Interaction of Static Nucleons*

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The Heitler-London method has been applied to the calculation of the energy of interaction of two nucleons, as given by the fixed-source model. Numerical results are also given for the normalization of the state vector and the number of mesons in the cloud, for the states which comprise the deuteron. Particular attention is given to the rate of convergence of the expansions and the influence of the excited states of the nucleons. The relation to the Tamm-Dancoff method is also discussed in detail. It is shown that in $T=0$ states, the interaction does not appear to differ in any significant way from the one-meson exchange term. In $T=1$ states, on the other hand very large contributions are obtained from the higher order terms. These contributions have such a nature as to suggest that even at low energies it is improper to apply the fixed-source model, with the assumption that the meson cloud follows adiabatically the motion of the nucleons, to $T=1$ states.

I. INTRODUCTION

IN this paper we present the results of some calculations of the properties of two-nucleon states, as obtained from the static model. There have been many earlier calculations based on this model,[1–5] but the method used in this paper is somewhat different from those used by previous investigators, and the numerical results also differ in certain particulars from those previously reported, as we shall discuss below.

The model used herein is a very crude one, and great care must be used in attempting to relate properties of real two-nucleon states to the predictions of this model. Nevertheless, since the model quite successfully reproduces many of the observed properties of the meson cloud of a single nucleon, it may be hoped that the meson exchange effects predicted by the model will provide a useful first approximation to the actual effects of meson exchanges, especially when the nucleons are far apart. Many effects, which are most important at small distances, are left out of the model. These are meson-meson interactions and other nonlinear meson effects, the effects of the relative motion of the nucleons, and the influence of all virtual particles except the pion. These effects can be simulated, approximately, by making appropriate phenomenological modifications of the interaction at small distances; in this way, as many authors have pointed out, the model might be used to correlate data on the two-nucleon system at low energies.[6–11]

In some of the earlier calculations of the nucleon potential, it was assumed that the nucleons were point sources; in others, extended sources were assumed. At the distances at which one may have confidence in the predictions of the model, there is very little difference in the results. However, the view adopted here is that an appropriate cutoff should be used. We are primarily interested in exploring the mathematical properties of a particular model, and this can be done in a self-consistent way only if the same cutoff is used in all parts of the calculation. Furthermore, we may expect that if the parameters of the model are chosen to reproduce as closely as possible the observed properties of the meson cloud of a single physical nucleon, the additional phenomenological modifications which are introduced at small distances will be more significant.

(a) Properties of One-Nucleon States

Before we proceed with the discussion of two-nucleon states, we shall examine some of the known properties of one-nucleon states. This is done in order to introduce the notation, and to obtain some numerical results which will be referred to later. We also wish to see how much the two-nucleon interaction is affected by a reasonable change in the parameters of the model (that is, a change allowed by the uncertainty in the properties of one-nucleon states).

We wish to discuss the eigenstates of the Hamiltonian

$$H=\sum_k Ka_k^*a_k-W_0(\alpha_x^*\alpha_x+\alpha_y^*\alpha_y)+\sum_k (a_k+a_{-k}^*)[V_{xk}+V_{yk}], \quad (1)$$

where

$$V_{xk}=\alpha_x^*[f_0v_k(2K)^{-\frac{1}{2}}\tau_{xk}i\boldsymbol{\sigma}_x\cdot\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{x})]\alpha_x.$$

We use natural units ($\hbar=c=M_\pi=1$), and express by the corresponding capital the energy of a meson whose momentum is denoted by a given lower case letter, e.g.,

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[1] Taketani, Machida, and Onuma, Progr. Theoret. Phys. (Kyoto) **7**, 45 (1952).
[2] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).
[3] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).
[4] S. Machida and T. Toyoda, Progr. Theoret. Phys. (Kyoto) Suppl. No. 3, 106 (1956).
[5] Konuma, Miyazawa, and Otsuki, Progr. Theoret. Phys. (Kyoto) **19**, 17 (1958).
[6] Iwadari, Otsuki, Tamagaki, and Watari, Progr. Theoret. Phys. (Kyoto) Suppl. No. 3, 32 (1956).
[7] P. S. Signell and R. E. Marshak, Phys. Rev. **106**, 832 (1957); **109**, 1229 (1958).
[8] de Swart, Signell, and Marshak, Nuovo cimento **6**, 1189 (1957).
[9] J. J. de Swart and R. E. Marshak, Phys. Rev. **111**, 272 (1958).
[10] S. Otsuki, Progr. Theoret. Phys. (Kyoto) **20**, 171 (1958).
[11] W. Watari, Progr. Theoret. Phys. (Kyoto) **20**, 181 (1958).

$K=(1+k^2)^{\frac{1}{2}}$. The source function v_k is normalized so that $v_k=1$ when $K=0$.

Some relations between the renormalization constants and the meson-nucleon scattering amplitudes have been derived by Cini and Fubini[12]:

$$\begin{aligned}1&=Z_2[1+(4\alpha_3/9)+(4\alpha_2/9)+(\alpha_1/9)],\\ \rho_1^{-1}&=Z_2[1-(2\alpha_3/9)+(\alpha_2/9)+(\alpha_1/9)],\\ Z_2^{\frac{1}{2}}&=Z_2[1+(\alpha_3/9)-(2\alpha_2/9)+(\alpha_1/9)],\\ \pi\gamma\alpha_i&=3\int_1^\infty \sin^2\delta_i(k)\, v_k^{-2}k^{-3}dK.\end{aligned} \tag{2}$$

We shall assume that the parameters of the model are chosen so that the low-energy meson-nucleon scattering is approximately reproduced; $\gamma=f^2/4\pi$ must then be about 0.08, and if the source function v_k is supposed to differ very little from unity in the resonance region, one finds that $\alpha_3\sim 2.6$ (if $\gamma=0.08$). If one assumes that $\alpha_2=\frac{1}{4}\alpha_3$ (which is consistent with both the strong- and weak-coupling limits), one then obtains from Eq. (2) values for α_1, Z_2, and ρ_1 as shown in Table I. It may be noted that the value of α_1 so obtained is much larger than α_3.[13]

The isotopic vector part of the nucleon magnetic moment[14] and the resonance energy of the meson-nucleon scattering[13,15] depend on the quantity γM, where $M=\int_0^\infty v_k^2k^4K^{-4}dk$. If $\gamma=0.08$, M should be about 3.5. The product γM is determined very roughly by these comparisons, but a suitable way to compare source functions with different shapes is to require that γM be invariable. For instance, $M=3.5$ corresponds to a sharp cutoff at $K_{\max}=5.5$; however, a more convenient analytical form is given by $v_k=(\Lambda^2-1)/(\Lambda^2+k^2)$, for which $\Lambda=7$.

A measure of the average energy of the virtual mesons in the cloud is given by $\omega=M^{-1}\int_0^\infty v_k^2k^4K^{-3}dk$. The average number of mesons may be shown to be approximately

$$N_0=\frac{3\gamma M\omega}{\pi}\left(1+\frac{4\alpha_3\omega^2}{9(\omega+\omega_0)^2}+\frac{4\alpha_2\omega^2}{9(\omega+\omega_2)^2}+\frac{\alpha_1\omega^2}{9(\omega+\omega_1)^2}\right),$$

TABLE I. Numerical data for one-nucleon states.

$f^2/4\pi$	0.08	0.10
α_3	2.6	2.24
α_1	5.4	4.9
Z_2^{-1}	3.0	2.8
ρ_1^{-1}	0.36	0.40
M	3.5	2.8
Λ	7	6
ω	6.1	5.5
N_0	3.0	2.5

[12] M. Cini and S. Fubini, Nuovo cimento **3**, 764 (1956).
[13] See also G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957).
[14] H. Miyazawa, Phys. Rev. **101**, 1564 (1956).
[15] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

where ω_0 is the resonance energy, and ω_1 and ω_2 are typical energies for scattering in the other states; $\omega_1\sim\omega_2\sim\omega$. Values for some quantities of interest are given in Table I.[16]

(b) One-Meson Exchange Energy

The states of two interacting nucleons may be classified according to their total isotopic spin, but there is a preferred direction in space (the axis joining the two nucleons) and we must distinguish between states with different components of angular momentum ($m=\pm1$ or 0) with respect to this axis. The states with $m=\pm1$ and a given isotopic spin I are degenerate —we shall refer to these as the *parallel* spin states ($\mathcal{P}_I$). There are two distinct states with $m=0$; the one which is symmetric in the spins of the nucleons, corresponding to the remaining member of the triplet, we shall call the *orthogonal* state ($\mathcal{O}_I$); the other, antisymmetric $m=0$ state, is the *singlet* state ($\mathcal{S}_I$). The Hamiltonian (1) therefore has in general six distinct discrete energy levels, although if the two nucleons are superposed, the energies of the $\mathcal{P}_1$ and $\mathcal{O}_1$ states coincide, as do also the $\mathcal{P}_0$, $\mathcal{O}_0$, and $\mathcal{S}_1$ energies. The relation to the central and tensor potentials is discussed in the Appendix.

