Exact Three-Variable Solutions of the Field Equations of General Relativity*†

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In order to trace out with more understanding the consequences of general relativity it is advantageous to have exact solutions of Einstein's field equations which show more detail than the familiar solutions with their high symmetry. In the present investigation, based on the method of separation of variables, all solutions of the field equations for empty space have been found which (1) have the "linked pair" form $g_{ij}=\pm\delta_{ij}A_i^2(x^0,x^1)B_i^2(x^0,x^3)$, and which (2) are nondegenerate—so far as could be determined—in the sense that all the g_{ij} cannot be reduced to functions of only two variables. Other solutions have been obtained from the solutions of the above form by interchange of variables. Explicit expressions are given for all twenty nondegenerate solutions, all apparently new. Of degenerate solutions, ten are presented, not all of them new. All thirty solutions are examined with respect to possible physical and geometrical interpretations.

1. NEED FOR ADDITIONAL EXACT SOLUTIONS

Recent work increases the suspicion that general relativity has many unusual consequences not yet brought to light nor assimilated into the rest of physics. It is reasonable to believe that it will help in reading out these consequences to have a substantial increase in the number and variety of exact solutions of Einstein's field equations which are available for analysis. This paper presents twenty solutions that depend upon three variables, all apparently unknown previously. They are offered as raw material for future studies about the nature of singularities, about the topology of space-time, and about the kind of events that can happen in a universe of pure "geometrodynamics."

The most familiar of exact solutions of Einstein's field equations known today are (1) the Schwarzschild solution for a point mass[1]; (2) the Reissner-Nordstrom solution for the combination of a point mass and a point charge[2]; (3) the solutions of Friedmann (pure dust), and Tolman (pure radiation), and Gamow (mixture) for the dynamics of a universe uniformly curved in space[1]; (4) the cylindrically symmetric gravitational waves of Einstein and Rosen[3]; (5) the plane gravitational waves of Bondi[4]; (6) the pure source-free time-symmetric and axially symmetric gravitational waves which, as Brill has proven, have positive definite mass and which—endowed with sufficient strength—curl up the metric into a closed space[5]; (7) multiply connected spaces endowed with many "wormholes" in which source-free electric lines of force are trapped to produce *classical* electric charges—solutions for which Misner and Wheeler[6] have given exact expressions at a moment of time symmetry; and (8) Bertotti's exact solution[7] for a space permeated with a uniform or covariantly constant electromagnetic field.

Reasons for wanting additional exact solutions of Einstein's field equations are readily apparent from a look at four sample areas of inquiry having to do with (1) the question whether singularities are *always* bound to appear, (2) departures of gravitational fields from spherical symmetry, (3) interactions of gravitational waves, and (4) snap-over from one topology to another.

(1) Is there any general answer as to the fate of a solution of Einstein's field equations as time evolves? Only for a finite proper time is the familiar spherically symmetric Schwarzschild solution free of singularity.[8] Only for a finite time is an isotropic homogeneous closed universe free of singularity, regardless whether the mass arises from a uniform distribution of dust (Fredmann), or radiation (Tolman), or any mixture of the two (Gamow). Is this finiteness of the time a property of these special models, or is it a feature to be expected quite generally? Additional special solutions will give more background for considering this question. It does not help in securing a general answer, although it is most interesting, to learn the general theorem[9] that no set of time-like geodesic curves can be continued without intersection or other singularity for all time orthogonal to an initial space-like surface unless that surface is flat. What one wants is information about the intrinsic geometry of the 4-space, not about a particular coordinate system.

(2) One knows that the electric potential near a point

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[1] See for example L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Inc., Cambridge, 1951).
[2] See for example, H. Weyl, *Space, Time, and Matter* (Dover Publications, New York, 1922), p. 261.
[3] See for example J. Weber and J. A. Wheeler, Revs. Modern Phys. **29**, 509 (1957).
[4] H. Bondi, Nature **179**, 1072 (1957); *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. 22.
[5] Dieter Brill, Ann. Phys. (N.Y.) (to be published).
[6] C. W. Misner and J. A. Wheeler, Ann. Phys. (N.Y.) **2**, 590 (1957).
[7] B. Bertotti (to be published).
[8] M. Kruskal (to be published).
[9] A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955); **106**, 172 (L) (1957). A. Komar, Phys. Rev. **104**, 544 (1956).

charge can be expanded in the form

$$V=e/r+\sum_{l,m} C_{l,m}Y^{(l,m)}(\theta,\varphi)r^l.$$

What are the analogous departures from the Schwarzschild solution? Can one find exact solutions for certain sample cases of a Schwarzschild metric as modified by a nonsymmetrical environment? How do these modifications affect the orbital motion of a planet or an idealized test particle?

(3) How do gravitational waves interact with each other and with Schwarzschild centers of mass? Can one find exact solutions[10] that describe special cases of either type of interaction?

(4) When gravitational waves implode into a limited region of space and create exceedingly strong local curvatures, or when strong fluctuations in the metric develop from other causes, can the topology snap over from one connectivity to another? What can one do to find out what types of snap-over are possible? How does the metric vary in the immediate neighborhood of the critical point? This question recalls the problem in hydrodynamics[11] to analyze fluid flow as the neck between two masses of liquid breaks during fission. The topology of the droplet changes with time in the example, yet the continuum equations of hydrodynamics suffice to analyze what goes on right up to and right after the moment of fission.

2. SIMILARITY SOLUTIONS AND LINKED PAIR SOLUTIONS

This investigation began with the hope to investigate the analogous fission process in geometrodynamics, near the point of fission. The equations in both cases are of course nonlinear. Therefore Wheeler suggested that solutions of the field equations of general relativity could be sought by the similarity methods familiar in hydrodynamics. Consider, for example, a shock wave that strikes the corner of an object and is reflected from it (Fig. 1); then it is found that, close to the corner, the ratio of the radius r of the reflected wave to the time t since impact is a constant.[12] Thus, $r/t=k$, or $r-kt=0$. That the similarity principle behind this experimental result and others can be used to solve the nonlinear equations of hydrodynamics has long been known.[13] A partial differential equation invariant under a similarity transformation admits solutions that are also invariant under the transformation. For example, the nonlinear equation

$$(\partial u/\partial t)^3=\partial^2u/\partial r^2$$

is unchanged by the similarity transformation $r'=\alpha r$, $t'=\alpha^b t$, $u'=\alpha^{(3b-2)/2}u$. Thus the two variables v

$$v=u/r^{(3b-2)/2} \quad \text{and} \quad x=r^b/t$$

are also unchanged by this transformation. It follows that the trial solution given by

$$v=F(x)$$

is also invariant under the transformation. Substituting this trial solution u into the partial differential equation, one reduces it to an ordinary differential equation for a function of a single variable.

Guided by this example, we considered metrics that depend only upon selected combinations of certain coordinates, thus

$$g_{ij}=A_{ij}(f(r,z)/g(t))B_{ij}(t), \tag{1}$$

and

$$g_{ij}=A_{ij}(r/a(t),\quad z/b(t))B_{ij}(t). \tag{2}$$

It should be mentioned, however, that the ratios of the functions of the coordinates appearing here are not necessarily invariants in the sense discussed above. When these proposed similarity expressions for the metric were substituted into the equations of general relativity, it was found that the equations to be solved were too difficult to work with, and thus it was found desirable to search for a simpler form. The search proceeded as in the following outline.

