Evaluation of Dispersion Relations*†

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A new evaluation of the dispersion relations for mesonic phenomena is proposed. The method, which utilizes the comparison function procedure, makes explicit use of crossing symmetry and allows for an exact treatment of nuclear recoils. For the case of meson-nucleon scattering at low energies, an expansion of a first order solution is made in inverse powers of the nucleon mass and agreement with the results of a previous evaluation is obtained. The extension of the method to other processes is briefly discussed.

I. INTRODUCTION

AN evaluation of the dispersion relations for pion-nucleon scattering and meson photoproduction at moderately low energies has been given by Chew, Goldberger, Low, and Nambu.[1,2] These authors were motivated and guided by the *P*-wave fixed source theory[3] and accordingly made use of an expansion in inverse powers of the nucleon mass. More specifically, a partial-wave decomposition of the dispersion relations was made. The resulting infinite set of coupled equations was made tractable by keeping the first two terms of the inverse nuclear mass expansion and by use of the fact that the (3,3) state alone yields the most important contribution to the dispersion integrals. In the case of meson photoproduction, for example, the resulting integral equations were then approximately solved, and the results have been shown to be in good agreement (maximum deviations $\leqslant 15\%$ for energies below the resonance) with experiment.[4]

The present study was undertaken to see if it would be possible to further improve this agreement by attempting a new evaluation which did not make the nucleon velocity expansion. In other words, the results of reference 1 leave open the question whether any of the residual disagreement with experiment is due to the nucleon mass expansion, the neglect of the small phase shifts or possibly some, as yet obscure, high-energy behavior. The basic premise to be tested by the present work is that it is only the first of these which is important. In addition, we make all of the same basic assumptions of reference 1 with regard to the high-energy behavior of the matrix elements and with regard to the dominance of the (3,3) phase shift. Furthermore, in lowest order, we make the stronger assumption that the (3,3) state exhausts the dispersion integrals at low energies. The latter approximation for the case of meson photoproduction means that the results derived in this way can be expected *a priori* to be valid only in the immediate neighborhood of the resonance. The extent to which the present formulas hold outside of this energy region will require further detailed evaluations. However, the agreement obtained between the results of CGLN and the static limit of our formulas would seem already to argue for their validity at threshold also.

In outline, the procedure which is proposed for evaluating the dispersion relations is as follows. For any given process, we take one of the components of the amplitude which describes this reaction, and through some means split this term into two parts. One of these parts is a known comparison function which is selected in such a way that the second term has in some approximation a known (experimentally or otherwise) phase. Then by a slight extension of the methods of Omnes,[5] the modulus of this unknown function is obtained by solving the complete dispersion equations, thus yielding eventually the entire amplitude. The method proposed contains sufficient generality to enable one to make convenient use of partial experimental information as well as enabling one to improve, in principle, any given solution by repeated iteration. Even at fairly relativistic energies the method is very simple from the viewpoint of mathematical simplicity and, further, all final results are most easily and conveniently written in covariant form. On the other hand, the reduction of the formulas to a form suitable for quantitative comparison to experiment is still fairly complicated in that numerical integrations are required.

In Sec. II we solve the dispersion equations in the sense discussed above, and indicate the important role played by the crossing symmetry in removing most ambiguities from the solution. Also contained in this

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[1] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337 (1957). Hereafter referred to as CGLN.

[2] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1345 (1957).

[3] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); and Phys. Rev. **101**, 1579 (1956).

[4] Uretsky, Kenny, Knapp, and Perez-Mendez, Phys. Rev. Letters **1**, 12 (1958). F. L. Goldwasser, *Proceedings of the Seventh Annual Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, Inc., New York, 1957).

[5] R. Omnes, Nuovo cimento **8**, 316 (1958). See also, N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, Groningen, 1953).

section is a discussion of uniqueness of the solution as well as a brief indication of possible procedures for the required splitting up of the amplitude. The results thus obtained are applied in Sec. III to the problem of pion-nucleon scattering. Following CGLN,[1] it is possible to determine all phases in terms of δ_{33}, and to obtain an "identity" in δ_{33} whose consistency measures the validity of the method. With a particular choice for the splitting up of the amplitude, agreement with those authors is obtained in the limit as the nucleon mass becomes large. No further progress on the location of the (3,3) resonance is obtained by this method. The extension of this approximation method to general processes, in which final-state interaction effects are important, is straightforward, and is briefly examined in Sec. IV with specific reference to photo- or meso-disintegration of the deuteron. And finally, we make some concluding remarks.

II. SOLUTION OF THE DISPERSION RELATIONS

The dispersion relations for pion-nucleon scattering,[1] photoproduction of mesons,[2] and electroproduction of mesons[6] are all very similar in structure and the equation for a typical invariant function which makes up the scattering amplitude may be written in the form

$$A(x,\nu_B)=B(x,\nu_B)+\frac{1}{\pi}\int_{x0}^{\infty} dy[\operatorname{Im}A(y,\nu_B)] \times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right]. \quad (1)$$

Here the ($\pm$) sign refers to evenness and oddness under crossing; that is, we have

$$A(-x,\,\nu_B)=\pm A(x-2\nu_B,\,\nu_B). \quad (2)$$

The known inhomogeneous term $B(x,\nu_B)$ is the Born approximation with similar behavior under crossing as A; ν_B is essentially the momentum transfer which is a fixed constant in the dispersion relations; x is an invariant which is connected to the energy in the center-of-mass system W by a relation of the form

$$x=(W^2-M^2)/2M,$$

and x_0 is some fixed positive lower limit which is simply related to the masses. Once the A's are known, the scattering amplitude for the given process is obtained from these by multiplying by appropriate invariants involving the Dirac γ matrices. The solution of Eq. (1) ultimately yields, in this way, the entire matrix element.

