Modified Analysis of Nucleon-Nucleon Scattering. II. Completed Analysis of p-p Scattering at 310 Mev*

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The application of a recently suggested modified method of analysis of nucleon-nucleon scattering experiments to p-p scattering at 310 Mev has been completed. The results are summarized and compared with the work of Gammel and Thaler and of Signell and Marshak. The analysis is carried out on several levels, with varying number of angular momentum states being described by phenomenological phase shifts while the higher angular momentum states are represented by the one-pion exchange contribution. It is found that the inclusion of the high angular momentum states in this manner makes a significant improvement in the analysis. The pion-nucleon coupling constant is also determined from the data with a fair accuracy. The five "best" sets of phase shifts of the conventional analysis are reduced to two sets, corresponding to Solution 1 and 2 of the conventional analysis. Some slight evidence favors Solution 1 over Solution 2. It is shown that a very satisfactory fit can be obtained with nine parameters instead of the 14 parameters of the conventional analysis. Some remarks are made about the extent to which Solutions 1 and 2 are distinct. Experiments are suggested which could resolve the remaining ambiguity due to the existence of two sets of phase shifts.

INTRODUCTION

IN recent papers[1,2] a modified method of analysis of nucleon-nucleon scattering experiments was suggested. The general theory of the method as well as a partial application to the p-p data at 310 Mev has already been given[2,3] (in this paper reference 2 will be designated as I). In the present paper we wish to summarize the results of the completed analysis of p-p scattering experiments at 310 Mev and state the conclusions that we believe can be drawn from these results.

In Sec. I we summarize the results. Tables and graphs are given showing the values of the goodness-of-fit parameter, χ^2, the phase shifts obtained in our analysis as well as predictions for the values of the various experimental quantities, and the corresponding error matrices. In Sec. II these results are discussed with reference to our original objectives in adopting the modified method of analysis. A comparison is also given with other recent work on the nucleon-nucleon interaction, particularly with the work of Gammel and Thaler,[4] and of Signell and Marshak.[5-7] Finally, Sec. III lists some of the conclusions.

I. SUMMARY OF RESULTS

Some partial results have been given in I. There the exact relativistic amplitude of the one-pion exchange contribution (henceforth referred to as OPEC) was used to represent the contributions in the angular momentum states beyond the H wave, and a serarch was carried out on the 14-phase shifts up through the H waves. The result of this work will not be reiterated here.

Our next step was to use the OPEC to represent also the G and H wave contributions, and carry out a search on the nine phase shifts up through F waves. Just as in

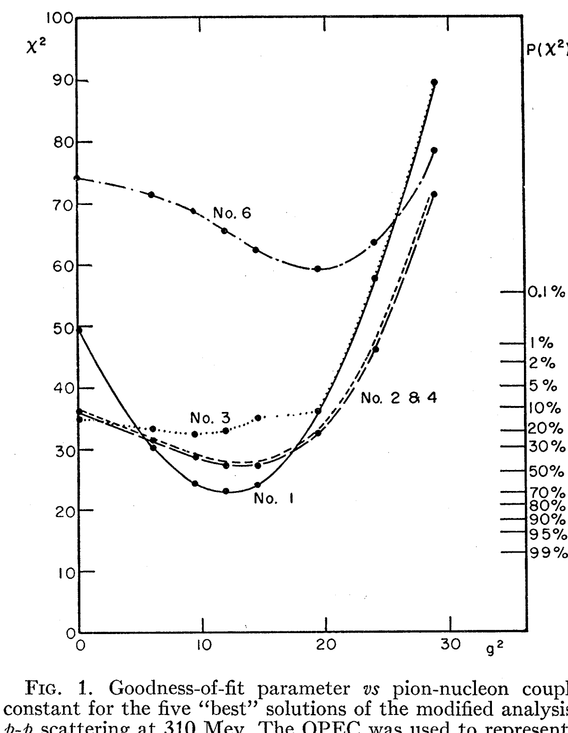

FIG. 1. Goodness-of-fit parameter *vs* pion-nucleon coupling constant for the five "best" solutions of the modified analysis of p-p scattering at 310 Mev. The OPEC was used to represent all angular momentum states from G waves on up. The right ordinate gives the percentage probability of obtaining a χ^2 value larger than the corresponding χ^2 on the left ordinate.

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* Work done under the auspices of the U. S. Atomic Energy Commission.

[1] M. J. Moravcsik, University of California Radiation Laboratory Report UCRL-5317-T, August, 1958 (unpublished).

[2] Cziffra, MacGregor, Moravcsik, and Strapp, Phys. Rev. **114**, 880 (1959).

[3] Moravcsik, Cziffra, MacGregor, and Stapp, Bull. Am. Phys. Soc. Ser. II, 4, 49 (1959).

[4] J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).

[5] P. S. Signell and R. E. Marshak, Phys. Rev. **109**, 1229 (1958). The tables in this and the next references giving the phase shifts at 310 Mev contain serveral errors.

[6] Signell, Zinn, and Marshak, Phys. Rev. Letters **1**, 416 (1958).

[7] We are indebted to Professor Marshak and Dr. Signell for an illuminating private communication.