(1) As was assumed implicitly above, the demand was made that all the metric coefficients g_{ij} should be independent of the coordinate x^2. If we require, in addition, that the metric be invariant under the transformation $x^2=-x^{2\prime}$, then it follows that (2) the metric can be diagonalized. We note first that the coefficients g_{ij} of the differential terms dx^idx^2 $(i\neq 2)$ vanish; this is an immediate result of the condition stated in the previous sentences. We then make a coordinate transformation of three coordinates x^0, x^1, x^3:

$$x^0=f(x'^0,x'^1,x'^3),\quad x^1=g(x'^0,x'^1,x'^3),\quad x^3=h(x'^0,x'^1,x'^3).$$

We impose on the three functions f, g, and h the three conditions that each of the remaining nondiagonal metric coefficients g_{01}, g_{03}, and g_{13} should vanish. This completes the argument.

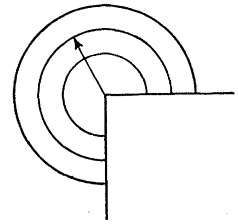


FIG. 1. Shock wave reflected from a corner.

[10] Bondi and Robinson give an exact solution which they interpret as describing the interaction of two *plane* gravitational waves. See also reference 3.

[11] See E. Power and J. A. Wheeler, Revs. Modern Phys. **29**, 480 (1957), for a table of analogies between general relativity and hydrodynamics.

[12] This result has been found experimentally in this laboratory: W. Bleakney and A. H. Taub, Revs. Modern Phys. **21**, 584 (1949).

[13] For a summary see for example G. Birkhoff, *Hydrodynamics: A Study in Logic, Fact, and Similitude* (Dover Publications, New York, 1955), especially Chap. IV.

If forms (1) and (2) are assumed to be diagonal, then it is still found that the field equations are too complicated to work with easily. Accordingly, an alternative requirement was made. (3) It was demanded that the metric should have the possibility to represent a wave traveling in the x^1 direction or in the x^3 direction or both, in this sense, that the g_{ij} should have the "linked pairs" form

$$g_{ij}=\delta_{ij}e_iA_i^2(x^0,x^1)B_i^2(x^0,x^3). \tag{3}$$

Here $e_0=-1$, $e_1=e_2=e_3=1$. Of course, it is not said, nor is it true, that every solution of this form represents a wave. It is true, however, that form (3) may admit similarity effects; a special case of (3) is a metric of the form

$$g_{ij}=\delta_{ij}e_iA_i^2\left(\frac{f(x^1)}{a(x^0)}\right)B_i^2\left(\frac{g(x^3)}{b(x^0)}\right)C_i^2(x^0). \tag{4}$$

Another special case of (1)–(3) is a solution of the field equations in which one of the variables x^0, x^1, or x^3 does not appear. Such solutions are here called degenerate. Degenerate solutions were discarded (1) because solutions in two variables have been quite thoroughly treated in the literature,[14] and (2) because a great saving in work resulted. However, special work was required to identify solutions, ostensibly dependent on three variables, which actually reduced to functions of two variables; several of these were found. All ten degenerate solutions are solutions which at first looked as if they were functions of three variables. The twenty nondegenerate solutions do not, apparently, reduce to functions of two variables.

3. THE FIELD EQUATIONS FOR A DIAGONAL METRIC

The Riemann curvature tensor R_{ijkl} and the contracted or Ricci curvature tensor R_{ik}—which is to be zero in regions where space is empty—both take particularly simple forms when the metric is diagonal and is expressed as an exponential:

$$g_{ik}=\delta_{ik}e_ie^{2f_i}, \tag{5}$$

where $e_0=-1$, $e_1=e_2=e_3=1$. The reciprocal of the metric tensor is

$$g^{kl}=e_k\delta_{kl}e^{-2f_k}. \tag{6}$$

The Christoffel symbols

$$\Gamma^i{}_{kl}=\tfrac{1}{2}g^{im}(g_{mk,l}+g_{ml,k}-g_{kl,m}) \tag{7}$$

reduce to the form

$$\Gamma^i{}_{kl}=\delta_{ki}f_{i,l}+\delta_{il}f_{i,k}-\delta_{kl}e_ke_ie^{2(f_k-f_i)}f_{k,i}, \tag{8}$$

where the comma denotes ordinary (not covariant) differentiation. All components of the Riemann curvature tensor

$$R^i{}_{klm}=\Gamma^i{}_{km,l}-\Gamma^i{}_{kl,m}+\Gamma^i{}_{nl}\Gamma^n{}_{km}-\Gamma^i{}_{nl}\Gamma^n{}_{km} \tag{9}$$

vanish except those of the following form:

$$\begin{aligned}R^l{}_{ilk}&=R^l{}_{kli}=-R^l{}_{ikl}=-R^l{}_{kil}=-e_ie_le^{2(f_i-f_l)}R^i{}_{llk}\\&=e_ie_le^{2(f_i-f_l)}R^i{}_{lkl}\\&=f_{l,k}f_{k,i}+f_{i,k}f_{l,i}-f_{l,i}f_{l,k}-f_{l,ik},\\&\qquad(i\neq l,\ l\neq k,\ k\neq i)\end{aligned} \tag{10a}$$

and

$$\begin{aligned}R^l{}_{ili}&=-R^l{}_{iil}=e_ie_le^{2(f_i-f_l)}R^i{}_{lil}\\&=f_{i,i}f_{l,i}-f_{l,i}^2-f_{l,ii}+e_ie_le^{2(f_i-f_l)}(f_{l,l}f_{i,l}-f_{i,l}^2\\&\quad-f_{i,ll})-\sum_{m\neq i,l}e_ie_me^{2(f_i-f_m)}f_{i,m}f_{l,m},\quad(i\neq l).\end{aligned} \tag{10b}$$

The contracted curvature tensor

$$R_{ik}=R^j{}_{ijk} \tag{11}$$

has diagonal components

$$\begin{aligned}R_{ii}=\sum_{l\neq i}\Big[&f_{i,i}f_{l,i}-f_{l,i}^2-f_{l,ii}+e_ie_le^{2(f_i-f_l)}\\&\times(f_{l,l}f_{i,l}-f_{i,l}^2-f_{i,ll}-f_{i,l}\sum_{m\neq i,l}f_{m,l})\Big],\end{aligned} \tag{12a}$$

and off-diagonal ($i\neq k$) components

$$R_{ik}=\sum_{l\neq i,k}(f_{l,k}f_{k,i}+f_{i,k}f_{l,i}-f_{l,i}f_{l,k}-f_{l,ik}),\quad(i\neq k). \tag{12b}$$

4. APPLICATION OF TECHNIQUE OF SEPARATION OF VARIABLES TO OBTAIN LINKED-PAIR SOLUTIONS

We now specialize to a metric of the linked-pair form

$$g_{ij}=\delta_{ij}e_iA_i^2(x^0,x^1)B_i^2(x^0,x^3), \tag{13}$$

and find that of the 10 independent components of R_{ik}, three (R_{02}, R_{12}, and R_{32}) vanish automatically. The demand of Einstein's field equations that all ten components shall vanish in empty space therefore gives us seven partial differential equations, each of the form

$$\sum_{i=1}^{n}F_i(x^0,x^1)G_i(x^0,x^3)=0. \tag{14}$$

Here the F_i and G_i are built out of the A_i and B_i and their derivatives. By the method of separation of variables we have split up these equations into simpler differential equations. The split-up process itself is algebraic. Thus, we suppress the variable $x^0=T$ and call $x^1=r$, $x^3=z$ and arrive at the typical equation

$$\sum_{i=1}^{n}F_i(r)G_i(z)=0. \tag{15}$$

[14] For example: H. Weyl, Ann. Physik **54**, 117 (1917); A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).