Suppose for the moment that the function A of Eq. (1) had a certain known phase[7] $\delta(x)$. Then, following Omnes,[5] we define a function $F(z)$ of the complex variable z by the relation

$$e^{\Delta(z)}F(z)=\frac{1}{2\pi i}\int_{x0}^{\infty} dy\,\operatorname{Im}A(y,\nu_B) \times\left[\frac{1}{y-z}\pm\frac{1}{y+z+2\nu_B}\right], \quad (3)$$

where now $\Delta(z)$ is given by[8]

$$\Delta(z)=\frac{1}{\pi}\int^{\infty} dy\,\delta(y)\left[\frac{1}{y-z}+\frac{1}{y+z+2\nu_B}\right]. \quad (4)$$

We observe directly from Eq. (4) that as z approaches the real axis from above (+) or from below (−), we may write for $x>x_0$

$$\Delta_\pm(x,\nu_B)=\rho(x,\nu_B)\pm i\delta(x), \quad (5)$$

where

$$\rho(x,\nu_B)=\frac{P}{\pi}\int_{x0}^{\infty} dy\,\delta(y)\left[\frac{1}{y-x}+\frac{1}{y+x+2\nu_B}\right]. \quad (6)$$

The symbol "P" stands for principal value integral. By making use of Eqs. (1), (3), (4), (5), we find the relations

$$2iF_+(x,\nu_B)e^{\rho+i\delta}=|A(x,\nu_B)|e^{i\delta}-B(x,\nu_B), \quad (7)$$

and

$$e^{\rho}[F_+(x,\nu_B)e^{i\delta}-F_-(x,\nu_B)e^{-i\delta}]=|A(x,\nu_B)|\sin\delta. \quad (8)$$

Finally, eliminating the unknown function $|A|$ between these two equations, we obtain the result

$$F_+(x,\nu_B)-F_-(x,\nu_B)=e^{-\rho(x,\nu_B)}\sin\delta(x)B(x,\nu_B), \quad (9)$$

from which it is clear that except for the addition of a function which is continuous across the cuts along the real axis, $F(z)$ is given by

$$F(z)=\frac{1}{2\pi i}\int_{x0}^{\infty} dy\,B(y,\nu_B) \times\sin\delta(y)e^{-\rho(y,\nu_B)}\left[\frac{1}{y-z}+\frac{g}{y+z+2\nu_B}\right]. \quad (10)$$

The function g is so far arbitrary except to the extent that the integral containing it be finite. If one were to substitute Eq. (10) into the preceding formulas to obtain A, it becomes evident that only by choice $g=\pm1$ will satisfy the crossing symmetry as expressed by Eq. (2) and thus this function is determined. It is noteworthy that, even though in practice the crossing term is often "small" under the dispersion integrals, one still

[6] Fubini, Nambu, and Wataghin, Phys. Rev. **111**, 329 (1958).

[7] Although in all applications which have so far been contemplated, δ always turns out to be independent of momentum transfer, this restriction is not in any way essential.

[8] It goes without saying, of course, that we assume the existence of all the integrals which we write down. Thus, for example, in Eq. (4), $\delta(y)$ must approach zero as y increases. However, for this particular case, this restriction will be removed in later discussions.

needs the crossing symmetry to remove this troublesome ambiguity. Thus, the solution to Eq. (1) under the assumption that the phase of A is known can be found from Eqs. (10), (3), and (1) to be

$$A(x,\nu_B)=B(x,\nu_B)+\frac{e^{\rho+i\delta}}{\pi}\int_{x_0}^{\infty} e^{-\rho(y)}\sin\delta(y)B(y,\nu_B)\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right]dy, \quad (11)$$

which may be written in the equivalent form

$$A(x,\nu_B)=B(x,\nu_B)e^{i\delta}\cos\delta+\frac{e^{\rho+i\delta}}{\pi}P\int_{x_0}^{\infty} dy\, e^{-\rho(y)}\sin\delta(y)B(y,\nu_B)\times\left[\frac{1}{y-x}\pm\frac{1}{y+x+2\nu_B}\right]. \quad (11')$$

It must be borne in mind that to the solution (11) we may still add the term $fe^{\rho+i\delta}$, where f is completely arbitrary except that it has no branch cuts anywhere in the complex plane. Since it is easily seen that this added term is a solution of the homogeneous form of Eq. (1) (i.e., with $B=0$) it can only be selected by some sort of a boundary condition. For the present cases of interest, we should like to argue that f is a polynomial which in the case of pion-nucleon scattering and meson photoproduction vanishes identically. This follows from the fact that by writing down the unsubtracted form of the dispersion relations Eq. (1), we have made a definite assumption with regard to the behavior of A at infinity. Since with $f=0$, the solution (11) has all the correct poles and branch cuts which it must have according to Eq. (1), and since f must be continuous across the cuts along the real axis, f can at worst be a polynomial. If in addition we assume that A approaches zero at infinity then this polynomical must clearly vanish. The solution given by Eq. (11) corresponds to this case.[9] If A does not approach zero sufficiently rapidly at infinity, then of course f will assume a different form which will be dictated by the conditions of the given situation. In any event, for the present cases of interest, we make the choice, $f=0$, and take (11) to be the appropriate solution of Eq. (1).

