}\bar{N}^m\frac{d^m}{d\bar{N}^m}V(\bar{N})\frac{1}{m!}\left(\frac{N-\bar{N}-\langle\sum_q n_q\rangle_{\bar{N}}}{\bar{N}}\right)^m,\qquad(4.\,9)\end{aligned}$$

FIG. 22. Second order diagram. (Middle large circle is the shaded area of Fig. 21 or Fig. 19, which is the same.)

and by (4. 7) and (4. 8)

$$\text{Square} \approx O(1)\frac{\sum_q n_q}{\bar{N}} + V(\bar{N})\frac{O(1)}{\bar{N}}. \qquad (4.\ 9')$$

Hence the sum of the second-order terms, which can be represented by Fig. 22, contributes to the energy as a connected diagram composed of an interaction of order $V(\bar{N})$ with an extra factor $(1/\bar{N})^2$, and so is of order $1/\bar{N}$.

In the first order diagram Fig. 16(a), even if we apply (4. 7) and (4. 8), there remains one term, namely

$$\Delta E_{\bar{N}}^{(1)} = \mu\langle\sum_k n_k\rangle_{\bar{N}} + O(1), \qquad (4.\ 10)$$

and so the energy of the system is

$$E_0 = E_{\bar{N}} + \mu\langle\sum_k n_k\rangle_{\bar{N}} \qquad (4.\ 11)$$

to the order $\bar{N}$. In these discussions we have discarded all the doubly or multiply connected graphs generated by the interaction H_I of the type illustrated in Fig. 23, because the contribution of these to E_0 is at most of order one, as was discussed in paragraph 2.

The above considerations concerning the evaluation of the ground-state energy can be extended to the evaluation of the low-lying excited state energies and conditions (4. 7) and (4. 8) are sufficient to get the excitation energy to order $O(1)$ by the following formula:

$$E_{\text{excitation}} = E_{\bar{N}}^{\text{excited}} - E_{\bar{N}}^{\text{ground}}, \qquad (4.\ 12)$$

because the excitation energy of zero order [regarding H_I (4. 1) as perturbation] is of order one and the perturbation caused by the interaction H_I becomes at most of order $1/\bar{N}$ by conditions (4.7) and (4. 8).

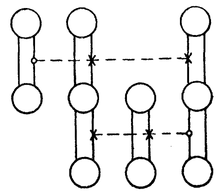

FIG. 23. Diagrams which contribute to ground state energy of order one, and of the same nature as the diagrams shown in Fig. 2.

The formulation we got above leads to the same formulation as that of Beliaev.[3]

5. CONCLUSIONS

We have shown that for the low-density Bose gas the energy of the ground state can be evaluated without a full knowledge of the potential energy between two particles up to the order $N\rho\alpha\times\rho\alpha^3\ln(\rho\alpha a^2)$, if we know the scattering length α at zero energy between two particles and the soft-core radius a (or the characteristic length of the potential) and, at the same time, the effect of the large population of zero-momentum particles can be replaced by a C-number weight of the interaction energy between excited particles and a change of the energy of the excited particles, and these are determined by Eqs. (4. 7) and (4. 8).

The fluctuation of the energy from the energy obtained by the prescription of paragraph 4 seems to be sufficiently small (for the ground-state energy whose main part is of order N, the fluctuation appears as a correction of order 1).

We have not yet investigated the effect of a different "shape" of the potential energy in detail; a particularly interesting one will be the case where scattering length $\alpha\to 0$ does not mean potential energy $\to 0$, in which case the expansion (1. 6) will probably lose its usefulness.

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