In the general treatment of Eq. (15), we consider two possibilities:

I. $F_1=0$.—This automatically reduces Eq. (15) from an equation of n terms to an equation of $n-1$ terms.

II. $F_1\neq 0$.—We divide by F_1 and differentiate with respect to r. Again, Eq. (15) is reduced to $n-1$ terms. If we continue this procedure, we obtain 2^n mutually exclusive cases of separation of Eq. (15). These cases provide a complete separation of variables for Eq. (15). (If one were applying the method of separation of variables to a problem of lesser generality, such that one or more of the F_i or G_i were known in advance to be constant, then further simplifications could be achieved by differentiating with respect to the missing variable.)

Using this method of analysis, we can show, for example, that Eq. (15) for the case $n=3$,

$$F_1(r)G_1(z)+F_2(r)G_2(z)+F_3(r)G_3(z)=0, \tag{16}$$

possesses the following eight mutually exclusive solutions:

1. $F_1=F_2=F_3=0$.
2. $F_1=F_2=G_3=0;\quad F_3\neq 0$.
3. $F_1=G_2=G_3=0;\quad F_2\neq 0$.
4. $F_1=0;\quad F_3=kF_2;\quad G_2=-kG_3;\quad F_2\neq 0;\quad G_3\neq 0$.
5. $G_1=G_2=G_3=0;\quad F_1\neq 0$.
6. $G_2=0;\quad F_3=kF_1;\quad G_1=-kG_3;\quad F_1\neq 0;\quad G_3\neq 0$.
7. $F_2=k_1F_1;\quad F_3=k_2F_1;\quad G_1=-k_1G_2-k_2G_3;$ $F_1\neq 0;\quad G_2\neq 0$.
8. $F_2=-k_1F_1-k_2F_3;\quad G_1=k_1G_2;\quad G_3=k_2G_2;$ $F_1\neq 0;\quad G_2\neq 0;$

$$F_3\neq CF_1 \text{ (any constant } C). \tag{17}$$

Each of these possible algebraic solutions represents in actuality a set of differential equations yet to be solved. Moreover, the variable x^0 has to be restored to the equations. If some of the F_i and G_i depend upon x^0, then some or all of the "constants" of separation may also depend upon x^0.

In principle, this split-up could be carried out for all seven equations $R_{ik}=0$. In practice, it was sufficient to break down in this way only the equation $R_{13}=0$. The information so obtained about the A_i and B_i was then substituted into the other six equations to define more fully the allowable dependence of the A_i (and B_i) upon x^0 and x^1 (and x^0 and x^3).

Each case was analyzed in terms of the special techniques appropriate to that case. As an example we cite one type of equation which occurred often in the analysis:

$$Cx^2+G_1(y)x+G_2(y)=F_1(x)[F_2(x)y+F_3(x)]^k. \tag{18}$$

We require that neither expression should equal zero. Here C, k, the F_i, and the G_i are all to be determined. k and C are constants. If $k=0$, we see quickly that G_1 and G_2 are constants. If $k\neq 0$, we take the $(1/k)$th power of both sides and differentiate twice with respect to y. Equating the coefficients of the powers of x to zero, we get a set of differential equations which can be solved easily. Six mutually exclusive solutions are found.

In the present work every appropriate simplification was made. For example, when a function $[F(x^i)]^2$ appeared as a multiplicative factor in g_{ii}, a change of variables, $x^{i*}=x^{i*}(x_i)$, was made to eliminate this factor:

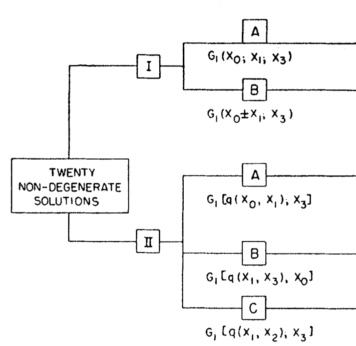
$$dx^{i*}=F(x^i)dx^i.$$

Furthermore, degenerate solutions—those which were functions of only two of the three variables x^0, x^1, and x^3—were all discarded, with the exception of a few which appear later in the paper. All metrics for which $R^i{}_{jlk}=0$, being flat, were at once discarded.

Finally, some solutions were obtained from others by complex coordinate transformations.[15] For example, the transformation $x_0\rightarrow ix_2$, $x_2\rightarrow x_0$, $l\rightarrow il$ changes the solutions later designated as III–7 and III–8 into III–9 and III–10.

It should be mentioned that the function $\sinh\theta$ (θ a function of the coordinates), wherever it appears, can be replaced by $\cosh\theta$ or e^θ; these functions cannot be obtained from $\sinh\theta$ by simple transformations.

Notation is as follows: all ϵ's signify ± 1. ϵ_1 and ϵ_2, where appearing, can independently take the values ± 1. l and λ are always constants and may have either sign. The e_i are given by $e_0=-1$, $e_1=e_2=e_3=1$. The x^i are written with subscripts (x_i) to facilitate writing.



i	a	b	c	d	e	f	g	h
1		✓				✓		1
2		✓					✓	3
1	✓	✓					✓	6
2	✓	✓				✓		2
3	✓	✓						4
4	✓	✓						4
1	✓	✓	0	1				2
2	✓	✓	0	3				16
3	✓	✓	1	3				16
4	✓	✓	0,1	3	✓			4
5	✓	✓	0,1	3	✓			4
6	✓	✓	0	3	✓			2
7	✓	✓	1	3	✓			2
1		✓	1	3				16
2		✓	1,3	3	✓			4
3		✓	1	3	✓			2
1			2	1				2
2			2	3				16
3			1,2	3	✓			4
4			2	3	✓			2

FIG. 2. Classification of the twenty nondegenerate three-variable solutions of Einstein's field equations. A check in response to a question means yes; a blank, no. (i) refers to the number of the solution in its particular category of classification. (a) Wave-like appearance? (b) Time (x_0)-dependent? (c) i for which the metric is invariant under $x_i\rightarrow -x_i$. (d) Number of functions of the single variable in a typical g_{ii} (classification II only). (e) Contains functions of unknown analytical form. (f) Does a sine appear? (g) Does a hyperbolic sine appear? (h) Total number of solutions obtainable from this one by considering all possible changes in the signs of the ϵ's and by substituting other functions for $\sinh\theta$. The number does not include variations due to changes in coordinates. G_1 throughout this figure should read G_i.

[15] Following the suggestion of Dr. Hans Buchdahl.

5. CLASSIFICATION OF SOLUTIONS

There are twenty separable nondegenerate solutions of the form $g_{ij}=e_i\delta_{ij}A_i^2(x_0,x_1)B_i^2(x_0,x_3)$ or derived therefrom by an interchange of coordinates (Fig. 2). Ten degenerate solutions are included. Each solution receives two classificatory labels [for example, I-A-1 and $G_1(x_0;x_1;x_3)$].

I. Typical metric component is a product of three factors, each factor being a function of a single coordinate or of a linear combination of coordinates.

A. The factors are functions of single coordinates. Two solutions are included; the notation is

$$G_i(x_0;x_1;x_3),\quad i=1,2.$$

B. One of the factors is a function of a single coordinate (x_3); at least one of the others is a function of $x_0\pm x_1$. Four solutions are included; the notation is $G_i(x_0\pm x_1;x_3)$, $i=1$ to 4.

II. Typical metric component is a product of two factors, one of which is a function of one variable and the other of which is a function of two variables, quadratic in at least one.

A. One is a function of x_3; one is a function quadratic in x_0 or x_1 or both. Seven solutions are included; the notation is $G_i[q(x_0,x_1);x_3]$, $i=1$ to 7.

B. One is a function of x_0; one is a function quadratic in x_1 or x_3 or both. Three solutions are included; the notation is $G_i[q(x_1,x_3);x_0]$, $i=1$ to 3.

C. One is a function of x_3; one is a function quadratic in x_1 or x_3 or both. Four solutions are included; the notation is $G_i[q(x_1,x_2);x_3]$, $i=1$ to 4.