One last point concerning the solution as given by Eq. (11) is that there still is the possibility of adding arbitrary integral multiples of π in arbitrary energy intervals to δ.[10] As will be directly shown, this ambiguity is due to the possibility of adding solutions of the homogeneous equation to the given solution. Such ambiguities, which are very similar to those considered by Castillejo, Dalitz, and Dyson,[11] may be thought of as contributions from bound states whose connection with the phase shift in this way is very suggestive of Levinson's theorem[12] in potential scattering. Suppose now that in the solution given by Eq. (11) we make the replacement

$$\delta\rightarrow\delta+\begin{cases} n_1\pi, & x_0\leqslant x\leqslant a \\ n_2\pi, & a\leqslant x\leqslant b \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{cases} \quad (12)$$

where n_1, n_2, $\cdots$ are even integers or zero, and the collection a, b, $\cdots$ is an arbitrary monotonic increasing set of points. Since this replacement leaves the original Eq. (1) invariant, a similar invariance should be obtained in the solution given by Eq. (11). However, because of the e^ρ factors in that solution, this invariance seems to be missing. We will now show that a transformation of the form given by Eq. (12) corresponds to the addition of solutions of the homogeneous form of Eq. (1) to Eq. (11) and thus these appendages may be deleted by making a suitable choice for the function f in the manner discussed in the preceding paragraph. To see this in detail, we note that the replacement Eq. (12) influences the solution given by Eq. (11) by multiplying the second term in this equation by the factor

$$\left|\frac{x'^2-a'^2}{x'^2-x_0'^2}\right|^{n_1}\left|\frac{x'^2-b'^2}{x'^2-a'^2}\right|^{n_2}\cdots\times\left|\frac{y'^2-x_0'^2}{y'^2-a'^2}\right|^{n_1}\left|\frac{y'^2-a'^2}{y'^2-b'^2}\right|^{n_2}\cdots, \quad (13)$$

where the primed symbols are defined by $y'=y+\nu_B$, $b'=b+\nu_B$, etc., and where, of course, y is still the dummy variable of integration. Consider now the identities

$$\frac{y'^2-a'^2}{x'^2-a'^2}\left(\frac{1}{y'-x'}\pm\frac{1}{y'+x'}\right)=\frac{1}{y'-x'}\pm\frac{1}{y'+x'}+\frac{1}{x'-a'}\binom{y'/a'}{1}-\frac{1}{x'+a'}\binom{y'/a'}{-1}, \quad (14a)$$

[9] This assumption neglects π–π scattering effects which may be very important and would necessitate a subtraction in the dispersion relations. This is neglected here because we are primarily interested in P-wave rescattering effects which are relatively insensitive to this subtraction.

[10] We are grateful to Professor Dalitz and Professor Nambu for pointing out this possibility to us as well as for some clarifying conversations with regard to this point.

[11] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956). See also R. Haag, Nuovo cimento **5**, 203 (1956).

[12] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949).

and

$$\frac{x'^2-a'^2}{y'^2-a'^2}\left(\frac{1}{y'-x'}\pm\frac{1}{y'+x'}\right)$$
$$\equiv\frac{1}{y'-x'}\pm\frac{1}{y'+x'}+\frac{1}{y'+a'}\binom{-1}{x'/a'}-\frac{1}{y'-a'}\binom{1}{x'/a'},\qquad(14b)$$

which are appropriate to the case for $n_1-n_2=2$. These identities make clear, that this particularly simple version of the transformation (12) only adds solutions of the homogeneous equation to the one given by Eq. (11). And as in the discussion above, we may remove these spurious appendages by choosing a function f which will cancel them out. Thus the result obtained from Eq. (14a) must be subtracted off because it introduces new and unphysical singularities into the physical region. Similarly, the addition of (14b) is ruled out because of the assumed behavior for A at infinity. Furthermore, it is clear that if we multiply Eq. (14a) or Eq. (14b) by additional factors of the form

$$[(y'^2-b'^2)/(x'^2-b'^2)]^{\pm1},$$

which corresponds to more complicated versions of the transformation (12) than we have just considered, then we generate only new solutions of the homogeneous equation. These can again be eliminated in the manner just discussed. We see, therefore, that this last ambiguity in the solution to Eq. (1) is removed by the observation that the transformation given by Eq. (12) merely appends solutions of the homogeneous equation. Therefore, Eq. (11) as it stands is the solution to Eq. (1) provided the behavior of A at infinity is correct as assumed. For any other energy dependence at infinity, solutions of the homogeneous equation must be added.

Now in order to apply the result given by Eq. (11) to the cases of interest in pion physics, some modifications are obviously required since in practice the phase of A is usually not independently known. Thus, in order to make practical use of the present result, it is necessary to rewrite Eq. (1) in such a way that this phase condition may be approximately satisfied. Suppose, therefore, that we have obtained a comparison function $C(x,\nu_B)$ which is determined by the requirement that in a certain energy interval the function $D(x,\nu_B)$ which in turn is defined by

$$D(x,\nu_B)=A(x,\nu_B)-C(x,\nu_B),\qquad(15)$$

has to a good approximation a known phase. For example, in the case of meson photoproduction at moderately low energies, C might be the contribution to the amplitudes from all the states with small phase shifts, in which case the phase of D would be simply δ_{33}. As will become evident shortly, the functional properties of C, such as poles and branch cuts, are relatively immaterial except in that certain integrals which involve C are convergent and a certain domain of analyticity in the variable ν_B exists. Because of the freedom so afforded, in general C may be chosen in various convenient ways. One such way might be to take a part of the Born approximation; a second might be to take partial results directly from experiment. And finally, the most interesting possibility, of course, is to select C from the results of a previous calculation. Indeed, once some approximate solution is available, one may iterate by taking the new C to be that part of the previous solution which will yield an improved knowledge of the phase of D.