It is well known that the one-meson exchange term of the interaction energy is obtained correctly from second order perturbation theory if the renormalized coupling constant is used:

$$V_1=-f^2\textstyle\sum_k(\boldsymbol{\tau}_x\cdot\boldsymbol{\tau}_y)\boldsymbol{\sigma}_x\cdot\mathbf{k}\boldsymbol{\sigma}_y\cdot\mathbf{k}v_k^2K^{-2}\times\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]. \tag{3}$$

As an introduction to the method which is found convenient for treating the higher order terms, we shall examine separately the contribution to (3) of the mesons with $m=0$ with respect to the axis of the nucleons, and the contribution of the mesons with $m=\pm1$. This is useful because the mesons with different azimuthal angular momenta give not only characteristic spin dependences, which can be related directly to the spin eigenfunctions, but different radial functions as well. Let

$$4\pi\textstyle\sum_k v_k^2K^{-2}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]=\sum_k f(k)\exp(i\mathbf{k}\cdot\mathbf{r})=F(r), \tag{4}$$

then

$$\textstyle\sum_k(k^0)^2f(k)\exp(i\mathbf{k}\cdot\mathbf{r})=-a_rF(r)=-Z(r), \tag{5a}$$

and

$$\textstyle\sum_k k^+k^-f(k)\exp(i\mathbf{k}\cdot\mathbf{r})=-b_rF(r)=-Y(r), \tag{5b}$$

where a_r is the operator d^2/dr^2, and $b_r=-r^{-1}d/dr$. The signs were chosen so that both Y and Z would be positive at large distances. Then Eq. (3) takes the form:

$$V_1=\gamma\boldsymbol{\tau}_x\cdot\boldsymbol{\tau}_y[\xi a_r+\eta b_r]F(r), \tag{6}$$

[16] These numerical values differ from those of reference 14 and from those of Halpern, Sartori, Nishimura, and Spitzer, Ann. Phys. (N. Y.) **7**, 154 (1959).

TABLE II. The one-meson exchange energy.

	ξ	η	$V_1(T=0)$	$V_1(T=1)$
$\mathcal{P}$	1	0	$-3\gamma Z$	γZ
$\mathcal{O}$	-1	-2	$3\gamma Z+6\gamma Y$	$-\gamma Z-2\gamma Y$
$\mathcal{S}$	-1	$+2$	$3\gamma Z-6\gamma Y$	$-\gamma Z+2\gamma Y$

where

$$\xi=\sigma_x^0\sigma_y^0 \quad \text{and} \quad \eta=+(\sigma_x^+\sigma_y^-+\sigma_x^-\sigma_x^+).$$

It is sometimes convenient, in the main body of this paper, to treat the charges of the mesons and their angular momenta in a similar way; on such occasions we shall write $\boldsymbol{\tau}_x\cdot\boldsymbol{\tau}_y=\alpha\tau_x^0\tau_y^0+\beta(\tau_x^+\tau_y^-+\tau_x^-\tau_y^+)$, where the "operators" α and β are $+1$ and -1, respectively. In Table II we show the eigenvalues of ξ, η, and V_1, in the eigenstates $\mathcal{P}$, $\mathcal{O}$, and $\mathcal{S}$.

If the nucleon is taken as a point source ($v_k\equiv 1$), then $Y(r)=(r^{-3}+r^{-2})e^{-r}$, and $Z(r)=(2r^{-3}+2r^{-2}+r^{-1})e^{-r}$; note that Z is considerably larger than Y, especially at large distances. However, if the nucleon core has a finite extension, $Z(r)$ changes sign at a distance about equal to the core radius, and $Z(0)=-Y(0)$.

The average momentum of the virtual mesons is very large, and hence the interaction energy is very sensitive to the nature of the source function. This is illustrated in Fig. 1. At separations as large as one meson Compton wavelength, corrections to the point-source potential which are as large as 20% are obtained; at smaller distances ($r\sim 0.5$), the corrections are enormous, and while the potentials obtained with the different extended sources differ among themselves by a small percentage of the point-source potential, the absolute magnitude of the differences is large. In integrating the Schrödinger equation, short-wavelength fluctuations in the potential are smoothed over; nevertheless, we must conclude that the potential obtained from the static model has at most a qualitative significance at small distances. Figure 1 shows not only that the energy is sensitive to poorly determined parameters of the model, but, moreover, that the model predicts that virtual mesons with high momenta are very important, and we do not know how accurately the model reflects the actual properties of such mesons.

II. THE HEITLER-LONDON APPROXIMATION

(a) Definition and Normalization of the Basic State

In a previous paper,[17] the author described a way of relating properties of two-nucleon states to properties of isolated nucleons. The method used a twofold expansion: an expansion of the state vector in a Heitler-London representation, and an expansion of the matrix elements between Heitler-London states in terms of the number of mesons common to the clouds of both nucleons. In this section we shall calculate some of the properties of the basic Heitler-London state Φ_{xy}, which is obtained by multiplying together two operators, each of which creates a single physical nucleon. An approximation which corresponds to the usual Heitler-London method in the theory of molecular bonding is obtained by using the basic state Φ_{xy} as an approximation to the exact two-nucleon state Ψ_{xy}. This approximation gives an upper bound to the interaction energy. In the next section, we shall examine the more complicated Heitler-London states $\Phi_{xy,k}$ and $\Phi_{xy,kl}$ (in which one or both of the nucleons is excited) and their role in the exact eigenstate.

We shall examine in particular the $\mathcal{P}_0$ and $\mathcal{O}_0$ states, as these occur in the triplet, even states (e.g., in the deuteron), which are the most well understood empirically; the comparison of our results with other potentials is consequently most interesting for these states.

It was pointed out in I that there is a certain arbitrariness in constructing a Heitler-London representation, arising from the fact that the operators $\mathfrak{F}_x^*$ [see Eqs. (I,3) and (I,4)] may be taken to be functions of an arbitrary set of canonical variables. For instance, $\mathfrak{F}_x^*$ may depend on the φ_k, the a_k^*, or on the π_k—the first two cases were described in detail in I. In the present problem the expansions for the matrix elements converge much more rapidly when the a^* operators are used than when the φ operators are used. When π

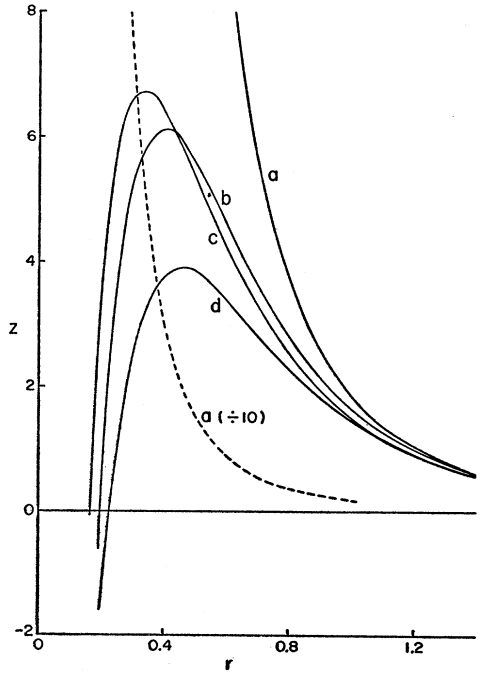

FIG. 1. The function $Z(r)$ obtained from different cutoff functions: (a) $v_k=1$ (point source); (b) $v_k=(\Lambda^2-1)(\Lambda^2+k^2)^{-1}$, with $\Lambda=7$; (c) $v_k=(\Lambda^2-1)^{\frac{1}{2}}(\Omega^2-1)^{\frac{1}{2}}(\Lambda^2+k^2)^{-\frac{1}{2}}(\Omega^2+k^2)^{-\frac{1}{2}}$, with $\Lambda=14$ and $\Omega=4.68$ [this gives the same M as case (b)]; (d) v_k has the same form as in (b), but $\Lambda=6$.

[17] R. E. Cutkosky, Phys. Rev. **112**, 1027 (1958). Hereafter this paper is referred to as I. A very similar approach was suggested by Novozhilov, references 18 and 20.

operators are used, the normalization of the basic state is almost exactly unity and the expansion which gives this quantity converges extremely rapidly. However, when an appropriate rearrangement of terms is made, the expectation values of operators in normalized states converge about equally rapidly in all cases. The best approximation to the exact state (in the sense of giving the lowest interaction energy) is obtained when the basic state is defined in terms of φ operators, and the poorest approximation is given by the π operators, but the basic states obtained in all methods give automatically both the second and fourth order perturbation theory results (with the renormalized coupling constant).