III. Degenerate solutions.

Ten are included; the notation is D_i, $i=1$ to 10.

List of Solutions

I-A-1; $G_1(x_0;x_1;x_3)$.§

$$ds^2=\sum_{i=0}^{3} e_ia_i^2A^{2n_i}B^{2m_i}C^{2l_i}dx_i^2;$$

$$A=x_1/l;\quad B=x_0/l;\quad C=\sin(x_3/l).$$

i	0	1	2	3
n_i	$\frac{3}{2}$	$\frac{3}{4}$	$-\frac{1}{2}$	7/4
m_i	$\frac{1}{2}$	1	$-\frac{1}{2}$	1
l_i	$\frac{3}{2}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{3}{4}$
a_i	λ	1	1	1

I-A-2; $G_2(x_0;x_1;x_3)$.§

$$ds^2=\sum_{i=0}^{3} e_ia_i^2A^{2n_i}B^{2m_i}C^{2l_i}dx_i^2;$$

$$A=x_1/l;\quad B=x_0/l;\quad C=\sinh(x_3/l).$$

i	0	1	2	3
n_i	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
m_i	$\frac{3}{4}$	$\frac{3}{2}$	$-\frac{1}{2}$	7/4
l_i	$\frac{3}{4}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{4}$
a_i	1	λ	1	1

I-B-1; $G_1(x_0\pm x_1;x_3)$.

$$ds^2=\sum_{i=0}^{3} e_iA^{2m_i}B^{2n_i}C^{2l_i}dx_1^2;$$

$$A=(x_0-x_1)/l;\quad B=(x_0+x_1)/l;\quad C=\sinh(2x_3/l).$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
m_i	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2-\sqrt{2}\epsilon)$
l_i	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$

I-B-2; $G_2(x_0\pm x_1;x_3)$.

$$ds^2=\sum_{i=0}^{3} e_iA^{2n_i}B^{2m_i}C^{2l_i}dx_i^2;$$

$$A=(x_1-x_0)/l;\quad B=(x_1+x_0)/l;\quad C=\sin(2x_3/l).$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
m_i	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2-\sqrt{2}\epsilon)$
l_i	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$

I-B-3; $G_3(x_0\pm x_1;x_3)$.

$$ds^2=\sum_{i=0}^{3} e_ia_i^2A^{2n_i}B^{2m_i}C^{2l_i}dx_i^2;$$

$$A=(x_0-x_1)/l;\quad B=x_0/l;\quad C=x_3/l.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon_1)$	$\frac{1}{2}(1+\sqrt{2}\epsilon_1)$	$-\frac{1}{2}\sqrt{2}\epsilon_1$	$\frac{1}{2}(2+\sqrt{2}\epsilon_1)$
m_i	$\frac{5}{2}+\epsilon_2-\sqrt{2}\epsilon_1(\frac{1}{2}+2\epsilon_2)$	$\frac{1}{2}+\epsilon_2-\frac{1}{2}\sqrt{2}\epsilon_1$	$-\epsilon_2+\frac{1}{2}\sqrt{2}\epsilon_1$	$3+\epsilon_2-\sqrt{2}\epsilon_1(\frac{1}{2}+2\epsilon_2)$
l_i	1	$\frac{1}{4}\sqrt{2}\epsilon_1(1+\epsilon_2)$	$\frac{1}{4}(1-\epsilon_2)+\frac{1}{8}\sqrt{2}\epsilon_1(-3+\epsilon_2)$	0
a_i	1	λ	1	$\frac{1}{4}(1-\epsilon_2)+\frac{1}{8}\sqrt{2}\epsilon_1(3\epsilon_2-1)$

§ *Note added in proof.*—Dr. F. A. E. Pirani has recently informed me that solutions I-A-1 and I-A-2 may be reduced from functions of three variables to functions of two variables. This may be accomplished by, for example, the transformations

$$x_3{}^*=x_1\cos\frac{x_3}{l},\quad x_1{}^*=x_1\sin\frac{x_3}{l}\quad\text{in I-A-1,}\qquad\text{and}\qquad x_3{}^*=x_0\sinh\frac{x_3}{l},\quad x_0{}^*=x_0\cosh\frac{x_3}{l}\quad\text{in I-A-2.}$$

I–B–4; $G_4(x_0 \pm x_1; x_3)$.

$$ds^2 = \sum_{i=0}^{3} e_i a_i^2 A^{2n_i} B^{2m_i} C^{2l_i} dx_i^2;$$

$$A = (x_0 - x_1)/l; \quad B = x_1/l; \quad C = x_3/l.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon_1)$	$\frac{1}{2}(1+\sqrt{2}\epsilon_1)$	$-\frac{1}{2}\sqrt{2}\epsilon_1$	$\frac{1}{2}(2+\sqrt{2}\epsilon_1)$
m_i	$\frac{1}{2}+\epsilon_2-\frac{1}{2}\sqrt{2}\epsilon_1$	$\frac{5}{2}+\epsilon_2-\sqrt{2}\epsilon_1(\frac{1}{2}+2\epsilon_2)$	$-\epsilon_2+\frac{1}{2}\sqrt{2}\epsilon_1$	$3+\epsilon_2-\sqrt{2}\epsilon_1(\frac{1}{2}+2\epsilon_2)$
l_i	$\frac{1}{4}\sqrt{2}\epsilon_1(1+\epsilon_2)$	1	$\frac{1}{4}(1-\epsilon_2)+\frac{1}{8}\sqrt{2}\epsilon_1(-3+\epsilon_2)$	0
a_i	λ	1	1	$\frac{1}{4}(1-\epsilon_2)+\frac{1}{8}\sqrt{2}\epsilon_1(3\epsilon_2-1)$

II–A–1; $G_1[q(x_0,x_1); x_3]$.

$$ds^2 = \sum_{i=0}^{3} e_i a_i^2 A^{2n_i} C^{2l_i} dx_i^2;$$

$$A = (x_0^2/l^2) + (x_1/l); \quad C = x_3/l.$$

i	λ	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
l_i	1	$(1/7)(1+2\sqrt{2}\epsilon)$	$(1/7)(2-3\sqrt{2}\epsilon)$	0
a_1	1	λ	1	$(1/7)(3-\sqrt{2}\epsilon)$

II–A–2; $G_2[q(x_0,x_1); x_3]$.

$$ds^2 = \sum_{i=0}^{3} e_i a_i^2 A^{2n_i} C^{2k_i} D^{2l_i} E^{2m_i} dx_i^2;$$

$$A = (x_1/l) - (x_0^2/16l^2); \quad C = (x_3/l) - \epsilon_1; \quad D = (x_3/l) + \epsilon_1; \quad E = (\epsilon_2 x_3/l) + \tfrac{1}{2}\sqrt{2}\epsilon_3.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
k_i	$\frac{1}{2}(2-\sqrt{2}\epsilon)$	$\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$
l_i	$\frac{1}{2}(4+3\sqrt{2}\epsilon)$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$	$-\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{3}{2}(1+\sqrt{2}\epsilon)$
m_i	$-3-\sqrt{2}\epsilon$	$-1-\sqrt{2}\epsilon$	$\sqrt{2}\epsilon$	$-4-\sqrt{2}\epsilon$
a_i	1	λ	1	$2\sqrt{2}$

II–A–3; $G_3[q(x_0,x_1); x_3]$.