More explicitly, let δ be the known phase which is to be associated with a given choice for C. By use of obvious additions and subtractions on Eq. (1), and by use of Eq. (15) one may derive a dispersion relation for D in the form

$$D(x,\nu_B)=B'(x,\nu_B)+\frac{1}{\pi}\int_{x0}^{\infty}dy[\mathrm{Im}D(y,\nu_B)]\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right],\qquad(16)$$

where B' is given by

$$B'(x,\nu_B)=B(x,\nu_B)-C(x,\nu_B)+\frac{1}{\pi}\int_{x0}^{\infty}dy[\mathrm{Im}C(y,\nu_B)]\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right].\qquad(17)$$

One notes directly that B' is purely real and further, since the integral in Eq. (17) is assumed to exist, that the function $B'+C$ has the identical crossing symmetry and analyticity properties[13] as are enjoyed by A. Now since Eq. (16) for D is of the same form as Eq. (1), and since by our choice for C, the phase of D is known, we may use the result given by Eq. (11) together with Eq. (15) to write the solution of Eq. (1) in the form

$$A(x,\nu_B)=B(x,\nu_B)+\frac{1}{\pi}\int_{x0}^{\infty}dy[\mathrm{Im}C(y,\nu_B)]\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right]+\frac{e^{\rho+i\delta}}{\pi}\int^{\infty}dy\,B'(y)e^{-\rho(y)}\sin\delta(y)\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x+2\nu_B}\right].\qquad(18)$$

[13] In this connection we think, of course, of the obvious analytic continuations of these functions which are here defined along a part of the real axis.

We emphasize again that the ambiguities relating to the crossing terms and those relating to the solutions of the homogeneous equations have been removed by means of the known crossing symmetry and the assumed behavior at infinity of the function A. An alternate and sometimes more convenient form for Eq. (18) may be obtained by regrouping the delta function parts of the integrals with the inhomogeneous term, with the result

$$\begin{aligned} A(x,\nu_B)=&C(x,\nu_B)\\ &+\cos\delta\, e^{i\delta}[B(x,\nu_B)+I(x,\nu_B)-\mathrm{Re}C(x,\nu_B)]\\ &+\frac{e^{\rho+i\delta}}{\pi}P\int_{x_0}^{\infty}dy\, e^{-\rho(y,\nu_B)}\sin\delta(y)\\ &\times[B(y,\nu_B)+I(y,\nu_B)-\mathrm{Re}C(y,\nu_B)]\\ &\times\left[\frac{1}{y-x}\pm\frac{1}{y+x+2\nu_B}\right], \end{aligned} \tag{19}$$

where $I(x,\nu_B)$ is given by

$$I(x,\nu_B)=\frac{P}{\pi}\int_{x_0}^{\infty}dy\,\mathrm{Im}C(y,\nu_B)\left[\frac{1}{y-x}\pm\frac{1}{y+x+2\nu_B}\right].$$

It is of some interest to examine the solution of Eq. (18) in terms of its dependence on the momentum transfer ν_B as well as the energy. This leads to a general restriction on possible comparison functions $C(x,\nu_B)$. Evidently, the solution given by Eq. (18) has a branch cut along the negative ν_B axis by virtue of the crossing terms. In the case of meson-nucleon scattering this corresponds to one of the branch lines which has been discussed by Mandelstam.[14] It is clear that the needed properties of the solution Eq. (18) can be obtained by requiring that the functions $\mathrm{Im}C(x,\nu_B)$ and $B'(x,\nu_B)$ have suitable analyticity properties in ν_B. In the case of meson-nucleon scattering, for example, the choice of a polynomial for C is certainly adequate. More specifically, if C is chosen to be the Born term minus a finite number of partial wave projection of this term,[15] then the proper analyticity is also maintained. Thus, the requirement of consistency with the Mandelstam representation somewhat restricts the choice for C.

Lastly we note one final simplifying approximation which may be made in Eq. (19) at a possible resonance energy x_r. Since the integrand in Eq. (19) is then the product of $\exp\{\rho(x_1,\nu_B)-\rho(y,\nu_B)\}$ and a function which is sharply peaked at this resonance, we may expand the former in a power series

$$e^{\rho(x_r,\nu_B)-\rho(y,\nu_B)}=1+a(y-x_r)+\cdots,$$

in which only the first term will make an appreciable contribution to the integral. Thus, the somewhat annoying e^{ρ} factors may be dropped at a sharp resonance with a consequent simplification of Eq. (19).

Therefore, we see that subject to the limitations discussed above, the solution of Eq. (1) may be obtained provided only that one can find a comparison function C as defined by Eq. (15) such that D has a known phase. The solution is then given by the two equivalent forms Eqs. (18) and (19).

