Uniform Electromagnetic Field in the Theory of General Relativity

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A cosmological solution of the Einstein-Maxwell's field equations, corresponding to the case of a uniform (that is, covariant constant) electromagnetic field, is derived by means of simple geometrical arguments; the Riemannian manifold it corresponds to is the product of two ordinary surfaces of constant curvature, whose type and radius depend on the values of the cosmological constant and the invariants of the electromagnetic field. The world-lines of charged test particles have also a very simple geometrical meaning.

1. The remarkably simple approach used in this paper is made possible by the work of Rainich, Misner, and Wheeler,[1] whose main conclusions are therefore befittingly recalled. The entire content of the source-free Maxwell's equations—at least in the case of the non-null field—can be expressed in terms of its energy-momentum tensor $T_{\mu\nu}$ alone. The matrix $T_\mu{}^\nu$ has the eigenvalues $(\rho, \rho, -\rho, -\rho)$, where the positive scalar ρ is defined in a locally Galilean frame in terms of the two invariants of the electromagnetic field by the relationship:

$$\rho^2 = (\mathbf{h}^2 - \mathbf{e}^2)^2 + (2\mathbf{e}\cdot\mathbf{h})^2. \qquad (1)$$

Two 2-planes ("blades") are therefore determined at each event by the eigenvectors of $T_\mu{}^\nu$. The orientation of the blades is subjected to a differential condition whose geometrical meaning has been recently clarified[2]; in the particular case, the only one of importance to us, in which the "complexion" of the field

$$\alpha = -\tfrac{1}{2}\arctan\frac{2\mathbf{e}\cdot\mathbf{h}}{\mathbf{h}^2 - \mathbf{e}^2} \qquad (2)$$

is a constant, they amount to the integrability conditions for both sets of blades: that is, they can be jointed together to form two ∞^2 families of surfaces. Lastly, as a consequence of the absence of charges, $T_\mu{}^\nu$ must obviously have a vanishing divergence. Vice versa, it can be proved that if these three conditions are satisfied, a skew field $f_{\mu\nu}$ can be recovered from $T_{\mu\nu}$, which satisfies both sets of Maxwell's equations. $f_{\mu\nu}$ is unique up to an arbitrary, but constant, contribution to the complexion α.

It is important to stress that these considerations do not contain *any* reference to the fact that $T_{\mu\nu}$ is proportional to the Ricci tensor relative to the metric we use; they can therefore be applied also to the more general field equations[3]

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \qquad (3)$$

including the cosmological constant Λ. The conservation law for $T_{\mu\nu}$ is, of course, still identically fulfilled as a consequence of (3).

It will be shown that any manifold which is the *product of two surfaces of constant curvature* is a solution of (3); the requirements of positive energy and of the proper signature will, however, rule out some cases. The main physical features of the solution will also be discussed and it will be shown that the electromagnetic field has vanishing covariant derivatives.[4]

2. If space-time is topologically equivalent to the product of two ordinary surfaces, Σ_+ (coordinates x^0 and x^1) and Σ_- (coordinates x^2 and x^3), a tensor field of arbitrary type is said to be *decomposable* if (a) its components with mixed indices vanish and (b) its components relative to Σ_+ (Σ_-) depend only on the coordinates x^0 and x^1 (x^2 and x^3). We demand that the fundamental tensor be of this kind:

$$g_{\mu\nu} = g_{+\mu\nu}(x_+) + g_{-\mu\nu}(x_-). \qquad (4)$$

As the notation suggests, a $+$ or a $-$ attached to a tensor shall annihilate, in a frame of reference adapted to the decomposition, the components whose indices do not all refer to Σ_+ or Σ_-, respectively. Space-time itself in this case is technically also called decomposable.[5] Its Riemann and Ricci tensors are decomposable and its components are just the corresponding tensors relative to Σ_+ and Σ_- [6]; hence we can write

$$R_{\mu\nu} = -K_+ g_{+\mu\nu} - K_- g_{-\mu\nu}, \qquad (5)$$

where K_+ and K_- denote here the Gaussian curvatures of Σ_+ and Σ_-, respectively. The matrix $R_\mu{}^\nu$ is therefore diagonal, with eigenvalues $(-K_+, -K_+, -K_-, -K_-)$.

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[1] C. W. Misner and J. A. Wheeler, Ann. Phys. **2**, 525 (1957). See also G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[2] G. Rosen, Phys. Rev. **114**, 1179 (1959); B. Bertotti, Phys. Rev. **115**, 742 (1959).