$$ds^2 = \sum_{i=0}^{3} e_i a_i^2 A^{2n_i} C^{2k_i} D^{2l_i} E^{2m_i} dx_i^2;$$

$$A = (x_0/l) + (x_1^2/16l^2); \quad C = (x_3/l) - \epsilon_1;$$
$$D = (x_3/l) + \epsilon_1; \quad E = (\epsilon_2 x_3/l) + \tfrac{1}{2}\sqrt{2}\epsilon_3.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
k_i	$\frac{1}{2}(2+\sqrt{2}\epsilon)$	$\frac{1}{2}(4+3\sqrt{2}\epsilon)$	$-\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{3}{2}(1+\sqrt{2}\epsilon)$
l_i	$\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$
m_i	$-1-\sqrt{2}\epsilon$	$-3-\sqrt{2}\epsilon$	$\sqrt{2}\epsilon$	$-4-\sqrt{2}\epsilon$
a_i	λ	1	1	$2\sqrt{2}$

In solutions II–A–4, II–A–5, II–B–2, and II–C–3, u is a solution of the equation

$$\frac{du}{dx} = \left(\frac{u^2-1}{4x}\right)\left[u\left(\frac{x+1}{x-1}\right)+\sqrt{2}\epsilon\right].$$

In solutions II–A–6, II–A–7, II–B–3, and II–C–4, v is a solution of the equation

$$\frac{dv}{dx} = \left(\frac{v^2-1}{4x}\right)\left[\frac{2v}{x-1}+\frac{4\sqrt{3}}{3}\epsilon\right].$$

The definition of x will be given in each case.

II–A–4; $G_4[q(x_0,x_1); x_3]$.

$$ds^2 = \sum_{i=0}^{3} e_i A^{2n_i} C^{2k_i} D^{2l_i} E^{2m_i} dx_i^2;$$

$$A = (\epsilon_1 x_0^2/16l^2) + (x_1^2/16l^2) + \lambda; \quad x = x_3/l;$$

$$C = \exp\left(\int \frac{u\,dx}{x}\right); \quad D = \epsilon_1 x; \quad E = \left(\frac{u^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
k_i	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$-\frac{1}{4}$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$
l_i	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{3}{4}$
m_i	0	0	0	1

II–A–5; $G_5[q(x_0,x_1);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=-(x_0^2/16l^2)-(\epsilon_1 x_1^2/16l^2)+\lambda;\quad x=x_3/l;$$

$$C=\exp\left(\int\frac{udx}{x}\right);\quad D=\epsilon_1 x;\quad E=\left(\frac{u^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
k_i	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$-\frac{1}{4}$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$
l_i	$\frac{1}{4}$	$-\frac{1}{4}$	0	$-\frac{3}{4}$
m_i	0	0	0	1

II–A–6; $G_6[q(x_0,x_1);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(\sqrt{3}x_1/3l)-(x_0^2/12l^2);\quad x=x_3/l;$$

$$C=\exp\left(\int\frac{vdx}{x}\right);\quad D=-x;\quad E=\left(\frac{v^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$\frac{1}{2}(2+\sqrt{3}\epsilon)$	$\frac{1}{2}(1+\sqrt{3}\epsilon)$	$-\frac{1}{2}\sqrt{3}\epsilon$	$\frac{1}{2}(3+\sqrt{3}\epsilon)$
k_i	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$-\frac{1}{2}$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$
l_i	$-\frac{1}{6}\sqrt{3}\epsilon$	$-\frac{1}{6}(3+\sqrt{3}\epsilon)$	$\frac{1}{6}\sqrt{3}\epsilon$	$-\frac{1}{6}(6+\sqrt{3}\epsilon)$
m_i	0	0	0	1

II–A–7; $G_7[q(x_0,x_1);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(\sqrt{3}x_0/3l)+(x_1^2/12l^2);\quad x=x_3/l;$$

$$C=\exp\left(\int\frac{vdx}{x}\right);\quad D=x;\quad E=\left(\frac{v^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{3}\epsilon)$	$\frac{1}{2}(2+\sqrt{3}\epsilon)$	$-\frac{1}{2}\sqrt{3}\epsilon$	$\frac{1}{2}(3+\sqrt{3}\epsilon)$
k_i	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$-\frac{1}{2}$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$
l_i	$-\frac{1}{6}(3+\sqrt{3}\epsilon)$	$-\frac{1}{6}\sqrt{3}\epsilon$	$\frac{1}{6}\sqrt{3}\epsilon$	$-\frac{1}{6}(6+\sqrt{3}\epsilon)$
m_i	0	0	0	1

II–B–1; $G_1[q(x_1,x_3);x_0]$.

$$ds^2=\sum_{i=0}^{3} e_i a_i^2 A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(x_3/l)-(x_1^2/16l^2);\quad C=(x_0/l)-\epsilon_1;$$

$$D=(x_0/l)+\epsilon_1;\quad E=(\epsilon_2 x_0/l)+\tfrac{1}{2}\sqrt{2}\epsilon_3.$$

i	0	1	2	3
n_i	$\frac{1}{2}(2+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$
k_i	$\frac{3}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(4+3\sqrt{2}\epsilon)$	$-\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
l_i	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}(2-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}\sqrt{2}\epsilon$
m_i	$-4-\sqrt{2}\epsilon$	$-3-\sqrt{2}\epsilon$	$\sqrt{2}\epsilon$	$-1-\sqrt{2}\epsilon$
a_i	$2\sqrt{2}$	1	1	λ

II–B–2; $G_2[q(x_1,x_3);x_0]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(\epsilon_1 x_3^2/16l^2)-(x_1^2/16l^2)+\lambda;\quad C=\exp\left(\int\frac{udx}{x}\right);$$

$$D=\epsilon_1 x;\quad E=\left(\frac{u^2-1}{x-1}\right)^{\frac{1}{2}};\quad x=\frac{x_0}{l}.$$

i	0	1	2	3
n_i	$\frac{1}{2}(2+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$
k_i	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$-\frac{1}{4}$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$
l_i	$-\frac{3}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$
m_i	1	0	0	0

II–B–3; $G_3[q(x_1,x_3);x_0]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(\sqrt{3}x_3/3l)-(x_1^2/12l^2);\quad x=x_0/l;$$

$$C=\exp\left(\int\frac{vdx}{x}\right);\quad D=x;\quad E=\left(\frac{v^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$\frac{1}{2}(3+\sqrt{3}\epsilon)$	$\frac{1}{2}(2+\sqrt{3}\epsilon)$	$-\frac{1}{2}\sqrt{3}\epsilon$	$\frac{1}{2}(1+\sqrt{3}\epsilon)$
k_i	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$-\frac{1}{2}$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$
l_i	$-\frac{1}{6}(6+\sqrt{3}\epsilon)$	$-\frac{1}{6}\sqrt{3}\epsilon$	$\frac{1}{6}\sqrt{3}\epsilon$	$-\frac{1}{6}(3+\sqrt{3}\epsilon)$
m_i	1	0	0	0

II–C–1; $G_1[q(x_1,x_2);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i a_i^2 A^{2n_i}C^{2l_i}dx_i^2;$$

$$A=(x_1/l)-(x_2^2/l^2);\quad C=x_3/l.$$

i	0	1	2	3
n_i	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
l_i	$(1/7)(2-3\sqrt{2}\epsilon)$	$(1/7)(1+2\sqrt{2}\epsilon)$	1	0
a_i	1	λ	1	$(1/7)(3-\sqrt{2}\epsilon)$