III. APPLICATION TO MESON-NUCLEON SCATTERING

In order to study the proposed method in more detail, let us now consider a specific application to the case of meson-nucleon scattering at low energies. The necessary selection of a suitable function $C^{\pm}$ can be simply made in this energy region because of the well-known (3,3) resonance, which, around its maximum, dominates all other states. Thus, in lowest order (in the sense of the iterative scheme discussed above) we shall assume that all states except this resonant one, are correctly given by the Born approximation. In other words, since the Born approximation is purely real, we are assuming that only the (3,3) state makes an appreciable contribution to the dispersion integrals—an approximation which is certainly reliable near the resonance. This means that if we make the choice

$$C^{\pm}(x,\kappa^2)=B^{\pm}(x,\kappa^2)-B_{33}{}^{\pm}(x,\kappa^2), \tag{20}$$

where $B^{\pm}$ here is the relevant part of the Born approximation and $B_{33}{}^{\pm}$ is that part of the (3,3) projection of the Born approximation which is to be associated with the given $B^{\pm}$, then the phase of $D^{\pm}(x,\kappa^2)$ as defined by Eq. (15) is given by δ_{33}. We note that the function $C^{\pm}$ as defined in Eq. (20) is purely real for sufficiently small momentum transfer and thus the solution given by Eqs. (18) and (19) will be conveniently simplified.

Before writing down the detailed solution, let us, for the sake of completeness review briefly some of the kinematics. We shall follow the notation and arguments of reference 1 as closely as possible. Let p_1, p_2, q_1, q_2 be the initial and final momenta of the nucleons and mesons respectively and define the three independent four-vectors P, Q, and κ by the relations

$$\begin{aligned} P&=\tfrac{1}{2}(p_1+p_2),\\ Q&=\tfrac{1}{2}(q_1+q_2),\\ \kappa&=\tfrac{1}{2}(q_1-q_2). \end{aligned} \tag{21}$$

Out of these three vectors, one can construct only two independent invariants which may be taken to be κ^2 and x which is defined by the relation

$$x=-(1/M)P\cdot Q+(1/M)\kappa^2.$$

In the center-of-mass system, κ^2 is one-half the three momentum transfer to the nucleon and x is connected to

[14] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[15] Although the extension of the amplitude to unphysical values of the momentum transfer by means of a Legendre polynomial expansion is open to question, for sufficiently small momentum transfers considerations due to H. Lehmann, Nuovo cimento **10**, 579 (1958), would seem to cast suspicion on the validity of this procedure.

the total energy W in this frame by the relation $x=(W^2-M^2-1)/2M$ and is also the incident meson energy in the laboratory frame. The T matrix for meson-nucleon scattering may be written in the form

$$T=-A+i\gamma\cdot QB, \tag{22}$$

where A and B are matrices in isotopic spin space and are functions of the two invariants x and κ^2. The assumption of charge independence reduces this complexity and allows only four independent functions which may be defined by

$$A_{\alpha\beta}=A^+\delta_{\alpha\beta}+A^-\tfrac{1}{2}[\tau_\beta,\tau_\alpha],$$

and similarly for $B_{\alpha\beta}$, where α and β are the initial and final charge states of the pion. Finally, in terms of the functions $A^\pm(x,\kappa^2)$ and $B^\pm(x,\kappa^2)$, the scattering amplitude $f^\pm$ in the center-of-mass system may be written

$$f^\pm=f_1{}^\pm+(\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1/q^2)f_2{}^\pm, \tag{23}$$

where q is the magnitude of the meson momentum in the c.m. system and where

$$f_1{}^\pm=\frac{E+M}{2W}[A^\pm+(W-M)B^\pm],$$

and (24)

$$f_2{}^\pm=\frac{E-M}{2W}[-A^\pm+(W+M)B^\pm],$$

and where lastly E is the total nucleon energy in this system.

The dispersion relations, for fixed κ^2, of the amplitudes $A^\pm$ and $B^\pm$, on the assumption that no subtractions are required, may be written in terms of the variable x and are

$$A^\pm(x,\kappa^2)=\frac{1}{\pi}\int_1^\infty dy[\mathrm{Im}A^\pm(y,\kappa^2)]\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x-2\kappa^2/M}\right], \tag{25a}$$

and

$$B^\pm(x,\kappa^2)=-\frac{g^2}{2M}\left(\frac{1}{x+1/2M}\pm\frac{1}{x-1/2M-2\kappa^2/M}\right)+\frac{1}{\pi}\int_1^\infty dy[\mathrm{Im}B^\pm(y,\kappa^2)]\times\left[\frac{1}{y-x-i\epsilon}\mp\frac{1}{y+x-2\kappa^2/M}\right]. \tag{25b}$$

Here $g\sim4$ and is the unrationalized pseudoscalar coupling constant. In terms of the variables x and κ^2, the crossing symmetry is expressed by the relations

$$A^\pm(-x,\,\kappa^2)=\pm A^\pm(x+2\kappa^2/M,\,\kappa^2),$$

and (26)

$$B^\pm(-x,\,\kappa^2)=\mp B^\pm(x+2\kappa^2/M,\,\kappa^2).$$

Now in accordance with the choice for $C^\pm$, Eq. (20), we must project out b the (3,3) part of the Born approximation which is given by the product of $i\gamma\cdot Q$ and the inhomogeneous term in Eq. (25b). The isotopic spin projecting is easily done and in the center-of-mass frame the result may be written in the form

$$b_{I=\frac{3}{2}}=\frac{g^2}{2M}\frac{1}{\cos\theta+a}\left[\frac{W-M}{E-M}+\frac{W+M}{E+M}\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1}{q^2}\right],$$

where θ is the angle of scattering of the meson and $1/a=q^2/(E\omega_q-\frac{1}{2})$ is essentially the product of the nucleon and meson velocities. The meson energy in the center-of-mass system is ω_q. The $P_{\frac{3}{2}}$ angular momentum state is easily projected from this formula, and we obtain the result

$$b_{33}=\frac{g^2}{2Mq^2}\left[\frac{W-M}{E-M}\alpha-\frac{1}{2}\frac{W+M}{E+M}\gamma\right](3\mathbf{q}_2\cdot\mathbf{q}_1-\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{q}_1),$$

where α and γ are given by

$$\alpha=1-\frac{a}{2}\ln\frac{a+1}{a-1}\cong-\frac{1}{3}\frac{q^4}{E^2\omega_q{}^2},$$

and

$$\gamma=3a+\tfrac{1}{2}(1-3a^2)\ln\frac{a+1}{a-1}\cong-\frac{4}{15}\frac{q^6}{E^3\omega_q{}^3}.$$