TABLE I. Blatt and Biedenharn (BB) and nuclear bar (NB) phase shifts in degrees for p-p scattering at 310 Mev. The angular, momentum states from G waves on are represented by the OPEC. The coupling constant for Solutions 1, 2, and 6 were $g^2=12.0$, 13.3 and 18.7, respectively. Also shown are the phase shifts from the OPEC (one-pion exchange contribution) alone with $g^2=14.4$, and those of Gammel and Thaler,[a] of Signell and Marshak,[b] and of Signell, Zinn, and Marshak.[c]

	Solution 1		Solution 2		Solution 6		OPEC		Gammel and Thaler		Signell and Marshak		Signell, Zinn and Marshak	
	BB	NB	BB	NB	BB	NB	BB	NB	BB	NB	BB	NB	BB	NB
$^1S_0$	−8.92	−8.92	−28.99	−28.99	−5.26	−5.26	−70.0	−70.0	−9.40	−9.40	1.3	1.3	1.3	1.3
$^1D_2$	12.82	11.87	5.73	4.78	9.37	8.42	3.35	2.40	15.27	14.32	12.5	11.5	12.5	11.5
$^1G_4$	2.08	0.77	2.16	0.85	2.50	1.19	2.23	0.92	2.55	1.24	2.8	1.5	2.8	1.5
$^3P_0$	−10.64	−11.27	−27.29	−27.92	−65.96	−66.59	70.6	70.0	−11.95	−12.58	10.4	9.8	−0.6	−1.2
$^3P_1$	−26.86	−27.49	−8.08	−8.71	−14.17	−14.80	−35.2	−35.8	−26.70	−27.33	−19.4	−20.0	−26.9	−27.5
$^3P_2$	17.44	16.65	24.01	21.05	11.54	10.88	11.99	5.67	16.33	15.54	9.5	7.3	10.7	8.5
ϵ_2	−5.94	−1.55	−18.48	−7.55	3.48	0.50	−40.10	−6.59	−3.67	−1.01	−20.8	−4.3	−20.1	−4.2
$^3F_2$	2.21	1.21	−1.66	−0.49	3.27	2.14	−1.39	2.98	0.29	−0.81	−3.6	−3.2	−2.5	−2.2
$^3F_3$	−2.37	−3.53	1.13	−0.03	−0.86	−2.02	−3.66	−4.82	−4.84	−6.00	−3.7	−4.9	−3.5	−4.7
$^3F_4$	5.28	3.54	5.23	3.33	5.72	3.11	3.75	0.95	5.07	3.16	8.6	7.1	7.7	6.5
ϵ_4	−22.67	−1.40	−25.52	−1.55	−32.67	−2.26	−44.15	−1.69	−22.89	−1.78	−10.2	−1.5		
$^3H_4$	1.35	0.49	1.24	0.54	0.74	0.75	0.37	0.57	0.11	−0.58	−0.1	−1.2		
$^3H_5$	0.32	−1.12	0.20	−1.24	−0.31	−1.75	0.10	−1.35	−1.66	−3.10	0.8	−0.6		
$^3H_6$	1.65	0.21	1.67	0.23	1.77	0.33	1.69	0.25	0.60	−0.84	−0.1	−1.5		

[a] See reference 3. [b] See reference 4. [c] See reference 5.

I, we used as starting solutions the five "best" solutions of Stapp, Ypsilantis, and Metropolis[8] (henceforth referred to as SYM), and the search was carried out for various values of the pion-nucleon coupling constant g. The resulting values of χ^2 are shown in Fig. 1. The values of the phase shifts at the minimum values of χ^2 for Solutions 1, 2, and 6 are given in Table I. These minima are at $g^2=12.0$ for Solution 1, at $g^2=13.3$ for Solution 2, and $g^2=18.7$ for Solution 6.

The error matrices for Solutions 1 and 2 at the above values of the coupling constant are given in Tables II and III. They were calculated following the procedure outlined by Anderson *et al.*[9] and also followed by SYM.[8] Finally, the predicted values of the physically observable quantities for Solutions 1 and 2 at the above values of the coupling constant are given in Figs. 2 through 7.

TABLE II. Error matrix for the nuclear bar phase shifts in degrees squared of Solution 1 of p-p scattering at 310 Mev. The angular momentum states from G waves on are represented by the OPEC. The coupling constant was chosen to be 12.0.

	$^1S_0$	$^1D_2$	$^3P_0$	$^3P_1$	$^3F_3$	$^3P_2$	$^3F_2$	ϵ_2	$^3F_4$
$^1S_0$	3.72	−0.39	0.53	−1.13	0.80	0.05	0.27	−0.50	−0.08
$^1D_2$		0.24	0.05	0.22	−0.04	0.04	−0.12	0.05	0.02
$^3P_0$			2.81	−0.05	0.64	0.49	−0.44	−0.12	−0.13
$^3P_1$				1.19	−0.33	0.21	−0.04	0.24	0.05
$^3F_3$					0.44	0.04	−0.08	−0.10	−0.00
$^3P_2$						0.33	0.02	0.03	−0.01
$^3F_2$							0.49	0.02	0.18
ϵ_2								0.19	0.03
$^3F_4$									0.12

TABLE III. Error matrix for the nuclear bar phase shifts in degrees squared of Solution 2 of p-p scattering at 310 Mev. The angular momentum states from G waves on are represented by the OPEC. The coupling constant was chosen to be 13.3.

	$^1S_0$	$^1D_2$	$^3P_0$	$^3P_1$	$^3F_3$	$^3P_2$	$^3F_2$	ϵ_2	$^3F_4$
$^1S_0$	4.67	−0.29	−0.92	−1.12	−0.25	−0.45	0.78	−0.62	0.37
$^1D_2$		0.29	−0.19	0.31	−0.01	−0.05	−0.09	0.07	−0.01
$^3P_0$			6.43	0.09	0.54	1.84	−1.33	0.37	−0.63
$^3P_1$				1.21	0.04	0.38	−0.35	0.42	−0.33
$^3F_3$					0.15	0.18	−0.13	0.08	−0.03
$^3P_2$						0.92	−0.47	0.21	−0.31
$^3F_2$							0.79	−0.19	0.44
ϵ_2								0.26	−0.13
$^3F_4$									0.31

The favorable results of the above procedure (which will be discussed in Sec. II) prompted us to take the next step and use the OPEC to represent also the contributions from the two "uncoupled" F waves, that is, from the $^3F_3$ and $^3F_4$ states, and carry out the search on the seven phase shifts representing the S, P, and D states and the two other F parameters, that is, the phase shift in the $^3F_2$ state and ϵ_2. The five best SYM solutions were used as initial values in the search. The use of the final solutions from the 9-parameter search as initial values gives the same result. The results are shown in Fig. 8.