II–C–2; $G_2[q(x_1,x_2);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i a_i^2 A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(x_1/l)+(x_2^2/16l^2);\quad C=(x_3/l)-\epsilon_1;$$

$$D=(x_3/l)+\epsilon_1;\quad E=(\epsilon_2 x_3/l)+\tfrac{1}{2}\sqrt{2}\epsilon_2.$$

i	0	1	2	3
n_i	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
k_i	$\frac{1}{2}(1-\sqrt{2}\epsilon)$	$\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(2-\sqrt{2}\epsilon)$	$\frac{1}{2}(1-\sqrt{2}\epsilon)$
l_i	$-\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$	$\frac{1}{2}(4+3\sqrt{2}\epsilon)$	$\frac{3}{2}(1+\sqrt{2}\epsilon)$
m_i	$\sqrt{2}\epsilon$	$-1-\sqrt{2}\epsilon$	$-3-\sqrt{2}\epsilon$	$-4-\sqrt{2}\epsilon$
a_i	1	λ	1	$2\sqrt{2}$

II–C–3; $G_3[q(x_1,x_2);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(x_1^2/16l^2)-(\epsilon_1 x_2^2/16l^2)+\lambda;\quad x=x_3/l;$$

$$C=\exp\left(\int\frac{u\,dx}{x}\right);\quad D=\epsilon_1 x;\quad E=\left(\frac{u^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$-\frac{1}{2}\sqrt{2}\epsilon$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(1+\sqrt{2}\epsilon)$	$\frac{1}{2}(2+\sqrt{2}\epsilon)$
k_i	$-\frac{1}{4}$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$	$\frac{1}{4}(1+\sqrt{2}\epsilon)$
l_i	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{3}{4}$
m_i	0	0	0	1

II–C–4; $G_4[q(x_1,x_2);x_3]$.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}C^{2k_i}D^{2l_i}E^{2m_i}dx_i^2;$$

$$A=(\sqrt{3}x_1/3l)+(x_2^2/12l^2);\quad x=x_3/l;$$

$$C=\exp\left(\int\frac{v\,dx}{x}\right);\quad D=-x;\quad E=\left(\frac{v^2-1}{x-1}\right)^{\frac{1}{2}}.$$

i	0	1	2	3
n_i	$-\frac{1}{2}\sqrt{3}\epsilon$	$\frac{1}{2}(1+\sqrt{3}\epsilon)$	$\frac{1}{2}(2+\sqrt{3}\epsilon)$	$\frac{1}{2}(3+\sqrt{3}\epsilon)$
k_i	$-\frac{1}{2}$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$	$\frac{1}{6}(3+2\sqrt{3}\epsilon)$
l_i	$\frac{1}{6}\sqrt{3}\epsilon$	$-\frac{1}{6}(3+\sqrt{3}\epsilon)$	$-\frac{1}{6}\sqrt{3}\epsilon$	$-\frac{1}{6}(6+\sqrt{3}\epsilon)$
m_i	0	0	0	1

III–1; D_1.

$$ds^2=-(x_0/l)dx_0^2+(l/x_0)dx_2^2+(x_0^2/l^2)(dx_1^2+dx_3^2).$$

III–2; D_2.

$$ds^2=-l^2e^{2\xi+\frac{5}{2}u}du\,dw+l^2\lambda^2e^{3(\xi+u)}d\xi^2+l^2e^{-\xi-u}d\eta^2.$$

III–3; D_3.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}B^{2m_i}dx_i^2;$$

$$A=x_1/l;\quad B=\sin(x_3/l).$$

i	0	1	2	3
n_i	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
m_i	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$

III–4; D_4.

$$ds^2=\sum_{i=0}^{3} e_i A^{2n_i}B^{2m_i}dx_i^2;$$

$$A=x_0/l;\quad B=\sinh(x_3/l).$$

i	0	1	2	3
n_i	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$
m_i	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$

III–5; D_5.

$$ds^2=\sum_{i=0}^{3} e_i a_i^2 A^{2n_i}C^{2l_i}dx_i^2;$$

$$A=x_0^*/l,\quad (x_0^*=x_0+x_1);\quad C=x_3/l.$$

i	0	1	2	3
n_i	$\frac{1}{2}(1+\sqrt{2}\epsilon_1)$	$\frac{1}{2}(1+\sqrt{2}\epsilon_1)$	$-\frac{1}{2}\sqrt{2}\epsilon_1$	$\frac{1}{2}(2+\sqrt{2}\epsilon_1)$
l_i	$(1/7)[3+\epsilon_2+\sqrt{2}\epsilon_1(1-2\epsilon_2)]$	$(1/7)[3-\epsilon_2+\sqrt{2}\epsilon_1(1+2\epsilon_2)]$	$(1/7)(1-2\sqrt{2}\epsilon_1)$	0
a_i	λ	1	1	1

III–6; D_6.

$$ds^2=\sum_{i=0}^{3} e_i a_i^2 A^{2n_i}(x-\epsilon_2)^{2k_i}(x+\epsilon_2)^{2l_i}x^{2m_i}dx_i^2;$$

$$A=x_0^*/l,\quad (x_0^*=x_0+\tfrac{1}{3}\sqrt{3}x_1);\quad x=x_3/l.$$

i	0	1	2	3
n_i	$\frac{1}{2}(2+\sqrt{3}\epsilon)$	$\frac{1}{2}(1+\sqrt{3}\epsilon)$	$-\frac{1}{2}\sqrt{3}\epsilon$	$\frac{1}{2}(3+\sqrt{3}\epsilon)$
k_i	$\frac{1}{4}(1+\sqrt{3}\epsilon)$	$\frac{1}{4}$	$\frac{1}{4}(1-\sqrt{3}\epsilon)$	$-\frac{1}{4}$
l_i	$\frac{1}{4}(3+\sqrt{3}\epsilon)$	$\frac{1}{4}(3+2\sqrt{3}\epsilon)$	$-\frac{1}{4}(1+\sqrt{3}\epsilon)$	$\frac{1}{4}(1+2\sqrt{3}\epsilon)$
m_i	$-\frac{1}{2}(2+\sqrt{3}\epsilon)$	$-\frac{1}{2}(2+\sqrt{3}\epsilon)$	$\frac{1}{2}\sqrt{3}\epsilon$	$-\frac{1}{4}(4+\sqrt{3}\epsilon)$
a_i	λ	1	1	$\sqrt{3}$

III–7; D_7.

$$ds^2=\frac{1}{(1-x_3^2/l^2)^2}\left[-dx_0^2+\sinh^2\left(\frac{2x_0}{l}\right)dx_1^2+\frac{dx_3^2}{(1-x_3^2/l^2)^2}\right]+\frac{x_3^2}{l^2}dx_2^2.$$

III–8; D_8.

$$ds^2=\frac{1}{(1+x_3^2/l^2)^2}\left[-dx_0^2+\sin^2\left(\frac{2x_0}{l}\right)dx_1^2+\frac{dx_3^2}{(1+x_3^2/l^2)^2}\right]+\frac{x_3^2}{l^2}dx_2^2.$$

III–9; D_9.

$$ds^2=-\frac{x_3^2}{l^2}dx_0^2+\frac{1}{(1+x_3^2/l^2)^2}\left[\sinh^2\left(\frac{2x_2}{l}\right)dx_1^2+dx_2^2+\frac{dx_3^2}{(1+x_3^2/l^2)^2}\right]$$

III–10; D_{10}.

$$ds^2=-\frac{x_3^2}{l^2}dx_0^2+\frac{1}{(1-x_3^2/l^2)^2}\left[\sin^2\left(\frac{2x_2}{l}\right)dx_1^2+dx_2^2+\frac{dx_3^2}{(1-x_3^2/l^2)^2}\right].$$

7. ANALYSIS OF THE METRICS

There are several techniques which can be used to aid in analysis of metrics. Six will be briefly mentioned here.

(1) Singularities, Topology, and Other Gross Features of the Metric

The most obvious features of any metric are its singularities; they may be the most important features, as well. They have been interpreted by researchers as the locations of mass points or distributions, wave fronts, and a host of other physico-geometrical objects. Singularities can sometimes be mere quirks of the coordinate system of the metric, and it is often difficult to tell whether a singularity is of this type or is an indication of real physical meaning. The reader is, no doubt, familiar with the old controversy over the meaning of the Schwarzschild singularity, for example.