By use of Eqs. (23) and (24), we obtain

$$a_{33}{}^\pm(x,\kappa^2)=-\begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}\frac{g^2J(x)}{2W}\times\left[3\frac{W+M}{E+M}\left(1-\frac{2\kappa^2}{q^2}\right)+\frac{W-M}{E-M}\right],$$

and (27)

$$b_{33}{}^\pm(x,\kappa^2)=-\begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}\frac{g^2J(x)}{2W}\times\left[\frac{3}{E+M}\left(1-\frac{2\kappa^2}{q^2}\right)-\frac{1}{E-M}\right],$$

where J depends only on the energy and is given by

$$J(x)=\frac{W-M}{E-M}\alpha-\frac{1}{2}\frac{W+M}{E+M}\gamma.$$

Finally, by use of Eqs. (18) and (20) we obtain as the solution with the correct crossing symmetry and the assumed behavior at infinity,

$$A^{\pm}(x,\kappa^2)=\frac{e^{\rho+i\delta_{33}}}{\pi}\int_1^\infty dy\, e^{-\rho(y)}\sin\delta_{33}(y)a_{33}^{\pm}(y)\times\left[\frac{1}{y-x-i\epsilon}\pm\frac{1}{y+x-2\kappa^2/M}\right],$$

and

$$B^{\pm}(x,\kappa^2)=-\frac{g^2}{2M}\left[\frac{1}{x+1/2M}\pm\frac{1}{x-1/2M-2\kappa^2/M}\right]+\frac{e^{\rho+i\delta_{33}}}{\pi}\int_1^\infty dy\, e^{-\rho(y)}\sin\delta_{33}(y)b_{33}^{\pm}(y)\times\left[\frac{1}{y-x-i\epsilon}\mp\frac{1}{y+x-2\kappa^2/M}\right], \quad (28)$$

where, of course, $a_{33}^{\pm}$ and $b_{33}^{\pm}$ are given by Eq. (27). Thus, once δ_{33} has been obtained, say from experiment, one need only carry out the single integrals in Eq. (28) to obtain the total scattering amplitude in this approximation. We note at this point that since δ_{33} may be presumed to approach π for large energies, it seems necessary to perform a subtraction in ρ to make its integral convergent. It is convenient to perform one subtraction at $x=\kappa^2/M$ because this retains explicitly the crossing symmetry and does not add an arbitrary constant to the solution. However, one subtraction is actually automatic, since the integrals in question occur in the combination $\rho(x)-\rho(y)$.

It is perhaps worth emphasizing again, that the solution given by Eq. (28) is but an approximate one based on a selection of $C^{\pm}$ as given by Eqs. (20) and (27). The obvious choice to make in order to obtain an improved solution is to take for the new $C^{\pm}$ the difference between the solution given by Eq. (28) and its (3,3) projection. One might then be hopeful that the phase of the new $D^{\pm}$ as defined by Eq. (15) will be closer to δ_{33} than before. Then by repeated use of Eq. (18) the improved solution would be obtained.

For the present, let us content ourselves by comparing the results given by Eq. (28) to experiment by actually comparing them to the analogous results of reference 1. In order to do this, it is first of all necessary to make a partial wave expansion of Eq. (28). This procedure turns out to be fairly complex because of the κ^2 dependence in the crossing term of ρ that corresponds, roughly speaking, to the effects of the (3,3) state on the rescattering of other states. We should now like to argue that at low energies, to a good approximation, one may neglect this particular κ^2 dependence and thus obtain simplifications in the partial wave decomposition. As we have already argued above, in the neighborhood of the resonance, the two e^ρ factors in Eq. (28) tend to cancel each other and thus the κ^2 dependence in these exponentials is probably not too important. In passing, we note that in the crossed term under the integrals in Eq. (28) the κ^2 dependence may be partially neglected at moderately small momentum transfers because of the $1/M$ factor and because of the expectation that, in general, the singular denominator in the uncrossed term makes the large contribution to the integral.