We shall use operators defined as functions of the a_k^* in the discussion which follows, partly because the convergence rates are suitable, and partly because the graphical description of the matrix elements corresponds closely to the Feynman graphs obtained in the Tamm-Dancoff method, as we shall discuss in some detail below. This makes the relation to previous calculations and some effects of nonadiabatic corrections most easily understood when this method is used.[18]

We may write the expansion (I, 11′) for the norm of Φ_{xy} in the form

$$(\Phi_{xy},\Phi_{xy})=1+A_{xy}=\sum_{n=0}^{\infty} A_n,$$

where A_n represents the contribution of n exchanged mesons. There is a close relation between the norm and the average number of mesons in the state Φ_{xy}, as given by Eq. (I, 28). Let $\mathfrak{N}_{xy}=(\Phi_{xy},\mathfrak{N}\Phi_{xy})$. From (I, 28) it follows that

$$\mathfrak{N}_{xy}=2(1+A_{xy})N_0+\delta\mathfrak{N}_{xy}+\delta\mathfrak{N}_{xy}', \tag{7}$$

where N_0 is the average number of mesons in the physical nucleon,

$$\delta\mathfrak{N}_{xy}=\sum_{n=1}^{\infty} nA_n,$$

and $\delta\mathfrak{N}_{xy}'$ is a relatively small correction given by

$$\begin{aligned}\delta\mathfrak{N}_{xy}'=\sum_{n=1}^{\infty}\sum_{\nu=0}^{n}\sum_{k_1\cdots k_n}&\frac{1}{(\nu!)}\frac{1}{(n-\nu)!}\\ &\times[\langle x|a_1^*\cdots a_\nu^*(\mathfrak{N}-N_0)a_{\nu+1}\cdots a_n|x\rangle\\ &\times\langle y|a_n^*\cdots a_{\nu+1}^*a_\nu\cdots a_1|y\rangle\\ &\quad+\text{symmetrical term in }(x,y)].\end{aligned} \tag{8}$$

The average number of mesons in the state Φ_{xy} is then

$$2N_0+\delta N,\quad\text{where}\quad \delta N=(\delta\mathfrak{N}+\delta\mathfrak{N}')/(1+A).$$

[18] This definition of the basic state is the same as that discussed by Iu. V. Novozhilov, J. Exptl. Theoret. Phys. U.S.S.R. **32**, 1262 (1957) [translation: Soviet Phys. JETP **5**, 1030 (1958)]; J. Exptl. Theoret. Phys. U.S.S.R. **33**, 901 (1957) [translation: Soviet Phys. JETP **6**, 692 (1958)].

There is no need to calculate δN very accurately, so in Eq. (8) we shall neglect the off-diagonal matrix elements of $\mathfrak{N}-N_0$ (between physical states), and equate the diagonal elements to the number of incident mesons.

We might expect that when the two nucleons coincide, a large contribution to the expansions would come from the $2N_0$th order terms (i.e., $n\sim5$ or 6). Our present method of evaluating the matrix elements in terms of expansions is therefore likely to be useful only at large distances. We wish to find out how the rate of convergence depends on the separation; the expansion for the norm is typical so we shall examine the first few terms of this expansion in some detail.

The first term is

$$A_1=\sum_k f^2\boldsymbol{\tau}_x\cdot\boldsymbol{\tau}_y\boldsymbol{\sigma}_x\cdot\mathbf{k}\boldsymbol{\sigma}_y\cdot\mathbf{k}v_k^2K^{-3}\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{9}$$

Let $K^{-1}f(k)=g(k)$, and let $\sum_k g(k)\exp(i\mathbf{k}\cdot\mathbf{r})=G(r)$. We obtain for A_1 an expression which is similar to Eq. (6), with $F(r)$ replaced by $-G(r)$. In the lowest order approximation, $\delta N=A_1$. We may note an illustration of a general qualitative result, that an increase in the average number of mesons in the cloud is usually associated with an attraction between the two nucleons, and vice versa.

The second term in the expansion for the norm is

$$\begin{aligned}A_2=\sum_{kl}\{&\tfrac{1}{2}\langle x|[V_k(H+K)^{-1}V_l\\ &+V_l(H+L)^{-1}V_k]|x\rangle(K+L)^{-2}\\ &\times\langle y|[V_k^*(H+L)^{-1}V_l^*+V_l^*(H+K)^{-1}V_k^*]|y\rangle\\ &+[\text{symmetrical term in }(x,y)]\\ &+\langle x|V_k(H+K)^{-1}(H+L)^{-1}V_l^*|x\rangle\\ &\qquad\times\langle y|V_l(H+L)^{-1}(H+K)^{-1}V_k^*|y\rangle\};\end{aligned} \tag{10}$$

we have used some well-known identities to eliminate the creation and annihilation operators. We evaluate (10) by the standard closure expansion method—a sum over a complete set of eigenstates is inserted between the V operators in each one-nucleon matrix element. It is convenient to distinguish three parts of A_2: the "zero-meson" or perturbation-theory part, $A_2^{(0)}$, is obtained when we keep only the zero-meson intermediate states in each sum; the resonant correction, $A_2^{(R)}$, arises when the contribution of the $(\frac{3}{2},\frac{3}{2})$ resonant scattering states to one sum is included, and either the zero-meson states or resonant states are included in the other sum; the nonresonant correction, $A_2^{(NR)}$, which consists of all remaining terms, depends on so many unknown quantities that we cannot evaluate it accurately, but we shall be able to show that it is relatively small.

The examination of the spin dependence of the various parts of A_2 is easiest if one treats separately the "uncrossed" terms, in which the V_k and V operators occur in the same order in each matrix element, and the "crossed" terms, in which they occur in the opposite order. The zero-meson uncrossed part of A_2 may easily

be shown to have the form

$$A_U(r)=\lim_{r=s}A_U(r,s),$$

where

$$A_U(r,s)=(\gamma^2/8)U_IU_AG_U(r,s),$$

and where

$$G_U(r,s)=(4\pi)^2\sum_{kl}4v_k^2v_l^2K^{-2}L^{-2}(K+L)^{-2}\times\exp(i\mathbf{k}\cdot\mathbf{r}+i\mathbf{l}\cdot\mathbf{s}). \quad (11)$$

The operator U_A depends on the angular momentum state of the nucleons:

$$U_A(\mathcal{P})=a_ra_s,$$
$$U_A(\mathcal{O})=a_ra_s+2(a_rb_s+b_ra_s)+4b_rb_s.$$

There is no need to write down $U_A(\mathcal{S})$, because it can be obtained from $U_A(\mathcal{O})$ by changing the signs of b_r and b_s. This rule holds throughout the calculation, so we only need to give results for $\mathcal{P}$ and $\mathcal{O}$ states. The operator U_I, which depends on the isotopic spin, is easily obtained by replacing the a's and b's by α and β, respectively.

The crossed zero-meson part of A_2 is obtained in a similar way from

$$A_C(r,s)=(3\gamma^2/8)C_IC_AG_C(r,s),$$

where

$$G_C=(4\pi)^2\sum_{kl}v_k^2v_l^2K^{-3}L^{-3}(4/3)[1-KL(K+L)^{-2}]\times\exp(i\mathbf{k}\cdot\mathbf{r}+i\mathbf{l}\cdot\mathbf{s}), \quad (12)$$

and

$$C_A(\mathcal{P})=a_ra_s+4b_rb_s,$$
$$C_A(\mathcal{O})=a_ra_s-2(a_rb_s+b_ra_s);$$

C_I is again easily obtained from C_A.

To simplify the numerical calculations we approximate the integrands in Eqs. (11) and (12) by a product: we replace $(K+L)^{-2}$ by $1/4KL$. Then $G_U(r,s)\sim G_C(r,s)\sim G(r)G(s)$. This introduces an error of perhaps 10% or 20%, which is small compared with other, more unavoidable errors, and to the effect of changing the parameters of the model slightly. In all the remaining calculations, we also approximate all six or nine dimensional Fourier integrals by appropriate sums of similar products.

The contribution of the resonant intermediate states to the matrix element $\langle 0|V_kf(H)V_l^*|0\rangle$ is

$$4\pi X(4KL)^{-\frac{1}{2}}v_kv_l(\mathbf{k}\cdot\mathbf{l}-\tfrac{1}{3}\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{l})(\delta_{kl}-\tfrac{1}{3}\boldsymbol{\tau}_k\cdot\boldsymbol{\tau}_l),$$

where

$$X=(3/\pi)\int\sin^2\delta_3(p)\,v_p^{-2}p^{-3}f(P)dP.$$

Since the width of the resonance is small compared to the average energy of the virtual mesons which are exchanged between the two nucleons, we may use the isobar approximation, which consists of replacing P by an appropriate average. Then

$$X=\alpha_3\gamma f(\omega_0);$$

we choose ω_0 so that

$$\alpha_3\gamma/\omega_0=(3/\pi)\int\sin^2\delta_3(p)\,v_p^{-2}p^{-3}P^{-1}dP,$$

which gives $\omega_0=2.1$. This approximation is equivalent to replacing the scattering states by a discrete state of energy ω_0.