Finally we also carried out a phase shift search using the OPEC to regresent the contributions in all of the F states as well as in the higher angular momentum states, and search only on the S, P, and D waves. The results are not shown in a graph; the values of χ^2 for all the solutions for all values of the coupling constants were larger than 130.

In addition to the searches which used as initial values the phase shifts of SYM we also carried out searches with random sets of initial phase shifts. This was done in order to see if there might be any acceptable sets of phase shifts in addition to those that correspond to the eight solutions listed in SYM. We performed such random searches on the 7-parameter and on the 14-parameter level, for a coupling constant of $g^3=14.4$. On each level we carried out 30 searches. We found no new sets of phase shifts. It will be recalled

[8] Stapp, Ypsilantis, and Metropolis, Phys. Rev. **105**, 302 (1957).

[9] Anderson, Davidon, Glicksman, and Kruse, Phys. Rev. **100**, 279 (1955).

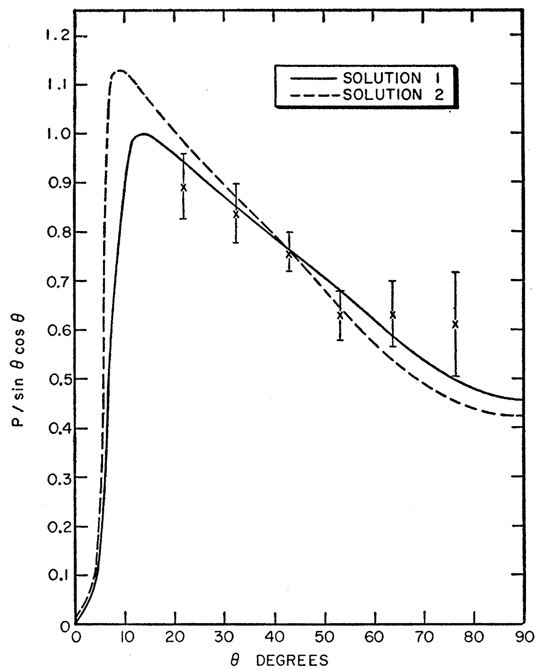

FIG. 2. Plot of P sinθ cosθ *vs* θ for Solutions 1 and 2 for those values of the coupling constant which give the minima in Fig. 1. Experimental values are shown in comparison.

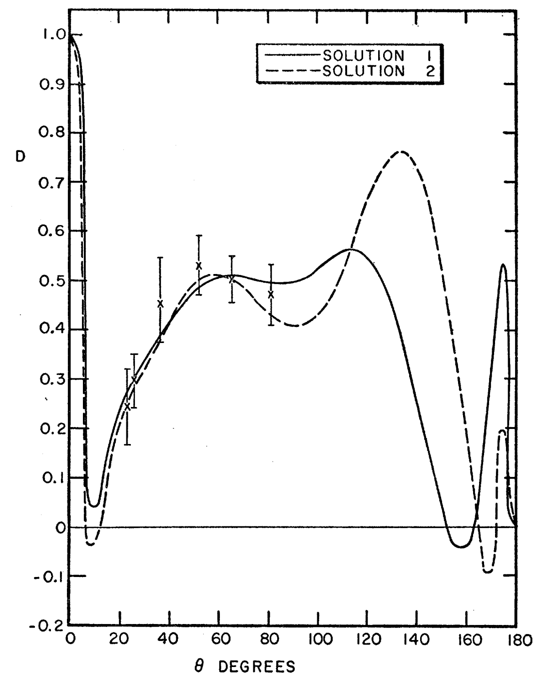

FIG. 3. Plot of D *vs* θ for Solutions 1 and 2 for those values of the coupling constant which give the minima in Fig. 1. Experimental values are given in comparison.

that a much more extensive random search procedure was carried out by SYM in finding their five best solutions. Our random searches furnish some additional evidence that the SYM search was indeed complete.

II. DISCUSSION OF RESULTS

The present method of analyzing nucleon-nucleon scattering data has four immediate, practical aims.[3] These are (a) the inclusion of the contributions of all higher angular momentum states, (b) the reduction of the number of parameters needed to represent the data, (c) the determination of the pion-nucleon coupling constant, and (d) the reduction of the number of sets of phase shifts. We will now discuss the results outlined in Sec. I in terms of these aims.

The original phase shift analysis of SYM used 14 parameters. We showed in I that, for Solutions 1 and 2, if the angular momentum states beyond H waves are represented by OPEC the same 14 parameters give, for reasonable values of the coupling constant, a better fit than SYM obtained. As we reduce the number of parameters and represent more and more of the high angular momentum states by OPEC the fit becomes less and less good as one would expect. Thus the question arises as to where one draws the line between an acceptable and unacceptable fit. We believe that our analysis in terms of 9 parameters plus the coupling constant (corresponding to Fig. 1) is certainly acceptable. Figures 1 and 8 also give, as a function of χ^2, the probability of obtaining a χ^2 value larger than the one in question. These probabilities are indicated at the right-hand edge of the figures. As one can see from