[3] The notation is as follows: Greek indices range and sum from 0 to 3; a metric of the type $-+++$ is adopted, and the Ricci tensor is defined as

$$R_{\mu\nu} = -\Gamma^\alpha{}_{\mu\nu,\alpha} + \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\mu}\Gamma^\beta{}_{\alpha\nu} - \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu}.$$

The comma indicates an ordinary derivative; the electromagnetic field $f_{\mu\nu}$ is measured in such units that the gravitational constant is equal to one; we have, then,

$$T_{\mu\nu} = 2(f_{\mu\alpha}f_\alpha{}^\nu + \tfrac{1}{4}g_{\mu\nu}f_{\alpha\beta}f^{\alpha\beta}).$$

[4] As a reference to the problem of the static electromagnetic field in general relativity, we quote only two papers by W. B. Bonnor, Proc. Phys. Soc. (London) **A66**, 145 (1953) and **A67**, 225 (1954), where additional bibliography can be found. The "uniform" electric fields calculated in the former [Section 3(a)] do not have vanishing covariant derivatives.

[5] On product spaces in Riemannian geometry, see for example J. A. Schouten, *Ricci-Calculus* (Springer-Verlag, Berlin, 1954), p. 285; F. A. Ficken, Ann. Math. **40**, 892 (1939).

[6] See, e.g., P. Jordan, *Schwerkraft und Weltall* (Friedrich Vieweg and Son, Braunschweig, 1955), p. 55, or Ficken's paper quoted in reference 5.

If we now want $R_\mu{}^\nu + \Lambda\delta_\mu{}^\nu$ to be proportional to the energy-momentum tensor of an electromagnetic field, we must demand

$$K_+(x_+) = \Lambda - \rho, \quad K_-(x_-) = \Lambda + \rho, \tag{6}$$

which implies that ρ, K_+ and K_- are all constant.

We now proceed to the explicit construction of Σ_+ and Σ_- by means of a well-known theorem of differential geometry.[7] In a three-dimensional flat space with a line element of the form

$$d\lambda^2 = e_1 du^2 + e_2 dv^2 + e_3 dw^2, \quad (e_1, e_2, e_3 = \pm 1), \tag{7}$$

the quadric

$$e_1 u^2 + e_2 v^2 + e_3 w^2 = \pm r^2 \tag{8}$$

has a constant Gaussian curvature $\pm r^{-2}$. According to the different choices of the signature of (7) (four possibilities: $+++$, $++-$, $+--$, $---$) and the sign of the curvature we have six cases altogether; only the choices described in Fig. 1 will, however, give the proper signature to Σ_+ $(-+)$ and to Σ_- $(++)$. The requirement of positive energy ($\rho > 0$) demands, moreover, that

$$K_- > K_+. \tag{9}$$

It remains to be shown that the other part of Rainich's conditions is satisfied. As explained in Sec. 1, in the case $\alpha = \text{const}$, it is enough to ascertain that the blades spanned by the eigenvectors of $R_\mu{}^\nu + \Lambda\delta_\mu{}^\nu$ are integrable; which in our case is obvious, since they are tangent to Σ_+ and Σ_-, respectively.

In the absence of the electromagnetic field ($\rho = 0$) we have the two alternatives: $\Sigma_+^{(1)}$ and $\Sigma_-^{(2)}$ ($K_+ = K_- > 0$) or $\Sigma_+^{(2)}$ and $\Sigma_-^{(1)}$ ($K_+ = K_- < 0$). The existence of this solution was already noticed by Kasner[8] who, however, did not make clear the problem of the signature and the two possibilities available. If $\Lambda > \rho$ we are faced essentially with the same choice (except for the fact that now the "spheres" may have different radius). $\Sigma_+^{(2)}$ and $\Sigma_-^{(2)}$ are instead to be taken when $\rho > \Lambda \geq 0$. A universe in which $\Sigma_+^{(2)}$ enters as a component has the remarkable property of having a *periodic time*, since the "bottleneck" of the hyperboloid is always time-like; $\Sigma_+^{(1)}$ is instead a surface closed in space, but open in time.

3. The geometrical properties of these solutions can be easily studied by contemplating separately Σ_+ and Σ_-. Let us first remark that all the geodesic lines of a "sphere" are obtained as sections with a plane through the origin[9]; in particular, the null geodesics of Σ_+ are the generators of the hyperboloid. The couple of null generators through a point P divide the quadric in four parts, two of which, in the case of $\Sigma_+^{(1)}$, are characterized as the sets of events which an observer at P can reach when moving toward the future or back into the past.[9] The "elliptic interpretation," which consists in identifying P and the point $\bar{P}$ which is symmetrical to P with respect to the origin, can then be consistently applied to $\Sigma_+^{(1)}$. For $\Sigma_+^{(2)}$, on the contrary, the "ken" of any observer is the whole hyperboloid.