In evaluating the Fourier transforms which are associated with $A_2^{(R)}$, we encounter the following integrals (with $n=0$, 1, or 2):

$$G_n(r)=\sum_k g(k)\exp(i\mathbf{k}\cdot\mathbf{r})K^n/(K+\omega_0)^n.$$

[In calculating the corresponding contribution to V_2 we also require $F_n(r)=\sum_k f(k)\exp(i\mathbf{k}\cdot\mathbf{r})K^n/(K+\omega_0)^n$.] These integrals, and the related functions $\mathcal{A}_n(r)=aG_n(r)$, $\mathcal{B}_n(r)=bG_n(r)$, etc., have been evaluated numerically, using the cutoff function $v_k=48/(49+k^2)$.[19]

After making the approximations described above, we obtain for the contribution to $A_2^{(R)}$ of the crossed

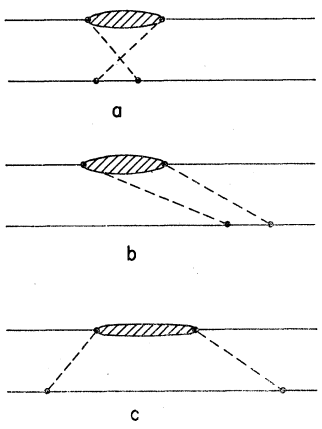

FIG. 2. Graphs (a) and (b) correspond to contributions to $A_2^{(R)}$ and $V_2^{(R)}$, but graph (c) is associated with the one-meson Heitler-London state. The cross-hatched areas represent the $(\frac{3}{2},\frac{3}{2})$ scattering state.

diagram, Fig. 2(a), the following expression:

$$A_2^{(R1)}=\tfrac{1}{4}\alpha_3\gamma^2R_{1I}R_{1A}[G_0(r)G_1(s)+2G_1(r)G_1(s)],$$

where

$$R_{1A}(\mathcal{P})=\tfrac{1}{3}[2a_ra_s+2b_rb_s],$$
$$R_{1A}(\mathcal{O})=\tfrac{1}{3}[2a_ra_s+2(a_rb_s+b_ra_s)+6b_rb_s].$$

The contribution of the uncrossed diagram, Fig. 2(b), which vanishes in $T=0$ states, is

$$A_2^{(R2)}=\tfrac{1}{4}\alpha_3\gamma^2R_{2I}R_{2A}G_0(r)G_1(s),$$

where

$$R_{2A}(\mathcal{P})=\tfrac{1}{3}[2a_ra_s+6b_rb_s],$$
$$R_{2A}(\mathcal{O})=\tfrac{1}{3}[2a_ra_s-2(a_rb_s+b_ra_s)+2b_rb_s].$$

Our numerical results also include the much smaller contribution to $A_2^{(R)}$ of the diagrams in which there is

[19] An IBM-650 electronic computor was used for these calculations. The author is grateful to Dr. S. H. Vosko for writing the program and explaining the operation of the machine.

a resonant meson scattering at each nucleon, but we shall not write the formulas.

An estimate of $A_2^{(NR)}$ may be obtained by using the renormalization constants which were discussed in Sec. I. In Eq. (10) we replace the Hamiltonian H, where it appears explicitly, by zero—that is, in the closure expansions we assume the energies of the virtual states are small in comparison with the average energy $\omega(r)$ of the mesons which are exchanged when the nucleons are separated by the distance r. This is very crude, especially at large distances, but we can partially compensate for the error at the end. With this approximation, and using the explicit form of V_k, we may evaluate A_2 in terms of Z_2 and ρ_1. We then subtract $A_2^{(0)}$ and the contribution of the resonant states ($A_2^{(R)}$ with ω_0 set equal to zero). We are left with an approximation to $A_2^{(NR)}$. In Fig. 3 it is shown that when ω_0 is neglected, $A_2^{(R)}$ is overestimated by a factor of about 2, for the distances of greatest interest. This is consistent with $\omega(r)$ being slightly smaller than ω. Considering the higher energy of the virtual states which give the nonresonant correction, it seems reasonable that the correct $A_2^{(NR)}$ might be about one-sixth that shown in Fig. 3; if this is correct, neglect of $A_2^{(NR)}$ is not serious.

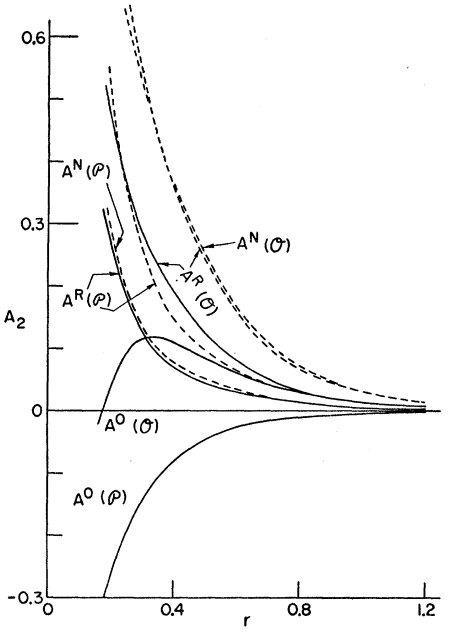

FIG. 3. The contribution of various terms to A_2 for the $\mathcal{P}_0$ and $\mathcal{O}_0$ states. The solid lines show the exact contributions, the dashed lines the approximations obtained by neglecting the energies of the one-nucleon states, as described in the text.

The calculation of the third order term A_3 requires the evaluation of matrix elements such as $\langle x|V_k f_1(H) V_l f_2(H) V_m|x\rangle$, where $f_1(H)$ and $f_2(H)$ stand for certain energy denominators. We use a double-closure expansion, although the corrections to the perturbation theory approximation are more important than in the second order term. The contribution of the resonant-scattering states cannot be calculated without additional assumptions, because it depends on the matrix element $\langle xp|V_l|xq\rangle$, which is related to meson production. We approximate this matrix element by $\delta_{pq}\langle x|V_l|x\rangle$. A combination of effects—the energy denominators, and the spin and isotopic spin matrix elements—makes the resulting contribution to $A_3^{(R)}$ of the terms which depend on this matrix element relatively small. Some typical diagrams which contribute to A_3 are shown in Fig. 4.

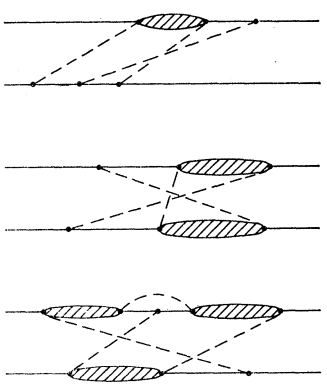

FIG. 4. Some typical graphs included in the calculation of A_3.

In Fig. 5 we show the contributions of the one-, two-, and three-meson exchange terms to $1+A$. The modifications of the norm are smaller for the $\mathcal{P}_0$ state than for the $\mathcal{O}_0$ state, and also appear to converge more rapidly. The $m=\pm 1$ mesons are less important for the $\mathcal{P}_0$ state, and the resonant-scattering corrections tend to cancel the zero order term, both in A_2 (as may be seen from Fig. 3) and in A_3. It appears that this feature persists in the higher order terms. In general, we may conclude that the norm $1+A$ could be obtained to a reasonable degree of accuracy (say, 15%) from the first few terms in the expansion at distances $r \gtrsim 0.2$ for the $\mathcal{P}_0$ state, and $r \gtrsim 0.4$ for the $\mathcal{O}_0$ state, although the accuracy to which we have actually obtained the terms is probably somewhat less than this in the present calculation.

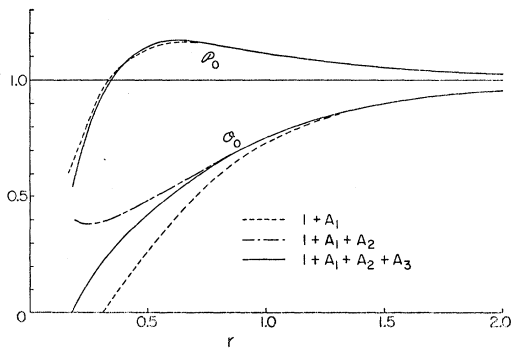

FIG. 5. The one-, two-, and three-meson contributions to the norm of the basic state. The two-meson curve is not shown for the $\mathcal{P}_0$ state in order to avoid overcrowding the graph.

(b) Interaction Energy and the Relation to the Tamm-Dancoff Method

The expansion for the "potential" $V_{xy}=(\Phi_{xy},H\Phi_{xy})$ is very closely related to the interaction in the Tamm-Dancoff method, in which case the "potential" $\mathcal{U}_{xy}(E)$ is an explicit function of the energy E of the state under consideration. Let us select from the terms in the Tamm-Dancoff expansion of $\mathcal{U}(E)$ the subset of "maximal" terms—those n-meson exchange terms which are represented by diagrams in which all n mesons are at some time simultaneously present in the field. In these maximal n-meson terms, we set the energy E equal to zero in all the energy denominators except those for the states which contain all n exchanged mesons. The resulting contribution to $\mathcal{U}(E)$ we call $\mathcal{U}_{\max}(E)$. It can be shown that $V=\mathcal{U}_{\max}(0)$ (i.e., E is set equal to zero in all the energy denominators) and $A=-[(d/dE)\mathcal{U}_{\max}(E)]_{E=0}$. This may be shown by considering the interaction representation, and the corresponding Feynman graph interpretation of Φ_{xy} and its matrix elements; the expression of matrix elements in terms of renormalized quantities was obtained by considering the Heisenberg representation. A relation between the renormalized expansions for A and V may be obtained as follows: In the one-nucleon matrix elements in (I, 11′), closure expansions over states n_x and n_y are inserted between the creation and destruction operators; if the summand is then multiplied by $E(n_x)+E(n_y)+\sum_i K_i$, the expansion for V is obtained.