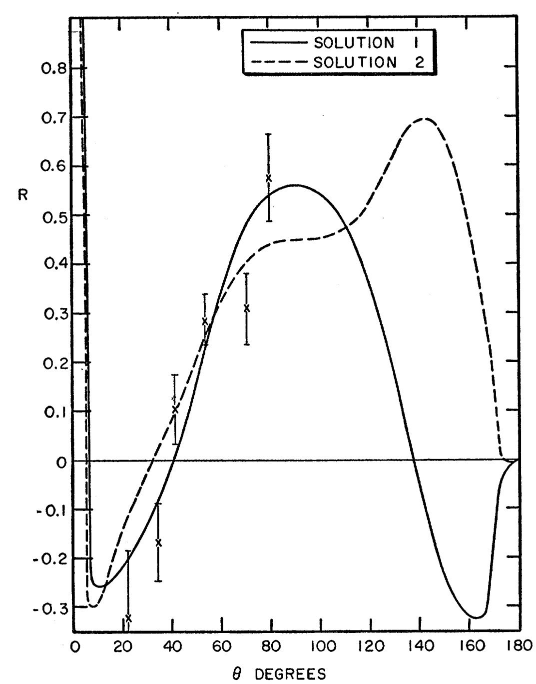

FIG. 4. Plot of R *vs* θ for Solutions 1 and 2 for those values of the coupling constant which give the minima in Fig. 1. Experimental values are given in comparison.

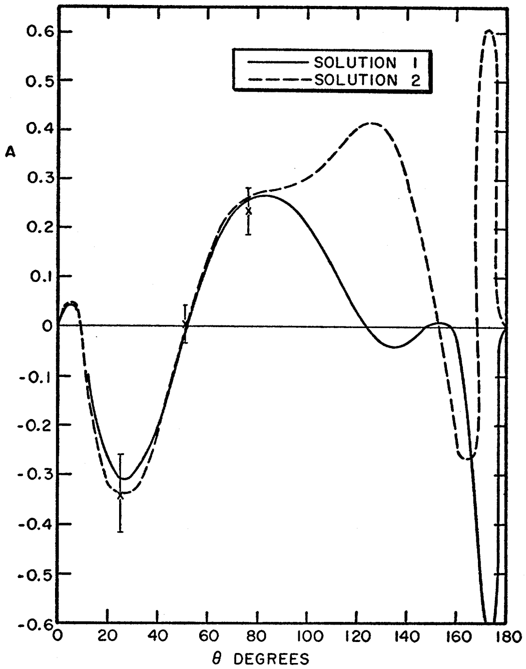

FIG. 5. Plot of A *vs* θ for Solutions 1 and 2 for those values of the coupling constant which give the minima in Fig. 1. Experimental values are given in comparison.

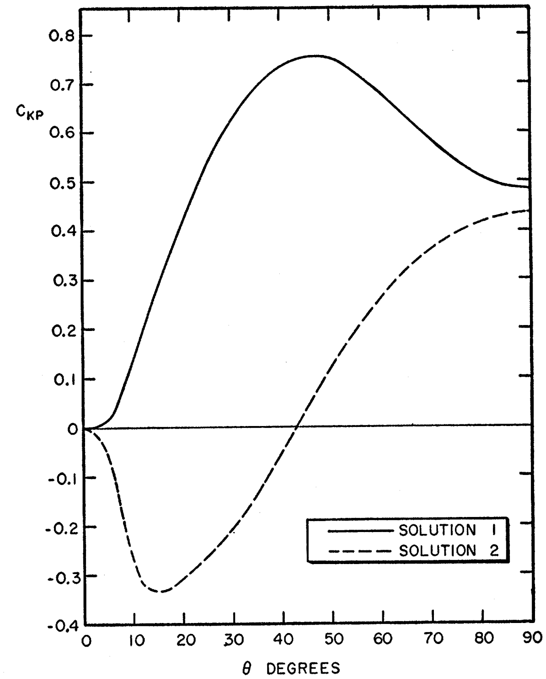

FIG. 6. Plot of C_{KP} *vs* θ for Solutions 1 and 2 for those values of the coupling constant which give the minima in Fig. 1.

Fig. 1, the minima of the curves corresponding to Solutions 1 and 2 are at the probability values of 70% and 45%, respectively. On the basis of this we consider these solutions quite acceptable. The corresponding probability values for the minima of the curves corresponding to Solutions 1 and 2 in Fig. 1 of I are 90% and 75%, respectively. Our confidence in this stage of our analysis is strengthened by the fact that the minima of the curves lie at very reasonable values of the coupling constant.

The analysis in terms of 7 parameters plus the coupling constant is more of a borderline case. There the minimum of the curve corresponding to Solution 1 is at a probability value of about 25%, while the rest of the solutions are all beyond the 0.1% probability level. The minimum of the curve corresponding to Solution 1 has its minimum at $g^2=7$ or so, which is quite different from the usually accepted value. In view of these facts we tend to disregard the quantitative results of this analysis in terms of 7 parameters, and believe that quantitatively it is not a good approximation to represent all of the F waves by OPEC.

The analysis in terms of 5 parameters gives what seem prohibitively large values of χ^2. In addition, the χ^2 curves of the various solutions show no minima at any positive value of g^2. Thus we believe that this 5-parameter analysis can be definitely rejected. It might be added, however, that if one judged the goodness of a fit by eye instead of the χ^2 test, even this 5-parameter fit would look like a reasonable fit. Its χ^2 values, which range from 130 on up, depending on the solution and the value of the coupling constant, compare favorably with the χ^2 value of around 1000 which represents the fit of Gammel and Thaler[4] at