Some of the techniques used in treatment of the Schwarzschild singularity may be of use in investigating other singularities. The reader is referred to Finkelstein,[16] Kruskal,[8] Einstein and Rosen,[17] and Lindquist,[18] all of whom have investigated the Schwarzschild singularity in one or another special way.

One method of investigating singularities in an invariant manner has been proposed by Wheeler and Brill.[19,5] Another such possible method would consist of inspecting the invariants of the metric; singularities appearing in the invariants would presumably be "real."

(2) Groups of Motions Admitted by the Metric; Other Symmetries

Groups of motions are of especial interest in general relativity, because they represent basic symmetries which are unaltered by coordinate transformations. This fact is illustrated by the requirement placed on a metric g_{ij} if the space of the g_{ij} is to admit a group: the fundamental form $ds^2=g_{ij}dx^idx^j$ must remain functionally invariant under all infinitesimal transformations of the group.[20] This requirement can be expressed mathematically through the Killing equations,[20]

$$\xi_{i;j}+\xi_{j;i}=0, \tag{19}$$

where the ξ_i are the infinitesimal generators of the group. An excellent application of using Eqs. (19) to restrict the g_{ij} to a simple form is given in a paper by Taub.[21]

On the other hand, we may investigate the converse problem: Given a metric g_{ij}, which groups of motions does it admit (if any)? In this case, in general, we must solve Eqs. (19) for the generators ξ_i. The question of integrability of Eqs. (19) is discussed in Eisenhart.[20]

A simple, though incomplete, method of searching for groups admitted by a metric is to inspect the metric for independence of some variable. If the g_{ij} are independent of x^k, then the space admits the one-parameter group having generators $\xi^k=\delta_0{}^k$. This group merely corresponds to the invariance of the metric under the translation $x'^k=x^k+\alpha$. The solutions $G_i(x_0;x_1;x_3)$, $G_i(x_0\pm x_1;x_3)$, $G_i[q(x_0,x_1);x_3]$, and $G_i[q(x_1,x_3);x_0]$ are all invariant under the transformation $x'^2=x^2+\alpha$, and the solutions $G_i[q(x_1,x_2);x_3]$ are invariant under $x'^0=x^0+\alpha$. A slight variation of this type of behavior is provided by D_5, which is invariant under the transformation $x'^0=x^0+\alpha$, $x'^1=x^1-\alpha$.

Kundt[22] has done a great deal of work on groups of motions in general relativity and has classified several solutions of the equations according to the groups of motions which they admit. It was thought desirable that the solutions of this paper should be similarly classified; however, Eqs. (19), when written out, are sufficiently complicated not to have been solved up to now.

[16] D. Finkelstein, Phys. Rev. **110**, 965 (1958).
[17] A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).
[18] R. W. Lindquist (unpublished).
[19] J. A. Wheeler, Australian J. Phys. (to be published).
[20] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1949).
[21] A. H. Taub, Ann. Math. **53**, 472 (1951).
[22] W. Kundt, Ph.D. thesis, University of Hamburg, 1958 (unpublished).

(3) The Curvature Tensor; Canonical Forms and Invariants

One especially fruitful method of investigating any physical system is to look at the invariants of the system. Closely associated with the invariants are the canonical forms, which represent mutually exclusive classes of the possible states of the system. The invariants and canonical forms of the Riemann tensor in the theory of general relativity have been well discussed by Komar,[23] Kundt,[22] Petrov,[24] and Pirani.[25] Pirani, in particular, has used the canonical forms to help outline a criterion for the existence of gravitational radiation. The metrics in this paper are all of canonical Type I; according to Pirani's criterion, this means that none of the metrics represent gravitational radiation. However, Pirani has recently informed me[26] that he feels the criterion to be too strong, and that he is studying possibilities of relaxing it. Hence, we must wait for these results before forming conclusions about the existence of radiation in these metrics.

(4) Can the Metric be Imbedded in Euclidean or Other Flat Space?

Imbedding of a metric, or a part of it, is actually only a mathematical convenience, useful for visualization. Because flat spaces have no particular significance in general relativity, imbedding has no physical meaning and is actually foreign to the spirit of the theory. It is mentioned here only as a possible tool for use in describing a given metric.

However, one finds that imbedding is difficult to use, even as a tool, because of the complexity of the equations. If the fundamental form of the lower dimensional space is

$$ds^2=g_{ij}dx^i dx^j, \quad i,j=1\cdots n$$

and if the fundamental form of the imbedding space is

$$ds^2=\sum_\alpha C_\alpha(dy^\alpha)^2, \quad \alpha=1\cdots m;\ m>n$$

where $C_\alpha=\pm 1$, then the following equations must be satisfied[20]:

$$\sum_\alpha C_\alpha \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\alpha}{\partial x^j}=g_{ij}. \tag{20}$$

These equations are much too difficult to solve in general. Several general theorems on imbedding exist, but do not aid the present problem. Nothing has been done on the problem at the present time.

[23] A. Komar, Ph.D. thesis, Princeton University, 1956 (unpublished); Proc. Natl. Acad. Sci. U.S. **41**, 758 (1955); Phys. Rev. **111**, 1182 (1958).
[24] A. Z. Petrov, Sci. Note Kazan State Univ. **114**, 55 (1954).
[25] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957).
[26] F. A. E. Pirani (private communication).

(5) The Geodesic Equations of Motion

Much of the physics inherent in any solution of the field equations lies in the equations of motion of a particle and the geodesics of the space under consideration. These equations are, as is well known,[1]

$$\frac{d^2x^i}{ds^2}+\Gamma^i{}_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}=0. \tag{21}$$

Much information about the space can be gleaned from a study of the geodesics. For example, a closed or quasi-closed geodesic, as exists for the Schwarzschild metric, may indicate a center of force. The concept of completeness—a geodesic is designated as *complete* if it can be extended indefinitely—can provide information about the boundaries of the space. It should be possible to investigate these features directly from Eqs. (21), although the author is not prepared to do so at the present.

Solution of Eqs. (21) for complicated metrics is usually quite difficult. Some enlightenment can be gained, however, from first integrals of motion. One of these always exists and is given by

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=1. \tag{22}$$

Other integrals of the motion exist if there are symmetries in the g_{ij}. We can see this by inspecting the Euler equations for the system[27]:

$$\frac{\partial L}{\partial x^i}-\frac{d}{ds}\frac{\partial L}{\partial(dx^i/ds)}=0, \tag{23}$$

where

$$L=\left(g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^{\frac{1}{2}}. \tag{24}$$

If $\partial g_{ij}/\partial x^k=0$, then $\partial L/\partial x^k=0$, and $\partial L/\partial(dx^k/ds)$ is a constant or first integral of the motion. If we evaluate this and use the result that $L=1$ from Eqs. (22) and (24), we find the following:

If $\partial g_{ij}/\partial k^k=0$, then

$$g_{ik}(dx^i/ds)=\text{constant}. \tag{25}$$

If the metric is diagonal, we get

$$dx^k/ds=\lambda/g_{kk}. \tag{26}$$

Equation (26) can be used to discuss types of singularities. It shows that, for $g_{kk}=0$, $dx^k/ds=\infty$, i.e., $ds=0$. Thus the points at the singularity in the x^k direction can perhaps be identified. For $g_{kk}=\infty$, $dx^k/ds=0$, and $x^k=$constant. This implies that there is no communication—travel is forbidden—along the

[27] C. Møller, *The Theory of Relativity* (Oxford University Press, London, 1955), p. 229.

singularity in the x^k direction. Note that, of course, these remarks only apply to a metric independent of x^k.