It is not hard to verify the remarks above by explicit calculation for the two- and three-meson exchange terms; for instance, diagrams (a) and (b) in Fig. 2 (which are to be interpreted as referring to renormalized quantities) give the same contribution to V as to $\mathcal{U}(0)$, while diagram (c), which is not a maximal diagram, is excluded from V. The contribution of diagram (c) to the interaction energy E is obtained when we include the one-meson Heitler-London state $\Phi_{xy,k}$ in the expansion of the eigenstate Ψ_{xy}. The contribution of the maximal three-meson diagram [Fig. 6(a)] is similarly included in V (the vertical dotted line denotes the intermediate state in which all three mesons are present) while diagram 6(b), which is excluded from V, is likewise included in the contribution of the one-meson state $\Phi_{xy,k}$.

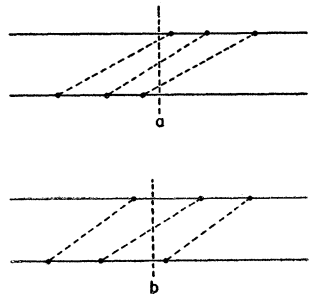

FIG. 6. Graph (a) is maximal, (b) is nonmaximal.

Since the terms included in the expansion of V_{xy} are just a subset of the terms which contribute to $\mathcal{U}_{xy}$, and in fact, are the terms in which the energy denominators are especially large, we may expect the expansion for V_{xy} (and that for A_{xy}) to converge considerably more rapidly than the expansion of the Tamm-Dancoff method. The terms which might cause the expansion for $\mathcal{U}(E)$ to converge slowly, or not at all, are separated off and treated by a different method. Furthermore, the complications which arise from the explicit energy dependence of $\mathcal{U}(E)$ are absent in the Heitler-London method, in particular, all nonadiabatic effects are associated with the contribution of the "excited" Heitler-London states, which, it may be anticipated, will make such effects easier to understand.

The relation of the Heitler-London to the Tamm-Dancoff method gives the interesting result, that the

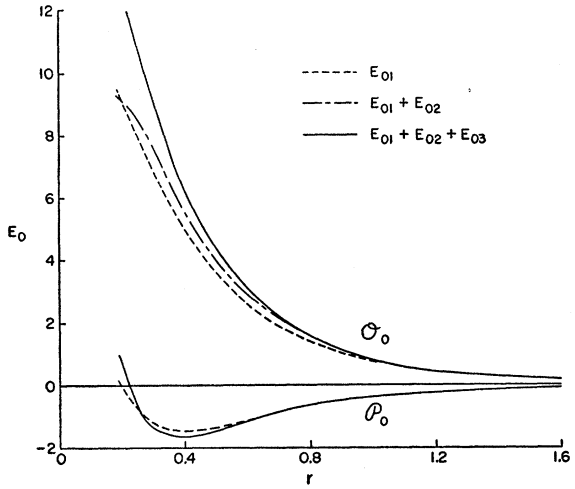

FIG. 7. The energy E_0, calculated from the basic Heitler-London states for the $\mathcal{P}_0$ and $\mathcal{O}_0$ states (which comprise the triplet even states). The two-meson curve is not shown for the $\mathcal{P}_0$ state.

approximation

$$\mathcal{U}(E)\approx\mathcal{U}_{\max}(0)+E[(d/dE)\mathcal{U}_{\max}(E)]_0$$

provides a variational approximation to the interaction energy E. We also note the importance of examining the E dependence of the Tamm-Dancoff energy denominators.

It is clear from the remarks above that the calculation of V_{xy} is completely analogous to the calculation of A_{xy}, so we do not need to discuss in detail either the method of calculation or the accuracy of the results. The expectation value of the energy in the state Φ_{xy} is $E_0(r)=V_{xy}/(1+A_{xy})$. When only a few terms of V or A are calculated, the various ways of representing the ratio might give somewhat different results, and we wish to choose the way which gives the most rapid convergence. Note that $V_n\approx-n\omega(r)A_n$, and in the present linear model, when n is large, $A_n\sim CB^n/n!$, where C and B are functions of the spin and position

variables. If the A_n and V_n have the forms cited, it appears that the most simple way of attaining rapid convergence is to expand $(1+A)^{-1}$ and group in the expanded ratio all terms which contain the same number of exchanged mesons, as follows:

$$\begin{aligned} &E_0=E_{01}+E_{02}+E_{03}+\cdots; \\ &E_{01}=V_1, \\ &E_{02}=V_2-A_1V_1, \\ &E_{03}=V_3-A_1(V_2-A_1V_1)-A_2V_1, \\ &\cdots . \end{aligned} \tag{13}$$

Successive approximations obtained from Eq. (13) are plotted in Figs. 7, 8, and 9. It can be seen that rather rapid convergence is obtained in $T=0$ states, especially in the $\mathcal{P}_0$ state. However, E_{02} and E_{03} are obtained as the differences of large numbers, and therefore are not

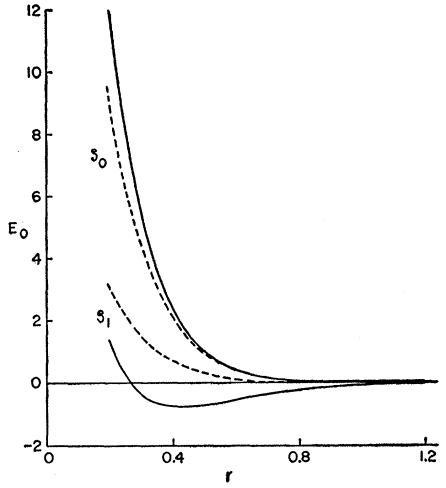

FIG. 8. The solid lines denote the one- and two-meson contribution to the energy E_0 of the singlet states. The dashed lines indicate the one-meson contribution.

given very reliably by the present calculation. Nevertheless, we note that the one-meson potential is much more important than the corrections which are obtained in the Heitler-London approximation. The uncertainties in the potential which arise from the uncertainty in the model are apparently more important than the higher order corrections to E_0, especially in the $\mathcal{P}_0$ state. In $T=1$ states, however, the convergence properties seem to be quite different; we note that the two-meson contributions are strongly attractive, particularly in the $\mathcal{P}_1$ state. It is likely that the three-meson terms are also important in $T=1$ states, but, as we shall discuss, other effects, which are harder to calculate, appear to be so large in $T=1$ states that it does not seem useful to carry further the calculation given above.

The close relation between the change in the average number of mesons in the field, the energy E_0, and the corrections to the norm, which was discussed above, is also seen in a comparison of Fig. 10 with Figs. 5 and 7.

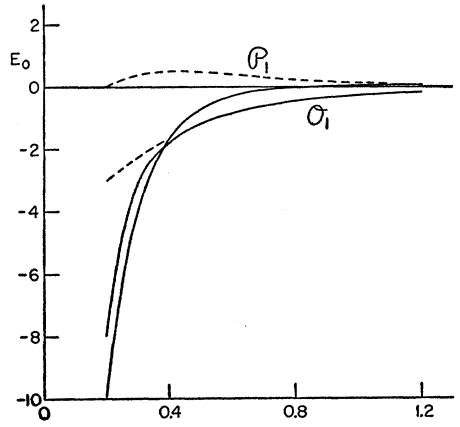

FIG. 9. Same as Fig. 8, for the $\mathcal{P}_1$ and $\mathcal{O}_1$ states (which comprise the triplet odd states).

Note that at the distances at which the expansions converge well, δN is much smaller than $2N_0$.

III. ROLE OF THE EXCITED STATES

In this section we shall examine the contribution of the excited configurations of the meson field to the state vector Ψ_{xy} and the energy E. In the Heitler-London representation these excited configurations are expressed in terms of operators which create nucleons in meson-scattering states.[20]

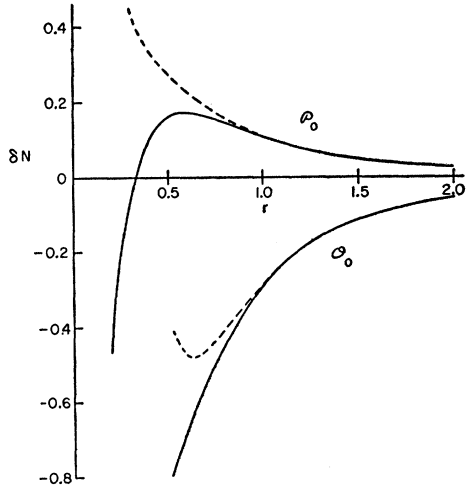

FIG. 10. The amount δN by which the average number of mesons in the $\mathcal{P}_0$ and $\mathcal{O}_0$ states differs from that of two isolated nucleons. The solid curves denote the value of δN calculated from the basic Heitler-London state; the dashed curves denote the result of including the two-meson excited state. Note that use of the basic Heitler-London state does not imply that $\delta N=0$. This is a consequence of the statistical correlations between the mesons in the individual clouds, which are properly included. Note also that δN is small compared with $2N_0$ for $r>0.3$, and that the sign of δN is related to the sign of E_0 as shown in Fig. 7.