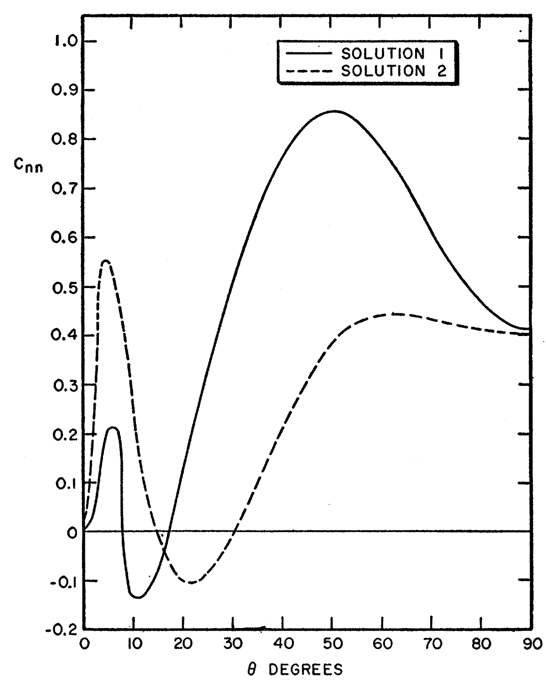

FIG. 7. Plot of C_{nn} *vs* θ for Solutions 1 and 2 for those values of the coupling constant which give the minima in Fig. 1.

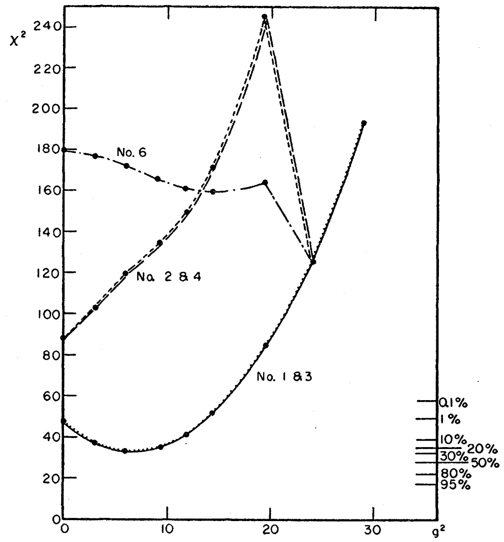

FIG. 8. Goodness-of-fit parameter *vs* pion-nucleon coupling constant for the five "best" solutions of the modified analysis of p-p scattering at 310 Mev. The OPEC was used to represent the $^3F_3$ and $^3F_4$ phase shifts as well as all angular momentum states from G waves on up. The right ordinate gives the percentage probability of obtaining a χ^2 value larger than the corresponding χ^2 on the left ordinate.

this energy. On the other hand, the potential of Gammel and Thaler was constructed to fit all data *up to* (and not only *at*) 310 Mev, and so one would expect a somewhat less good fit at a given energy.

In regards to points (a) and (b), the modified method therefore appears to have successfully included the contributions of higher angular momenta and reduced the number of parameters from 14 to 9 plus the coupling constant.

We have already said something about the determination of the coupling constant. This matter, however, requires further discussion. From the 9-parameter analysis, the values of the coupling constant obtained from Solutions 1 and 2, respectively, are $g^2=12.0\pm2.1$ and $g^2=13.3\pm3.3$, which correspond to $f^2=0.062\pm0.011$ and $f^2=0.069\pm0.017$. The errors quoted here were determined simply from the steepness of the χ^2 curve by the usual method[9] and do not include ρ^2, the factor representing the error due to the uncertainty in the knowledge of the functional form used for the fitting.[10] This latter error is included, however, in Table IV which gives the values and errors of the coupling constants as determined from Solutions 1 and 2 in the 14-, 9-, and 7-parameter analysis. One would expect that if too many free parameters are used one gets a good fit and hence ρ^2 is small. At the same time, the many free parameters obscure the component of the analysis which serves to determine the coupling constant, and hence the conventional statistical error is large. As one moves to fewer and fewer free parameters, ρ^2 will increase since the fit gets tighter and tighter, but at the same time the component which is the basis of the determination becomes more and more pronounced and hence the statistical error itself gets smaller and smaller. It should be mentioned, that while the statistical error is independent of the absolute value of χ^2, the quantity ρ^2 depends very sensitively on it. Now it is a rather striking feature of the SYM analysis, as well as our 14-parameter analysis, that the χ^2 values obtained for the best solutions are noticeably smaller than the expected value of χ^2 for the appropriate number of degrees of freedom. One possible reason for such a phenomenon could be the overestimation of the errors pertaining to the experimental data used in the analysis. We have no reason to believe that this is indeed the case, but if it were, all ρ^2 values would have to be increased, and hence probably more consistency would be evident with the value of the coupling constant obtained from the 7-parameter analysis.

We might also remark in connection with the coupling constants obtained from our analysis that they refer to the interaction of a proton and a neutral pion. The over-all value obtained from our analysis is slightly lower than the value $f^2\approx0.08$ customarily quoted as the average of various other determinations, but the difference cannot be taken too seriously in view of the above discussion of uncertainties.

Finally, we believe that this analysis has brought about a definite simplification in the picture of multiple phase shift solutions. First of all, the results show that Solutions 3 and 4 are very closely connected to Solutions 1 and 2, respectively. They evidently correspond to small local minima near the deeper minima of Solutions 1 and 2. In fact, the maximum separating the minima of Solutions 3 and 1 might disappear for certain reasonable values of the coupling constants. Similarly, Solutions 2 and 4 might merge under reasonable conditions. The similarity between the SYM Solutions 1 and 3 and between 2 and 4 already suggest a relationship of the type we have found.

Regarding the remaining set of phase shifts, corresponding to SYM 6, the new analysis greatly strengthens the argument for its rejection. In the reasonable

TABLE IV. Values of the pion-nucleon coupling constant, as obtained in the various stages of the modified analysis of p-p scattering at 310 Mev. For further details see the text.