Thus, assuming this neglect of the κ^2 dependence, which is generated solely by the crossing symmetry in ρ, to be justified, the task of projecting the various angular momentum states from Eq. (28) is simple. We note that $a_{33}^{\pm}(x,\kappa^2)$ and $b_{33}^{\pm}(x,\kappa^2)$ are linear functions of κ^2 and thus the integrals can only make an appreciable contribution to states of $J=\frac{3}{2}$ and $J=\frac{1}{2}$. Further, as shown in reference 1 the phases of the two $D_{\frac{3}{2}}$ states are both $\leqslant 1°$ up to the resonance energy and thus we need to consider only the rescattering corrections to the remaining S and $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states. Following CGLN, we obtain, using their normalization, the results

$$\begin{aligned} f_S^{\pm}&=f_1^{\pm}(0)+\tfrac{1}{2}q^2f_1^{\pm\prime}(0),\\ f_{P_{\frac{1}{2}}}^{\pm}&=f_2^{\pm}(0)-\tfrac{1}{6}q^2f_1^{\pm\prime}(0),\\ f_{P_{\frac{3}{2}}}^{\pm}&=-\tfrac{1}{6}q^2f_1^{\pm\prime}(0),\end{aligned} \quad (29)$$

where f_S is the amplitude of the S states, etc., and where $f_1^{\pm}(x,\kappa^2)$ and $f_2^{\pm}(x,\kappa^2)$ are given in terms of $A^{\pm}$ and $B^{\pm}$ by Eq. (24), and finally, where the primes stand for differentiation with respect to κ^2. Using the methods of reference 1, the amplitudes of the other states as well as the corrections to Eq. (29) due to these states may be obtained. For the present purpose, however, Eq. (29) will suffice.

The obtaining of explicit formulas for f_S, $f_{P_{\frac{1}{2}}}$ and $f_{P_{\frac{3}{2}}}$ in the two isotopic spin states, is now a simple matter. One computes $f_1^{\pm}(x,\kappa^2)$ and $f_2^{\pm}(x,\kappa^2)$ directly from Eqs. (24) and (28) and substitutes these into Eq. (29) to obtain the desired result. Fortunately, there is no need to present the resultant, fairly complicated formulas here explicitly since they may be directly obtained from the corresponding results of reference 1. Thus for the S states one takes Eqs. (3.19) while for the P states one takes Eqs. (3.29) and (3.32) of reference 1 and makes there the substitution,

$$\mathrm{Im}f_{33}^{\pm}(\nu_L')\rightarrow -\begin{pmatrix}\frac{2}{3}\\ -\frac{1}{3}\end{pmatrix}\frac{g^2J(\nu_L')}{2W'}e^{i\delta_{33}(\nu_L)}\times\sin\delta_{33}(\nu_L')e^{\rho(\nu_L)-\rho(\nu_L')}. \quad (30)$$

This procedure will reproduce the present results.

For later applications to meson photoproduction and for the sake of completeness, let us write down the greatly simplified version of these formulas which is obtained for P waves by making the expansion in inverse powers of the nucleon mass. Defining the variable $\omega=W-M$ and keeping only zeroth and first order in $1/M$, there results for the P waves

$$f_{33}=\frac{4}{3}\frac{f^2q^2}{\omega}+\tfrac{4}{3}f^2q^2\frac{e^{\rho(\omega)+i\delta_{33}(\omega)}}{\pi}\int_1^\infty\frac{d\omega'}{\omega'}\sin\delta_{33}(\omega')e^{-\rho(\omega')}\times\left[\frac{1}{\omega'-\omega-i\epsilon}+\frac{1}{M}+\frac{1}{9}\frac{1}{\omega'+\omega}\right],$$

$$f_{11}=-\frac{8}{3}\frac{f^2q^2}{\omega}+\frac{3f^2q^2}{M}+\frac{64}{27}\frac{q^2f^2}{\pi}e^{\rho+i\delta_{33}}\int_1^\infty\frac{d\omega'}{\omega'}\frac{\sin\delta_{33}(\omega')}{\omega'+\omega}e^{-\rho(\omega')}, \qquad (31)$$

$$f_{13}=f_{31}=\tfrac{1}{4}f_{11}-\frac{3}{4M}f^2q^2.$$

The very striking similarity of the results given by Eq. (31) to the corresponding formulas of reference 1 [Eq. (4.1)] is not too surprising if one considers that Eq. (31) may be obtained by use of the limiting form of the substitution Eq. (30). Using the type of arguments presented in the previous section where the solutions of the dispersion relations were obtained, it is immediately clear that f_{33} as given in Eq. (31) is a solution of the corresponding f_{33} equation of CGLN. In making this argument, we take the f_{33} equation of CGLN, divide it by q^2, and then assuming that we know the phase of f_{33}/q^2, a possible "solution" of this equation is given by our Eq. (31) if crossing is treated approximately. In this sense, the first equation agrees completely with the corresponding results of CGLN. The last set of equations in (31) is identical to the corresponding results of CGLN and thus also requires no further discussion. Lastly, the f_{11} equation can easily be shown to coincide with the results of CGLN at very low energies. Upon letting the e^ρ factors cancel each other, and upon neglecting δ_{33} in the exponential at low energies, we may use the effective range expansion for f_{33} under the integral in the f_{11} equation, and thusly reproduce the corresponding result. We see, therefore, that the choice for $C^\pm$ as given by Eq. (20) essentially yields the results of CGLN in the $1/M$ limit and, thus, it enjoys the corresponding agreement with experiment.

Finally, let us emphasize again that the purpose of solving meson-nucleon scattering in the present context is only with the viewpoint of illustrating the general method and of deriving results to be used in a discussion of pion photoproduction. The fairly remarkable agreement with the results of reference 1 is, of course, no accident in that the basic assumption of the dominance of the (3,3) state under all dispersion integrals is made there also. Indeed, from many points of view, the formulas of reference 1 are preferable since these, for example, involve dispersion integrals containing $\mathrm{Im}f_{33}$ rather than $\sin\delta_{33}$ and the former is much more convenient because of the sharper peak at resonance. On the other hand, our formulation of the solution is stated in a more general way and allows one to find an improved solution starting from any approximate one. Furthermore, we expect the present method to be applicable to other processes in which some phase will be a given piece of data rather than something which must be codetermined with the rest of the solution as in pion-nucleon scattering.