[20] The definition of the excited Heitler-London states given in I and used in this section differs from that of Novozhilov, reference 17, and also Iu. V. Novozhilov, J. Exptl. Theoret. Phys. U.S.S.R. 35, 742 (1958) [translation: Soviet Phys. JETP 35(8), 515 (1959)]. Vestnik Leningrad. Univ., No. 16, Ser. Fiz. i Khim., 21 (1958).

We approximate the exact state vector by

$$\Psi_{xy}\approx\chi_0\Phi_{xy}+\sum_k\chi(k)\Phi_{xy,k}. \quad (14)$$

An equation for $\chi(k)$ is found by minimizing the expectation value of the energy. It is convenient to use as base vectors for the one-meson states, states which are orthogonal to the basic state, and which are approximately normalized. Therefore we write

$$\chi(k)=\Gamma\phi(k),$$
$$\chi_0=\Gamma[\phi_0-\sum_k\Gamma A(;k)\Gamma\phi(k)], \quad (15)$$

where $\Gamma=(1+A)^{-\frac{1}{2}}$, and $A(;k)=(\Phi_{xy},\Phi_{xy,k})$, as defined in I. We use a matrix notation in which Γ, A, and $A(;k)$ are considered as 16×16 matrices. The coupled equations for the amplitudes are

$$E\phi_0=E_0\phi_0+\sum_k W(k)\phi(k),$$
$$E\phi(k)+E\sum_l B(k,l)\phi(l)=W^\dagger(k)\phi_0+(E_0+K)\phi(k)+\sum_l W(k,l)\phi(l). \quad (16)$$

In Eq. (16), E_0 is the expectation of the energy in the basic state, as calculated in the previous section. Furthermore, in terms of the quantities defined in I and above,

$$W(k)=\Gamma[V(;k)-E_0A(;k)]\Gamma,$$
$$B(k,l)=\Gamma[A(k;l)-A(k;)\Gamma^2A(;l)]\Gamma,$$
$$W(k,l)=\Gamma[V(k;l)+\tfrac{1}{2}(K+L)A(k;l)+A(k;)\Gamma^2E_0A(;l)-V(k;)\Gamma^2A(;l)-A(k;)\Gamma^2V(;l)]\Gamma. \quad (17)$$

From the general expansions given in I it can be shown that the one-meson exchange part of $W^\dagger(k)$ is

$$W_1^\dagger(k)=\sum_p(\langle xk|V_p^*|x\rangle\langle y|V_p|y\rangle+\langle x|V_p^*|x\rangle\langle yk|V_p|y\rangle)[P^{-1}+(K+P)^{-1}]. \quad (18)$$

This may be expressed in terms of the scattering amplitude for a meson with momentum k, and is therefore quite independent of many of the details of the model, although it is sensitive to the cutoff function. Note that we may always write $W^\dagger(k)$ as the sum of two terms, $W^\dagger=W_x^\dagger+W_y^\dagger$, where $W_x^\dagger$ contains the meson scattering bra $\langle xk|$, and can be considered to be the part of $W^\dagger$ which excites the nucleon at x into a one-meson scattering state. We may similarly write $W(k,l)$ and $B(k,l)$ as the sum of four terms. This suggests writing $\phi(k)=\phi_x(k)+\phi_y(k)$, where

$$(E-E_0-K)\phi_x(k)+E(B_{xx}+B_{xy})(\phi_x+\phi_y)=W_x^\dagger\phi_0+(W_{xx}+W_{xy})(\phi_x+\phi_y). \quad (19)$$

In Eq. (19) the meson momentum has been absorbed into the matrix notation. We also have

$$(E-E_0)\phi_0=W_x\phi_x+W_y\phi_y+W_x\phi_y+W_y\phi_x. \quad (20)$$

The solution of Eq. (19) is very complicated in general. At large distances we expect that

$$\phi_x(k)\sim-K^{-1}W_{1x}^\dagger(k)\phi_0, \quad (21)$$

but the multiple scattering effect gives corrections of relative order r^{-1}. Note that if we use Eq. (21), the last two terms of Eq. (20) give contributions to the energy E which have the form of a three-meson exchange, and which we must therefore drop, since we are making no attempt to calculate the three-meson terms. However, the integration over the intermediate meson momentum involves the scattering cross section; if the cross section had a narrow resonance, this would give an exponential falloff at large distances which was almost as slow as the e^{-2r} decrease obtained from the first two terms, and it is desirable to prove that such terms are actually spurious and are cancelled by terms coming from corrections to Eq. (21).

As a consequence of the multiple scattering effect, the states $\Phi_{xy,k}$ are very imperfectly orthogonal, and ϕ_x and ϕ_y are only in a very loose sense interpretable as amplitudes for excitation of a given nucleon. For instance, a typical term in $A(k;l)$ is

$$\sum_p(P-K\pm i\eta)^{-1}(P-L\mp i\eta)^{-1}\langle xk|V_p|x\rangle\langle y|V_p^*|yl\rangle,$$

which at large distances involves derivatives with respect to r of the singular factor

$$(L-K\pm i\eta)^{-1}r^{-1}(e^{\pm ilr}-e^{\mp ikr}).$$

Similar singularities occur in the higher order terms of $W(k)$. If we use as an approximate solution of Eq. (19) the following equation:

$$\phi_x\sim-K^{-1}(W_{1x}^\dagger+W_{2x}^\dagger)\phi_0+K^{-1}W_{1xy}K^{-1}W_{1y}^\dagger\phi_0, \quad (22)$$

we find, by using the Low scattering equation, that the contribution of the singular three-meson terms to the energy are cancelled and the remaining three-meson terms decrease with the exponential factor e^{-3r}. By adding additional terms to Eq. (22), this procedure of cancelling the long-range multiple scattering effects can be carried further.

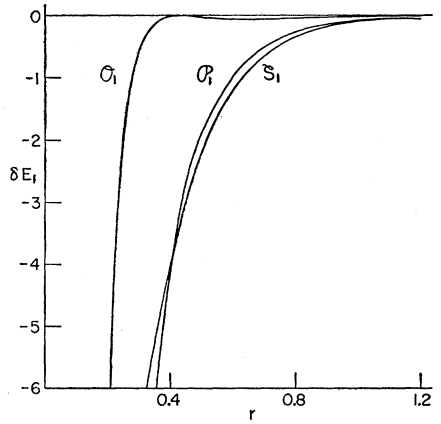

FIG. 11. The additional attraction obtained from the excited Heitler-London state in which one meson is being scattered by a nucleon. Only the $(\frac{3}{2},\frac{3}{2})$ resonant scattering was included in the calculation.

The discussion above shows that the appropriate perturbation solution of Eq. (19) gives the following estimate of the contribution of one-meson states to the energy:

$$\delta E_1 = (E - E_0) = -\sum_k K^{-1}[W_{1x}(k)W_{1x}^\dagger(k) + W_{1y}(k)W_{1y}^\dagger(k)]. \quad (23)$$

This is precisely the counterpart of diagram 2(c). In evaluating Eq. (23) we use the isobar-approximation described in Sec. II. The results are shown in Fig. 11. An estimate of the probability that one of the nucleons is excited into the resonant or "isobar" state is given by $P_1 = -\delta E_1/\omega_0$. Since we have used a perturbation method to calculate $\phi_x(k)$, Eq. (23) is not accurate unless $P_1 \ll 1$. From Fig. 12 it is seen that this restricts the validity of Eq. (23) to rather large distances, but we may certainly infer that the one-meson configuration must play an extremely important role in $T=1$ states at small distances.

In evaluating the contribution of two-meson states to the interaction energy, we neglect meson production in meson-nucleon collisions, so one meson must be scattered by each nucleon. The method applied above then gives

$$\delta E_2 = -\sum_{kl} W_1(k,l)(K+L)^{-1}W_1^\dagger(k,l), \quad (24)$$

where

$$W_1^\dagger(k,l) = \sum_p \langle xk|V_p^*|x\rangle\langle yl|V_p|y\rangle \times[(P+K)^{-1}+(P+L)^{-1}].$$

The probability of finding both nucleons excited into the resonant state is approximately $P_2 = -\delta E_2/2\omega_0$, which, for the $\mathcal{P}_0$ and $\mathcal{O}_0$ states, is also shown in Fig. 12. In $T=0$ states, if we neglect the scattering in all but the resonant $(\frac{3}{2},\frac{3}{2})$ state, we find $P_1=0$; by using the values of α_1 and α_2 given in Sec. I we may estimate that in the $\mathcal{P}_0$ state P_1 is about equal to the value of P_2 shown in Fig. 12, while in the $\mathcal{O}_0$ state P_1 is probably somewhat larger than P_2. These values are of course very rough. In the $T=1$ states, the other scattering amplitudes give a negligible contribution to P_1 except in the $\mathcal{O}_1$ state. The contribution of the two meson states to the average number of mesons in the cloud is indicated by a dashed line in Fig. 10.