	Number of parametrs	g^2	Random statistical error	ρ^2	Total statistical error
Solution 1	14	13.5	6.2	0.60	4.8
	9	12.0	2.1	0.85	1.9
	7	6.7	1.8	1.14	1.9
Solution 2	14	15.4	6.7	0.77	5.9
	9	13.3	3.3	1.01	3.3
	7	...	...	...	...

[10] For a more complete discussion of this question see, e.g., P. Cziffra and M. J. Moravcsik, University of California Radiation Laboratory Report UCRL-8523, October, 1958 (unpublished).

range of g^2 this solution remains in the $<0.1\%$ probability range whereas the other two solutions reach $>40\%$ probable values (see also I). The status of Solution 6 is important since it is the solution corresponding to the analysis of Feshbach and Lomon.[11] In the analysis of SYM, Solution 6 was the least favored of the five best but the difference was not nearly as pronounced.[12]

Thus the choice is narrowed down to two solutions, 1 and 2. We find that on the basis of our analysis of the presently available *p-p* data at 310 Mev we cannot decide between these two solutions with anything like the same degree of certainty as we could eliminate Solution 6. Just as in the case of the SYM analysis, Solution 1 looks slightly more promising, since it has a lower χ^2. Furthermore, Solution 1 exhibits somewhat more sensitivity to the value of the coupling constant than Solution 2, which might also be construed as slight evidence in favor of Solution 1. Finally the decided superiority of Solution 1 over Solution 2 in the 7-parameter analysis can also be used for evidence in favor of Solution 1. One might argue that the 7-parameter analysis shows, at least for some solutions, the qualitative behavior we expect, and hence that the OPEC gives the uncoupled F phase shifts at least qualitatively right. Thus we can rely on the 7-parameter analysis for qualitative predictions, one of which is that Solution 1 is to be preferred over Solution 2. In summary, therefore, we can say that we have some arguments for preferring Solution 1 over Solution 2, but we cannot consider the evidence as being conclusive.

It is natural to ask whether there is some experiment which could definitely distinguish between the two solutions. A glance at the predictions of the physical observables, particularly that of c_{KP}, shows that, if properly chosen, even a qualitative experiment could easily eliminate this ambiguity. These correlation experiments, however, are very difficult to carry out at the angles where the distinction is clear-cut. A study of the feasibility of measuring c_{KP} in the neighborhood of 45° is being carried out.[12]

It might be of interest to mention that according to Gammel and Thaler[4] Solution 1, which we tend to favor, is the one which can be approximated by a local potential. According to these authors neither Solution 2 nor 6 can be approximated in such a fashion.

Table I also gives the phase shifts of Signell and Marshak,[5,6] and of Gammel and Thaler,[4] as well as the phase shifts of the OPEC itself. The comparison with the OPEC phase shifts shows clearly that the quantitative results of the 7-parameter analysis turn out to be unsatisfactory mainly on account of the $^3F_3$ phase shift. One can also see that the Gammel-Thaler phase shifts are in general agreement with Solution 1 as expected since the Gammel-Thaler potential was based upon SYM Solution 1. One can also see that the Signell-Marshak phase shifts differ from either Solution 1 or 2 in most of the lower angular momentum states. It seems as if the modified version given by Signell, Zinn, and Marshak[6] gives slightly better agreement.

Another question which deserves some attention is to what extent Solutions 1 and 2 are separate solutions.[13] One can see from our results that in some extreme cases, for instance, in the 7-parameter analysis for values of the coupling constant which are larger than 24, even Solutions 1 and 2 merge. In general, however, the two solutions give quite different sets of phase shifts. In order to see in what way these two solutions differ at the more interesting stages of our analysis, we took the 9-parameter case at $g^2=14.4$, held the S phases fixed at various values between those given by Solutions 1 and 2, and minimized for the rest of the phase shifts. For the initial values of the rest of the phase shifts we used several points in the 8-dimensional space of these phase shifts. These points lay along the straight line connecting the points corresponding to Solutions 1 and 2. We found that we got two distinct solutions in this whole range of S phases, one corresponding to Solution 1 and the other to Solution 2. Whether, at a given fixed value of the S phase, we got one solution or the other depended on the set of initial values of the other phases. In particular, when the point representing the initial set of phase shifts on the straight line in the 8-dimensional space was closer to the end point representing Solution 1, we obtained a solution corresponding to Solution 1, while when the point representing the initial set was closer to the endpoint representing Solution 2 we obtained a solution corresponding to Solution 2. The values of χ^2 as a function of the S phase shift are shown in Fig. 9. While the above study is not quite exhaustive it gives a rather strong indication that Solutions 1 and 2 are indeed separate and that there is no simple "valley" which takes one solution into the other. A completely conclusive study of this problem would probably call for the very time-consuming procedure of carrying out an extensive random search procedure for each fixed value of the S phase shift lying between those given by Solutions 1 and 2.

In the course of our random search procedure we found some of those solutions of SYM which in their analysis had higher χ^2's than the five best solutions. We traced these "bad" solutions through various stages of our analysis and for several values of the coupling constant to see whether they remain "bad." We found that all of these solutions continued to be much worse than Solution 1 throughout the analysis, although several of them became better than Solution 6. They were also worse than Solution 2 except at the 7-parameter level where some of them were about the

[11] H. Feshbach and E. L. Lomon, Phys. Rev. **102**, 891 (1956).

[12] Solution 6 is now also excluded by some new experimental results. We are indebted to Professor A. Ashmore for valuable private communication on this and other matters.

[13] We are indebted to Dr. Gammel for arousing our interest in this particular question.