IV. CONCLUDING REMARKS

The main result of the present study may be summarized by noting that Eq. (18) is the solution to the dispersion relation given by Eq. (1) subject, of course, to our obtaining a suitable comparison function C. The generality possible in the choice of this comparison function makes it convenient to make use of partial experimental information and also to improve any given solution by iteration. Although the method is not basically restricted to any particular energy region, in the case of pion-nucleon scattering at low energies, the obvious zeroth order comparison function yields essential agreement with the results of CGLN. Furthermore, even though the present discussion has been consistently slanted towards simple processes which directly involve mesons, it is evident that the proposed method for solving the dispersion relation has wider generality. In particular, the extension of the present ideas to general processes involving strong interactions in the final state is fairly straightforward provided, of course, that one may write down a suitable form of the dispersion relation.

As an example of such processes let us consider briefly photo- or meso-disintegration of deuterium.[16] The dispersion relation at constant momentum transfer ν may be written in terms of the total center-of-mass energy x and is of the form

$$A(x,\nu)=V(x,\nu)+\frac{1}{\pi}\int dy[\mathrm{Im}A(y,\nu)]\frac{1}{y-x-i\epsilon},$$

where the unphysical and negative energy contributions have been absorbed into the inhomogeneous term. We assume that $V(x,\nu)$ is known at least approximately. Now at very low energies, assuming that only one partial wave l_0 is appreciably different from its Born approximation, we choose our lowest order comparison function C by the relation $C=V-V_{l_0}$ where V_{l_0} is the projection of V on the l_0th partial wave. The resulting equation is easily solved by use of the methods described above. An improved solution may now be obtained by taking the new comparison function

$$C=A-A_{l_0},$$

where A is the solution just obtained and A_{l_0} is its l_0th

[16] Similar work on photodisintegration has been independently carried out in more detail by Bunji Sakita, Bull. Am. Phys. Soc. Ser. II, 4, 267 (1959).

projection. Assuming that this choice for C will make another partial wave dominate, we proceed as before. Continuing in this way, and assuming further that each V_l is a slowly varying function of x, we find an approximate solution[17] in the form

$$A(x,\nu)=\sum_l V_l(x) \times P_l(\cos\theta) \exp\left\{\frac{1}{\pi}\int \delta_l(y)dy\frac{1}{y-x-i\epsilon}\right\}. \quad (32)$$

In writing down this solution, we have applied the boundary condition that each partial wave must have the scattering phase δ_l as demanded by unitarity, and further in the limit as this phase approaches zero, the amplitude reduces to the Born approximation. The possibility of making a subtraction in the exponential is, of course, always understood. In this approximation, we see that each partial wave has the correct phase according to unitarity and its modulus differs from the Born approximation only by the easily computed exponential factor. For the cases in which V_l is not a slowly varying function of the energy, the more general Omnes solution must be used. In any event, Eq. (32) would seem to offer a simple and convenient estimate of final state interaction effects.

In summary, the approach used here seems to allow a convenient evaluation of final state interaction effects in a variety of processes. The results for pion-nucleon scattering derived in Sec. III will be applied in a later paper to the problem of photoproduction of pions from nucleons.

[17] We are grateful to Professor Nambu for first suggesting this solution to us.

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Photoproduction of π Mesons*†

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The dispersion relations for meson photoproduction at moderately low energies are examined by means of the comparison function method which was proposed in an earlier paper. Assuming that only the (3,3) state is appreciably modified by rescattering effects, an approximate solution is obtained. Nucleon recoil and crossing symmetry are treated exactly. The static limit of this solution yields substantial agreement with the results of Chew, Goldberger, Low, and Nambu. It is hoped that an evaluation including effects of nucleon recoil will improve the agreement in the resonance region.

1. INTRODUCTION

A FAIRLY exhaustive discussion of the photoproduction of π mesons from nucleons in the low-energy range has been carried out by Chew, Goldberger, Low, and Nambu.[1] This work is based on the technique of dispersion relations, whose validity for this process was recently established from the general axioms of field theory by Oehme and Taylor.[2] The evaluation of the dispersion relations was effected by using the relation between meson photoproduction and pion-nucleon scattering demanded by unitarity,[3] and also by making an expansion in inverse powers of the nucleon mass. The latter procedure makes it conveniently possible to compare these results to those of the static model.[4] However, this expansion necessarily restricts the results to energies near and below the scattering resonance, where nucleon recoil effects are relatively unimportant.

In the present study the comparison function method which was proposed in a previous paper[5] is used together with unitarity to obtain explicit approximate solutions to the photoproduction dispersion relations. The essential approximation made in applying this method to the present case is the assumption that the (3,3) scattering phase shift is the only one which is important under the dispersion integrals. However, since the crossing symmetry is treated exactly, the final result does contain some rescattering corrections of the (3,3) state to all partial waves. Multi-meson production is

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[1] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1345 (1957). (Hereafter to be denoted by CGLN.) Dispersion relations for S and P waves to order $1/M$ have also been derived by L. D. Solov'ev, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 801 (1957) [translation: Soviet Phys. JETP **6**, 617 (1958)].

[2] R. Oehme and J. G. Taylor, Phys. Rev. **113**, 371 (1959).

[3] K. M. Watson, Phys. Rev. **95**, 228 (1954).

[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

[5] R. Blankenbecler and S. Gartenhaus, preceding paper [Phys. Rev. **116**, 1297 (1959)] (hereafter to be denoted by I).