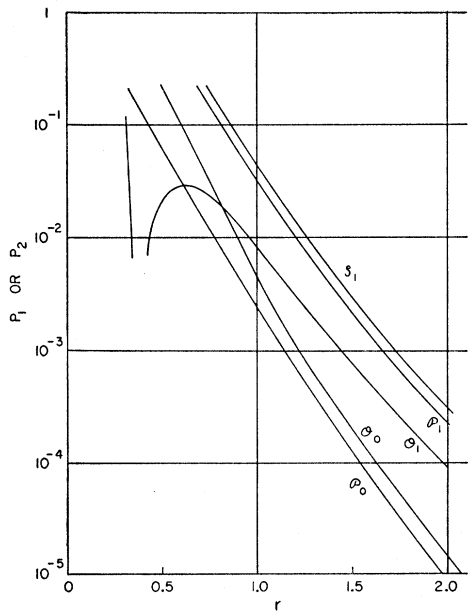

FIG. 12. The curves labeled $\mathcal{P}_1$, $\mathcal{O}_1$, and $\mathcal{S}_1$ give the probability P_1 of finding the one-meson excited state present in these eigenstates. The curves labeled $\mathcal{P}_0$ and $\mathcal{O}_0$ give the probability P_2 that the two-meson configuration is present. The $\mathcal{S}_0$ curve (not shown) is similar to those for $\mathcal{P}_0$ and $\mathcal{O}_0$.

At small distances, the perturbation theory method outlined above is not a convenient way to study the excited configurations. A more suitable approach is to use a variational method, with a simple trial function $\phi(k)$. The differences in the energies E_0 obtained from the basic state become larger than ω_0 at small distances, so in states in which a strong repulsion is obtained from the basic state, there is a tendency for $\Phi_{xy,k}$ or $\Phi_{xy,kl}$ to become a major component of the eigenstate; the variational method appears to be the only suitable way to treat such effects.

An interesting insight into the role of the excited states is obtained from a comparison with the results of Zharkov[21] and Matsumoto *et al.*[22] who used perturbation theory to treat a model in which the nucleon core has an excited state (the isobar). We have shown that for $T=0$ states it is indeed a good approximation to replace E_0 by the second order perturbation theory result; furthermore, the contribution of the excited configurations given by Eq. (23) and Eq. (24) can be seen to agree exactly with the contribution of the isobar to the corresponding terms of Zharkov's results, if one makes an appropriate identification of the coupling constants, and remembers that in the calculations reported here an extended source was used, while in those of Zharkov a point source was used. Zharkov's coupling constant for the isobar, g_1, is related to the parameters used here by the relation $g_1^2/g^2 = \alpha_3/6$. However, while Zharkov used approximately the same value for g^2 as we use for $f^2/4\pi$, his value for g_1^2 is twice as large as that which is obtained from the more fundamental approach of this paper, and his numerical results therefore seriously overestimate the effect of the isobar.

IV. DISCUSSION

The method suggested in I, the expansion of the state vector in Heitler-London states, provides a straightforward, unambiguous way of deriving automatically renormalized expressions for the interaction

[21] G. F. Zharkov, J. Exptl. Theoret. Phys. U.S.S.R. **34**, 1211 (1958) [translation: Soviet Phys. JETP **34**(7), 837 (1958)].

[22] Matsumoto, Hamada, and Sugawara, Progr. Theoret. Phys. (Kyoto) **10**, 199 (1953); T. Matsumoto and M. Sugawara, Progr. Theoret. Phys. (Kyoto) **12**, 553 (1954).

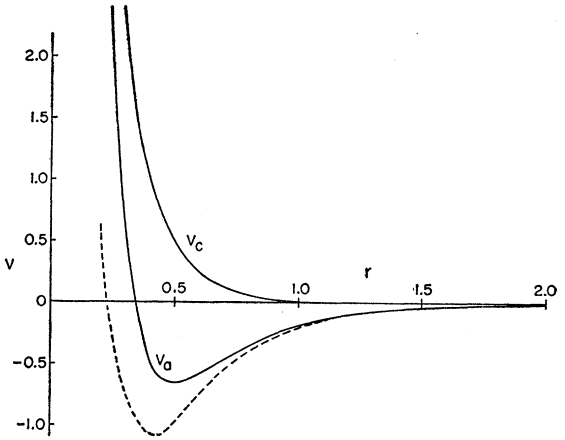

FIG. 13. The "radially adiabatic" potential V_a calculated by the method discussed in the Appendix. The solid curves show V_a as calculated from $E_0(\mathcal{P}_0)$ and $E_0(\mathcal{D}_0)$ and for comparison the corresponding central potential V_c. The dashed curve shows the result of including δE_2 in the calculation of V_a.

energy. The full utility of the method, however, lies in the fact that renormalized expressions for all matrix elements may be derived by similar techniques. In this paper we have only examined, as a by-product, the average number of mesons in the field (Fig. 10), although this quantity is of interest merely as an illustration of certain qualitative features of the structure of two-nucleon states. A more extensive examination of two-nucleon states, using a more realistic model, will be required before we may acquire confidence in our understanding of nuclear interactions. Our principal conclusion is that we may anticipate that the Heitler-London method will be useful for this purpose.

We may, nevertheless, also draw from the numerical results several interesting conclusions, which are different for $T=0$ and $T=1$ states. In $T=0$ states the corrections to the one-meson potential are mostly small and masked by the uncertainties in the form of the one-meson potential at small distances. While an additional attraction at small distances is obtained from the excited Heitler-London states (see Figs. 12 and 13), the use of the static approximation is quite suspect for these terms, and their contribution is not larger than other effects one might reasonably anticipate at the distances ($r<0.6$) at which they become important. We may therefore conclude that for $T=0$ states the only important qualitative features of the potential which may reasonably be inferred from the static model are just the features of the one-meson potential.

The principal characteristics of the one-meson potential in $T=0$ states are a strong, attractive tensor potential, and, in the $^1S_0$ state, a very weak attraction at large distances and a strong repulsion at smaller distances. It has been shown[3,6,21] that the one-meson potential gives a reasonable description of the properties of the deuteron. The uncertainty in the pion potential at small distances gives just enough freedom to allow one to fit the binding energy easily. Therefore, there is as yet no evidence for important nonpion effects in $T=0$ states. It must be noted that the theoretical arguments underlying the Gartenhaus[3] potential differ from those given in this paper in two important respects. In the Gartenhaus potential, all the scattering corrections were omitted, and, moreover, the terms which correct the expectation value of the energy for the improper normalization of the state vector [the terms involving the A_n in Eq. (13)] were omitted. Fortunately, these two corrections have opposite signs and so tend to cancel; furthermore, they are not as important as the $T=0$ one-meson potential.

The higher order corrections to the $T=1$ potentials appear to be very important (partly because the one-meson energy is only $\frac{1}{3}$ as great as in the corresponding $T=0$ states). The perturbation theory part of E_{02} is more attractive and does not cancel the attractive scattering corrections. In addition to E_0, there is an enormous attraction at small distances from the one-meson excited states (see Fig. 11), although our approximative calculation probably overestimates this effect. It is clear from Figs. 8, 9, and 11, that strongly repulsive nonpion effects (or pion effects which have not been included) must enter at very small distances, otherwise, agreement with our empirical knowledge of these states would not be possible.

It can be seen from Fig. 9 that the $T=1$ tensor potential is greatly reduced by the two-meson terms in E_0, and changes sign at $r\sim0.4$ (such a sign change also occurs in the Gartenhaus potential). If the energy δE_1 (Fig. 11) is added to E_0, the sign change occurs at $r\sim0.7$, and the tensor potential becomes strongly attractive at smaller distances. This contradicts one of the assumptions from which the existence of a strong spin-orbit potential was inferred;[7] so to the extent that the static part of the interaction may be deduced from the adiabatic approximation, these results tend to weaken the empirical evidence for a spin-orbit interaction.

The only published numerical results which are similar to those given here are those of Konuma, Miyazawa, and Otsuki.[5] The KMO potential is just the sum of our E_0, δE_1, and δE_2, with a small additional contribution from the S-wave pion-nucleon interaction; a point source was also used for the calculation so the results correspond to ours only for $r>0.8$. It has been shown that terms can be added to the static-model interaction Hamiltonian in such a way that the low-energy S-wave meson scattering is reproduced.[23] Our methods would then lead to the same S-wave contribution as that in reference 5 (except for a reduction due to the finite source). Since there is no reason to believe that such a description of the S-wave interaction can give better than a very rough estimate of the correct S-wave effects, and since it has been shown in refer-

[23] Drell, Friedman, and Zachariasen, Phys. Rev. **104**, 236 (1956).

ence 5 that the S-wave effects calculated in this way are very small, we have not attempted to include them.