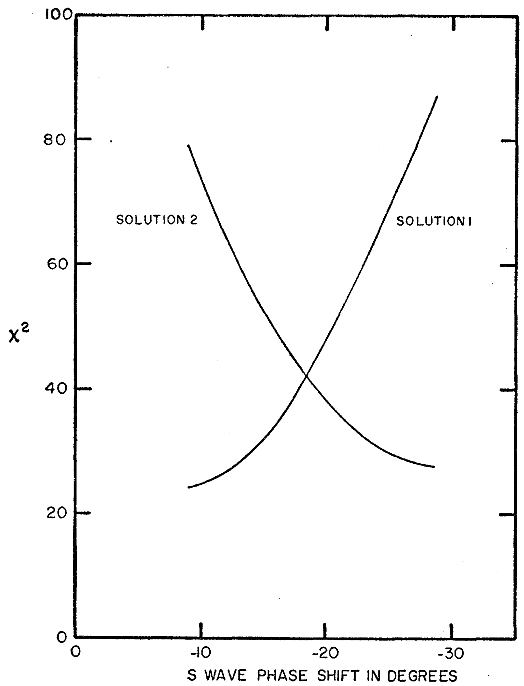

FIG. 9. Goodness-of-fit parameter *vs* fixed S nuclear bar phase shift, for the 9-parameter analysis at $g^2=14.4$. The two solutions, resembling Solutions 1 and 2, respectively, remain distinctly different throughout the whole range of S phase shifts. Any search using as initial values some sets of phase shifts intermediate between Solutions 1 and 2 results in either one or the other of these distinct sets.

same as Solution 2. This investigation strengthens our claim that the choice of the right set of phase shifts has been narrowed down to two solutions.

The present analysis is based on the observation that just outside the physical region in the complex $\cos\theta$ plane there is a pole where the scattering amplitude (and thus the differential cross section) goes to infinity. On the other hand, in the physical region the p-p differential cross section looks almost isotropic except for the forward and backward peaks due to the Coulomb interaction. In an attempt to exhibit the effect of the pole in the physical region, we computed the differential cross section without the Coulomb effects as given by Solutions 1 and 2. These results for Solution 1 at small angles are shown in Fig. 10. The curve for Solution 2 is very similar. We also computed the differential cross section without the Coulomb effect as predicted by the SYM Solutions 1 and 2 and found that even at the smallest angles they differ from our predictions only by a few percent. Thus the slight peaking must be produced by those higher angular momentum states which are already included in the SYM analysis. It is also interesting to note that the analogous n-p differential cross section at this energy has a much more pronounced peak in the forward direction, which therefore must be due to the triplet even and singlet odd states.

The fact that the pole has such a small visible effect on the p-p differential cross section at this energy would, at first sight, cast doubt on the whole approach which is the basis of our analysis. One should remember, however, that our analysis also includes data on other experimental quantities, such as polarization and triple scattering parameters, where the influence of the pole is not quite so hidden. To demonstrate this point we applied the method given by Chew[14] to determine the pion-nucleon coupling constant from the 14 pieces of data on differential cross section, which are among the data used in our analysis. We found that the resulting coupling constant has an error assigned to it which is of the order of ten times as large as the error we obtained in our analysis. Furthermore, the selection of the proper order of the extrapolating polynomial is quite ambiguous, and the value of the coupling constant, ranging from 4 to 40, depends very much on the order of the polynomial. To be sure, the difference in the success of the two methods of determination in this case is due in part simply to the increased number of data used in our analysis. In view of the success[15] of the Chew method in the case of n–p scattering, however, it is difficult to escape the conclusion that the net effect of the pole in the p-p differential cross section is anomalously small, and that therefore the data on the other physical observables are mainly responsible for the good and precise value of the coupling constant we obtain.

The predictions of Solutions 1 and 2 were also used to compute the so-called Wolfenstein amplitudes,[16] which are given in Table V.

Finally, we wish to draw attention to two particular aspects of our procedure. The first is that, as usual in

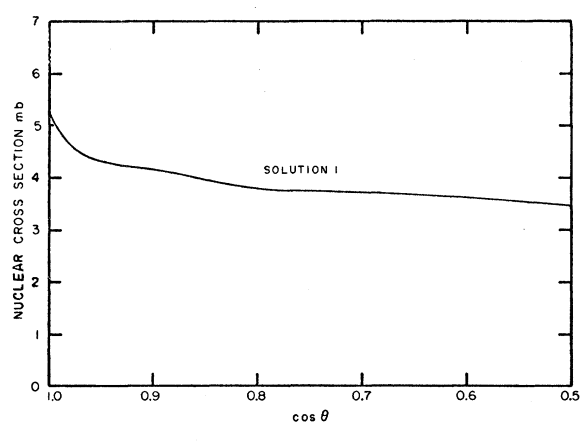

FIG. 10. The nuclear part of the differential cross section of p-p scattering at 310 Mev as predicted by Solution 1 corresponding to the minimum in Fig. 1. Even at the smallest angles the curve rises only slightly, although the pole is just outside the physical region.

[14] G. F. Chew, Phys. Rev. **112**, 1380 (1958).
[15] P. Cziffra and M. J. Moravcsik, Phys. Rev. **116**, 226 (1959).
[16] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954), see Eqs. 3.4 and 3.5.

TABLE V. The Wolfenstein parameters[a] corresponding to Solution 1 (with $g^2=120.$) and Solution 2 (with $g^2=13.3$) in the 9-parameter analysis.