Most solutions in this paper are independent of x^2; the others are independent of x^0. Thus, we have two integrals of the motion available for each solution. These are of most immediate interest in the singularity treatment just outlined.

Another method of employing Eqs. (21) consists of comparing them with the Newtonian equations of motion for a known mass distribution. If the two sets of equations are approximately equivalent, then the metric probably represents the gravitational field of the mass distribution. One example of this method is found in the geodesic equations for the Schwarzschild metric, which closely approximate the Newtonian equations of motion for a central gravitational field.[1] Another very good example is found in a paper by Wilson[28]; by such a comparison as described above, he shows that the metric he has represents the gravitational field of an infinite line mass. Unfortunately, this method seems to be useful in only a small number of cases.

(6) Topology and Structure of Space-Like Surfaces at Moments of Time Symmetry

Interest in such three-dimensional spacelike surfaces stems from the fact that they may be used as initial-value data in the problem of constructing a four-dimensional solution of the field equations. The topology and structure of such surfaces will then determine the topology and structure of the four-dimensional space. The initial value problem for general relativity has been studied extensively by Lichnerowicz[29] and Foures[30] and has been discussed by Misner and Wheeler.[6] Christakis[31] has discussed the initial value problem for the metrics of this paper.

Five of the solutions in this paper are time-symmetric with respect to the plane $x^0=0$. They are all indicated on the flow chart.

We will now proceed to a discussion of the individual solutions. Due to several circumstances, this discussion will be rather brief. It is hoped that future study of these metrics can produce more information about their physical interpretation.

Solutions I–A–1, I–A–2

It is not clear whether the coordinates in I–A–1 are cylindrical or spherical. I–A–2 is the analog of I–A–1, but with $\sinh(x^3/l)$ instead of $\sin(x^3/l)$. The solutions are quite similar for small x^3/l.

[28] W. Wilson, Phil. Mag. **40**, 703 (1920).

[29] A. Lichnerowicz, *Problèmes Globaux en Méchanique Relativiste* (Hermann et Cie, Paris, 1939); J. Math. Pures Appl. **23**, 37 (1944); Helv. Phys. Acta, Suppl. IV, 176 (1956).

[30] Y. Foures-Bruhat, Acta Math. **88**, 141 (1952); J. Rational Mech. Anal. **4**, 951 (1956).

[31] A. Christakis, Senior thesis, Princeton University, 1959 (unpublished).

Solutions I–B–1, I–B–2

As these solutions stand, they look remarkably like superimposed waves traveling in oppposite directions. They might even be standing waves, except for a slight asymmetry in the exponents. However, this behavior may only be a feature of the coordinate system; the substitution $u=x_0-x_1$, $v=x_0+x_1$, destroys the wave-like appearance, and it is not known which form is most "physically real." If the exponential form is used in I–B–1, instead of the hyperbolic sine, the function C can be simplified considerably by a change of variable.

Solutions I–B–3, I–B–4

These solutions also have plane-wave appearances. The presence of x_1 by itself in I–B–4 might give rise to possibilities of the attenuation needed in cylindrical waves.

Solutions II–A–1 to II–A–7

These seven solutions are conveniently discussed together. The other notation for them is $G_i[q(x_0,x_1);x_3]$, indicating the quadratic character of x_0 and/or x_1. Those solutions quadratic in x_0 seem to indicate some sort of wave reflecting from the plane $x_1=0$ at the time $x_0=0$; those symmetrical in x_1 show waves going in both directions from the plane $x_1=0$.

Solutions II–A–4 to II–A–7 contain functions of x_3 of unknown analytic form, and consequently, the x_3 singularities are not easily discussed. Solutions II–A–2 to II–A–4, on the other hand, have three obvious x_3 singularities. However, in order that the metric remain real, we must require that C, D, and E should always be positive. This requirement has the effect of always having $x_3/l>1$, so that the only effective singularity is at $x_3/l=1$.

Solutions II–B–1 to II–B–3

As is indicated by the alternative notation

$$G_i[q(x_1,x_3);x_0],$$

these solutions are quadratic in x_1 and/or x_3. If these are rectangular coordinates, as seems most likely, then the singularities of these solutions are elliptic, parabolic, or hyperbolic cylinders. It is not known what physical significance such surfaces have. The x_0 singularities are the same as the x_3 singularities in the set

$$G_i[q(x_0,x_1);x_3].$$

Solutions II–C–1 to II–C–4

These solutions, $G_i[q(x_1,x_2);x_3]$, combine the singular cylinders of the set II–B with the x_3 singularities of II–A. The solutions are purely static and are thus time-symmetric.

Solution III–1 hardly needs discussion, because it has such a simple form. III–2 is only a function of two variables but is nondiagonal. If we put $u=(x_0+x_1)/l$, $w=(x_0-x_1)/l$, we return to the diagonal form, which now is a function of three variables. Furthermore this form has a plane-wave appearance.

Solutions III–3 to III–6

These solutions are all degenerate, but have many similarities to some of the nondegenerate solutions. No new comments are necessary, except considering III–5 and III–6; in the x_0+x_1 form, these solutions indicate plane wave traveling in one direction. It is not certain, of course, which form of the metric is most physical.

Solutions III–7 to III–10

These solutions are most conveniently discussed as a body. It should be noted from earlier discussions that these are only related through *complex* coordinate transformations; in other words, each of these represents a different physical situation.

Solutions III–10 will have a more familiar look if we define

$$m=\tfrac{1}{4}l, \qquad T=x_0,$$
$$\theta=x_2/2m, \qquad r=2m/(1-x_3{}^2/16m^2),$$
$$\phi=x_1/2m.$$

Substitution of these quantities into III–10 yields the Schwarzschild solution. A similar transformation of III–9 yields the metric

$$ds^2=-\left(\frac{2m}{r}-1\right)dT^2+r^2(d\theta^2+\sinh^2\theta d\varphi^2)+\frac{dr^2}{2m/r-1}. \qquad (5\text{–}8)$$

Because of the $\sinh\theta$ in $g_{\phi\phi}$, this metric is not spherically symmetric. Instead, it seems to represent a pseudosphere. Eisenhart[32] presents a very nice discussion of imbedding the metric $d\theta^2+g_{\varphi\varphi}d\varphi^2$ in 3-space, where $g_{\phi\phi}$ is either $\sinh^2\theta$, $\cosh^2\theta$, or $e^{2\theta}$. Any of these functions, as pointed out in Sec. 4, is acceptable for use in (5–8). For small θ, metric (5–8) is approximately spherically symmetric; it may be possible to piece such a metric together with the Schwarzschild metric.

Solutions III–7 and III–8 are somewhat different from solutions III–9 and III–10 because of the different arrangement of the x_i and the g_{ii}. It seems quite obvious from g_{22} that we should define $x_3=r$, $x_2=l\theta$, where r and θ are cylindrical coordinates. III-7 has a pronounced singularity at the cylinder $r=l$.

8. SUMMARY AND ACKNOWLEDGMENTS

The technique of separation of variables is a powerful tool in obtaining solutions of the equations of general relativity. It can be used to find all solutions of a given class, or it can be used in many cases to check whether a certain type of solution exists.

The solutions given in this paper are presented as raw material for further research in general relativity. It is hoped that their analysis will lead to further insights.

Any reader who desires a more detailed treatment of the equations in this paper should consult the original thesis of the same title (Princeton University, 1959, unpublished). Microfilm copies may be obtained from University Microfilms, Ann Arbor, Michigan.

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[32] L. P. Eisenhart, *An Introduction to Differential Geometry* (Princeton University Press, Princeton, 1947), pp. 277–286.