The most obvious frame of reference one can choose for Σ_+ and Σ_- is a "polar" one. For instance, in the case of $\Sigma_+^{(2)}$, by putting

$$u = r_+ \sinh\chi, \quad v = r_+ \sin\tau \cosh\chi, \quad w = r_+ \cos\tau \cosh\chi, \tag{10}$$

we obtain

$$ds_+^2 = du^2 - dv^2 - dw^2 = r_+^2(-\cosh^2\chi d\tau^2 + d\chi^2). \tag{11}$$

With the substitution

$$t = r_+\tau, \quad x = r_+ \sinh\chi, \tag{12}$$

this line element reads

$$ds_+^2 = -\left(1 + \frac{x^2}{r_+^2}\right)dt^2 + \frac{dx^2}{1 + x^2/r_+^2}, \tag{13}$$

where the variables t and x have the usual meaning when $r_+ \to \infty$. Formulas analogous to (11)–(13) can be easily found for all the other "spheres"; in particular for $\Sigma_-^{(2)}$ we can write

$$ds_-^2 = r_-^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{14}$$

or

$$ds_-^2 = \left(1 - \frac{z^2}{r_-^2}\right)dy^2 + \frac{dz^2}{1 - z^2/r_-^2}. \tag{15}$$

Our solution enjoys a six-parameter group of motions (the "rotations" on Σ_+ and Σ_-); and, according to the Pirani-Petrov classification,[10] is of Type I, because of its static character [see (13) and (15)].

To calculate the electromagnetic field $f_{\mu\nu}$ determined by the geometry, a few elementary facts are needed.[11] A non-null bivector can always be expressed as a linear combination of two dual simple bivectors, which define uniquely at each event a pair of orthogonal blades. In terms of a basis of orthonormal vector $\xi_\mu{}^{(\alpha)}$ lying in the blades, we can write

$$f_{\mu\nu} = \rho^{\frac{1}{2}}[(\xi_\mu{}^{(0)}\xi_\nu{}^{(1)} - \xi_\nu{}^{(0)}\xi_\mu{}^{(1)}) \cos\alpha + (\xi_\mu{}^{(2)}\xi_\nu{}^{(3)} - \xi_\nu{}^{(2)}\xi_\mu{}^{(3)}) \sin\alpha]. \tag{16}$$

An elementary calculation shows that the values of ρ

[7] See L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1926), Sec. 61.

[8] E. Kasner, Trans. Am. Math. Soc. **27**, 101 (1925).

[9] Most of the considerations which can be applied to Σ^+ have an analog in the case of the De Sitter universe; for a more detailed study, see E. Schrödinger, *Expanding Universes* (Cambridge University Press, Cambridge, 1956), Chap. I and II.

[10] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957); A. Z. Petrov, Sci. Mem. Kazan State Univ. **114**, 55 (1954).

[11] As far as the geometry of the electromagnetic field at a given event is concerned, see J. L. Synge, *Relativity: The Special Theory* (North Holland Publishing Company, Amsterdam, 1956), p. 326.

and α deduced from (16) agree with their definitions (1) and (2).

If the vectors $\xi_\mu{}^{(\alpha)}$ are taken along the axes t, x, y, and z of (13) and (15), the only nonvanishing components of $f_{\mu\nu}$ turn out to be the *constants:*

$$\begin{aligned} f_{01} &= -f_{10} = (-g_{00}g_{11}\rho)^{\frac{1}{2}} \cos\alpha = \rho^{\frac{1}{2}} \cos\alpha, \\ f_{23} &= -f_{32} = (g_{22}g_{33}\rho)^{\frac{1}{2}} \sin\alpha = \rho^{\frac{1}{2}} \sin\alpha. \end{aligned} \tag{17}$$

Electric and magnetic fields are now *parallel.* At the origin ($t=x=y=z=0$), where the fundamental tensor has vanishing derivatives [see (13) and (15)], the covariant derivatives of $f_{\mu\nu}$ are nil; but since the origin is just an arbitrary point, this holds everywhere. Similar considerations can be made for all the other cases. The value of the constant α, which is left undetermined by the geometry, can always be chosen so as to have a pure electric or a pure magnetic field.