We have already remarked that the Heitler-London method, compared to an approach like that used in reference 5, has the advantage that a straightforward calculation of additional properties of two-nucleon states is easily made. In addition, the separation of the effects obtained from the basic Heitler-London state from those associated with the excited states is very useful. This separation makes it easier for us to judge the accuracy of the calculation and the validity of the adiabatic assumption. The expansion for the energy E_0 apparently converges quite rapidly, but while we may be sure that the excited states given an additional attraction, the discussion in Sec. III shows that it is very hard to calculate this attraction accurately.

Nonadiabatic effects are also particularly associated with the excited Heitler-London states, especially the one-meson states. As pointed out in Sec. II, the contribution of these states corresponds to the graphs with relatively small energy denominators. The importance of nonadiabatic corrections to δE_1 can be estimated by the ratio ϵ/ω_0, where ϵ, the kinetic energy of the nucleons in the interaction region, is always quite big.[24] The large attraction which was obtained from the one-meson excited states is therefore an indication that the static potential concept has limited utility in $T=1$ states. Only by including explicitly the excited configurations of the meson field can nonadiabatic effects, especially such nonadiabatic phenomena as real meson production, be adequately treated. A natural generalization of the usual semiphenomenological approach, as suggested by the Heitler-London method, would be to consider not a Schrödinger equation for a single wave function, but coupled equations for amplitudes $\varphi_0(r)$, $\varphi_1(r,k)$, and $\varphi_2(r,k)$, where $\varphi_0(r)$ would denote the amplitude for finding unexcited nucleons displaced by the amount r, and $\varphi_i(r,k)$ the amplitude for finding the ith nucleon scattering a meson. The numerical results obtained in Sec. III suggest that such an approach might lead to conclusions which differed considerably from those obtained by assuming that a static potential could correctly describe the true interaction for $r<1$ in $T=1$ states.

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[24] As a simple illustration, note that in the $^1S_0$ state of two protons, the intermediate isobar-nucleon system forms a $^5D_0$ state, in which the centrifugal energy alone is greater than ω_0 when $r<0.65$.

APPENDIX

In the adiabatic approximation the interaction energy E_{xy} is interpreted as the potential energy of a two-body Schrödinger equation. The conventional central and tensor potentials are obtained from the relations

$$3V_C=2E(\mathcal{P})+E(\mathcal{O}),$$
$$6V_T=E(\mathcal{P})-E(\mathcal{O}).$$

However, the representation of the wave function by orbital angular momentum eigenstates is not necessary, and is not particularly convenient unless the centrifugal energy is large compared with $6V_T$; in the deuteron, at distances where the interaction is strong, this is not the case. The introduction of radially adiabatic eigenstates makes the influence of the tensor force much more clear.[25]

We first note that the states with $L=J$ are pure $\mathcal{P}$ states, and that the $^3P_0$ state of two protons is a pure $\mathcal{O}_1$ state. In the other cases, we begin with the usual coupled equations for amplitudes $u(r)$ $(L=J-1)$ and $w(r)$ $(L=J+1)$:

$$\begin{aligned}
&-\frac{1}{M}u''+\frac{J(J-1)}{Mr^2}u+V_Cu-\frac{2(J-1)}{2J+L}V_Tu\\
&\qquad+\frac{6[J(J+1)]^{\frac{1}{2}}}{2J+1}V_Tw=Eu,\\
&-\frac{1}{M}w''+\frac{(J+1)(J+2)}{Mr^2}w+V_Cw-\frac{2(J+2)}{2J+1}V_Tw\\
&\qquad+\frac{6[J(J+1)]^{\frac{1}{2}}}{2J+1}V_Tu=Ew,
\end{aligned}\tag{A1}$$

where the primes indicate differentiation with respect to r. We transform these coupled equations by writing

$$\begin{aligned}
u(r)&=\bar{u}(r)\cos\alpha-\bar{w}(r)\sin\alpha,\\
w(r)&=\bar{w}(r)\cos\alpha+\bar{u}(r)\sin\alpha,
\end{aligned}\tag{A2}$$

where α is a function of r which is chosen to diagonalize, at a given separation of the two nucleons, the sum of the tensor and centrifugal potentials. Let

$$\rho_J=-3Mr^2V_T/(2J+1);\tag{A3}$$

ρ_J denotes the ratio of the splitting of the $\mathcal{P}$ and $\mathcal{O}$ states at a given distance r to the centrifugal splitting of the $L=J\pm1$ states at the same distance. Then

$$\tan2\alpha_J=2\rho_JJ^{\frac{1}{2}}(J+1)^{\frac{1}{2}}/(\rho_J+2J+1).\tag{A4}$$

[25] R. S. Christian and E. W. Hart, Phys. Rev. 77, 441 (1950). In this reference, the transformation given below was introduced in conjunction with the WKB approximation. We shall show that it may be formulated in a rigorous and generally useful way. The work "adiabatic" in the remainder of this Appendix refers to the spin-angular eigenfunctions, and should not be confused with the assumption that the meson field follows adiabatically the motion of the nucleons.

The coupled equations for $\bar{u}$ and $\bar{w}$ are

$$-\frac{1}{M}\bar{u}''+V_a\bar{u}+\frac{1}{M}\left(\alpha'\frac{d}{dr}+\frac{d}{dr}\alpha'\right)\bar{w}=E\bar{u}, \quad \text{(A5)}$$

and

$$-\frac{1}{M}\bar{w}''+V_b\bar{w}-\frac{1}{M}\left(\alpha'\frac{d}{dr}+\frac{d}{dr}\alpha'\right)\bar{u}=E\bar{w}. \quad \text{(A6)}$$

If a spin-orbit interaction is added to (A1), an appropriate modification of (A3) leads again to equations of the form of (A5) and (A6).

A small term from the radial kinetic energy is included in the adiabatic potentials V_a and V_b. If $\rho_J>0$ (as in the deuteron), V_a and V_b have the form

$$V_a=E(\mathcal{P})+[J(J-1)/Mr^2]+\Delta+\alpha'^2/M,$$
$$V_b=E(\mathcal{O})+[(J+1)(J+2)/Mr^2]-\Delta+\alpha'^2/M, \quad \text{(A7)}$$

where

$$\Delta=[J-J^{\frac{1}{2}}(J+1)^{\frac{1}{2}}\tan\alpha_J][E(\mathcal{O})-E(\mathcal{P})]/(2J+1).$$

When there is a very weak, attractive tensor potential ($\rho_J\sim 0+$),

$$\Delta=\frac{J}{2J+1}[E(\mathcal{O})-E(\mathcal{P})]\left[1-\rho_J\frac{J+1}{2J+1}+O(\rho_J^2)\right].$$

When $\rho_J\to\infty$, the $\mathcal{O}$ and $\mathcal{P}$ states become eigenstates, $\tan\alpha_J\to J^{\frac{1}{2}}(J+1)^{-\frac{1}{2}}$, and

$$V_a\sim E(\mathcal{P})+(J^2+J)/Mr^2,$$
$$V_b\sim E(\mathcal{O})+(J^2+J+2)/Mr^2. \quad \text{(A8)}$$

When $\rho_J\to-\infty$, the expressions (A8) for V_a and V_b are interchanged.

In the deuteron (where $J=1$), $\rho>1$ when $0.3<r<1.2$, and V_a therefore differs greatly from V_c, as is shown in Fig. 13. It should be noted that (A5) and (A6) can be derived from a variational principle; therefore if $\bar{w}$ is set equal to zero in (A5), the resulting equation for $\bar{u}$ gives an upper bound to E. The coupling between the adiabatic eigenstates depends on the product of α' and the radial velocity[26]; when $r>0.5$, $\alpha'\lesssim 0.2$, so the coupling is sufficiently small in the outer part of the deuteron that perturbation theory can be used to calculate its effect in this region. At small distances, the coupling term in (A5) is larger, but its effect is masked by the uncertainty in V_a. The quadrupole moment depends on α, $\bar{u}$, and $\bar{w}$ at large distances, and the contribution of $\bar{w}$ depends mainly on α and $\bar{u}$ at large distances, because V_b gives an extremely strong repulsion at small distances. This shows (in accordance with the conclusion of Iwadari *et al.*[6]) that the quadrupole moment of the deuteron depends primarily on the one-meson part of the interaction at large distances, where the size of the source has a small effect.

The error in the calculated V_a for the $T=0$, $J=1$ states depends primarily on the error in the calculation of $E(\mathcal{P})$ since the $\mathcal{P}_0$ state is the principal component of the radially adiabatic eigenstate. We have

$$\delta V_a=\delta E(\mathcal{P})\cos^2(\bar{\alpha}-\alpha)+\delta E(\mathcal{O})\sin^2(\bar{\alpha}-\alpha),$$

where

$$\bar{\alpha}=\tan^{-1}(2^{-\frac{1}{2}}).$$

[26] Thus when the radial velocity is assumed very small, as in the WKB approximation (reference 25), one has only Eq. (A5) to consider.