4. The motion of charged test-particles is governed by the equation:

$$m\frac{\delta v^\mu}{\delta s} \equiv m\left[\frac{dv^\mu}{ds} + \Gamma^\mu{}_{\rho\sigma} v^\rho v^\sigma\right] = ev^\rho f_\rho{}^\mu. \tag{18}$$

If the components with respect to Σ_+ and Σ_- are introduced, (18) splits into two pairs of equations for the vectors $v_+{}^\mu$ and $v_-{}^\mu$, respectively:

$$m\frac{\delta v_+{}^\mu}{\delta s} = m\left[\frac{dv_+{}^\mu}{ds} + \Gamma_+{}^\mu{}_{\rho\sigma} v_+{}^\rho v_+{}^\sigma\right] = ev_+{}^\rho f_{+\rho}{}^\mu, \tag{19}$$

and similarly for the other one. Physically, this means that if the particle moves initially on Σ_+, it will do so forever. Let us also remark that (19) has the first integral

$$g_{+\mu\nu} v_+{}^\mu v_+{}^\nu = ds_+{}^2/ds^2 = \text{const} = -P^2 \text{ (say)}, \tag{20}$$

and similarly for the other equation:

$$g_{-\mu\nu} v_-{}^\mu v_-{}^\nu = ds_-{}^2/ds^2 = \text{const} = Q^2 = 1 + P^2. \tag{21}$$

We will now prove that any line satisfying (19) is a *plane section* of the "sphere," and vice versa. Its geodesic curvature[12]

$$\kappa_+ = P^{-2}\left(g_{\mu\nu}\frac{\delta v_+{}^\mu}{\delta s}\frac{\delta v_+{}^\nu}{\delta s}\right)^{\frac{1}{2}} = P^{-2}\frac{e}{m}(v_+{}^\rho v_+{}^\sigma f_+{}^\mu{}_\rho f_{+\sigma\mu})^{\frac{1}{2}},$$

can be easily computed from (16) and (20), remembering (21) and the obvious relationship

$$g_{+\mu\nu} = -\xi_\mu{}^{(0)}\xi_\nu{}^{(0)} + \xi_\mu{}^{(1)}\xi_\nu{}^{(1)}. \tag{22}$$

It turns out to be a constant:

$$\kappa_+ = \rho^{\frac{1}{2}}(e/Pm)\cos\alpha. \tag{23}$$

Similarly for Σ_- we have, instead,

$$\kappa_- = \rho(e/Qm)\sin\alpha. \tag{24}$$

[12] See, e.g., L. P. Eisenhart, *Differential Geometry* (Princeton University Press, Princeton, 1947), Sec. 34.

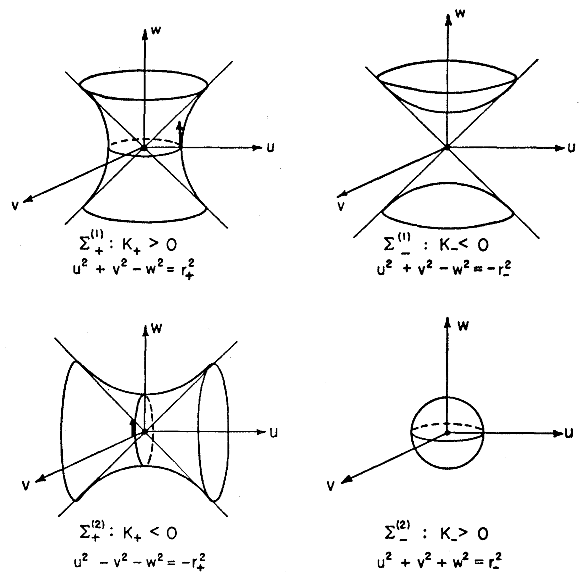

FIG. 1. The surfaces of constant curvature allowed for Σ_+ (signature $-+$) and Σ_- (signature $--$). The arrow on Σ_+ denotes a time direction.

In the case of $\Sigma_-{}^{(2)}$, an ordinary sphere, it is easily proved by elementary arguments that the lines of constant geodesic curvature are just (all) the plane sections. Since all the other cases are obtained from the sphere by multiplying one or more of the auxiliary variables u, v, and w by $(-1)^{\frac{1}{2}}$, without altering the value of the geodesic curvature, the same conclusion still holds good.

5. The main qualitative physical feature of the solution is the *anisotropy* of space: on any space-like hypersurface a privileged congruence of lines is defined as the intersections with Σ_+. They are the lines of force of the electric field for an observer whose time coordinate is orthogonal to the hypersurface. It is doubtful whether this abstract model has any cosmological application; but it certainly raises the problem of the peculiar topological structure induced by the presence of the electromagnetic field. Moreover, its fundamental simplicity suggests it as a basis for the understanding of more complicated situations; one could study, for instance, the perturbations induced by small deviations of the electromagnetic field from uniformity.

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Note added in proof.—Dr. I. Robinson kindly informed me that he has obtained essentially the same solution; his paper is being published in a Polish journal.