	$\theta_{\mathrm{c.m.}}$	$\bar{B}$[b]	B	$\bar{C}'$[c]	C'[c]	$\bar{G}$	G	$\bar{H}$	H	$\bar{N}$	N
Solution 1	1°	0.1167	−0.0294	0.0602	0.0018	−0.0101	−0.0861	0.0171	0.0167	0.0034	−0.0347
	2°	0.1162	−0.0293	0.0602	0.0018	−0.0097	−0.0861	0.0175	0.0167	0.0034	−0.0347
	5°	0.1126	−0.0289	0.0600	0.0018	−0.0075	−0.0858	0.0200	0.0165	0.0032	−0.0345
	15°	0.0878	−0.0259	0.0584	0.0018	0.0034	−0.0828	0.0345	0.0152	0.0013	−0.0330
	30°	0.0453	−0.0182	0.0535	0.0020	0.0095	−0.0721	0.0455	0.0128	−0.0039	−0.0286
	60°	−0.0259	0.0008	0.0400	0.0029	−0.0103	−0.0378	0.0350	0.0103	−0.0103	−0.0149
Solution 2	1°	0.0313	−0.0310	0.0599	−0.0007	0.0888	−0.0814	−0.0666	0.0056	0.0132	−0.0379
	2°	0.0309	−0.0310	0.0599	−0.0007	0.0890	−0.0814	−0.0661	0.0055	0.0109	−0.0379
	5°	0.0275	−0.0307	0.0597	−0.0007	0.0907	−0.0811	−0.0629	0.0051	0.0107	−0.0377
	15°	0.0065	−0.0295	0.0581	−0.0010	0.0970	−0.0779	−0.0421	0.0023	0.0093	−0.0355
	30°	−0.0210	−0.0273	0.0529	−0.0018	0.0826	−0.0665	−0.0155	−0.0044	0.0063	−0.0292
	60°	−0.0511	−0.0235	0.0388	−0.0034	0.0171	−0.0328	0.0124	−0.0186	−0.0021	0.0132

[a] See reference 16.
[b] The Wolfenstein parameters listed here are defined in Eqs. (3.4) and (3.5) of reference 16. The barred and unbarred quantities are real and imaginary amplitudes, respectively.
[c] The quantities $\bar{C}'$ and C' are related to the Wolfenstein $\bar{C}$ and C by the equations $\bar{C}=C'\sin\theta$ and $C=-\bar{C}'\sin\theta$.

phase shift analyses, we deal only with scattering amplitudes and not with potentials. In particular, the one-pion exchange contribution which we have included is completely covariant and is not based on the idea of a potential.

The second point is that our inclusion of the one-pion exchange contribution is exact. We use the term "one-pion exchange contribution" in the sense of dispersion theory which classifies contributions in terms of the singularities to which they belong.[17] This is stressed in order that the circumstance that the OPEC is, formally, identical with the Born approximation will not lead to the erroneous conclusion that our method depends on the validity of perturbation theory.

The basic philosophy underlying the present method of analysis is the attempt to work from the outside of the nucleon toward the inside, combining our theoretical understanding of the edge of the nucleon with a phenomenological description of the inside. This approach has long been used by various Japanese workers[18] who, for some years, have been contributing to the understanding of nuclear forces along these lines. Our work, though based on certain differences in view and method, is in accord with their general principles. We regret that a direct comparison of our results with their work cannot be made without a considerable amount of additional computation converting potential models into scattering amplitudes and phase shifts.

III. CONCLUSIONS

We will now summarize the main results of this analysis.

(a) The final phase shift solutions are expected to be more accurate by virtue of the inclusion of all higher partial waves.

(b) The pion-nucleon coupling constant has been determined fairly accurately from nucleon-nucleon scattering experiments.

(c) The number of sets of phase shifts has been reduced from the five given by SYM to two, with some additional evidence mildly favoring one of these two sets.

(d) The results indicate how far into the interior of the nucleon the one-pion exchange force is dominant.

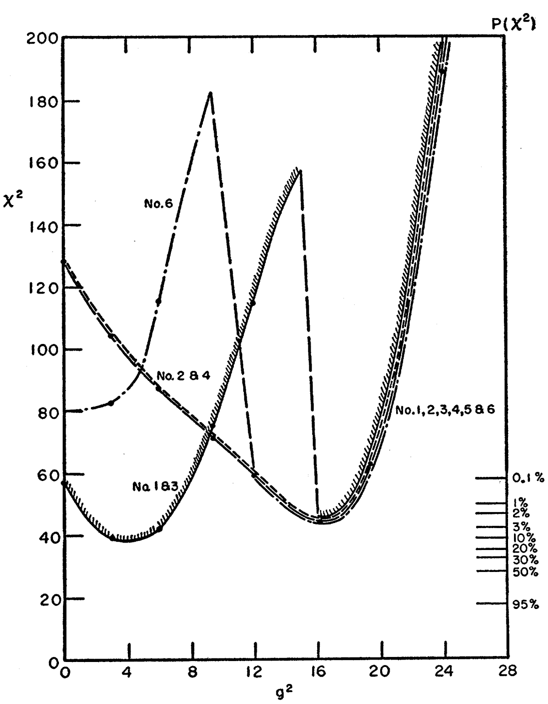

FIG. 11. Goodness-of-fit parameter *vs* pion-nucleon coupling constant for the five "best" solutions of the modified analysis of p-p scattering at 310 Mev. The OPEC was used to represent the ${}^3F_2$ and ϵ_2 phase shifts as well as angular momentum states from G waves on up. The right ordinate gives the percentage probability of obtaining a χ^2 value large than the corresponding χ^2 on the left coordinate.

[17] A more detailed discussion of this point is given by H. P. Noyes and D. Wong (to be published).

[18] For a summary of the Japanese contributions, see e.g., Progr. Theoret. Phys. (Kyoto), Suppl. III (1956).

(e) Some understanding has been obtained concerning the extent to which the remaining two solutions are different.

This concludes our analysis of present data on *p-p* scattering at 310 Mev. Further work on similar analyses of *p-p* scattering data in the range of 0–40 Mev, at 150 Mev, and at 200 Mev are in progress and will be reported in subsequent papers.

IV. ACKNOWLEDGMENTS

We are indebted to Professor Geoffrey F. Chew for several stimulating discussions. Frequent consultations with Dr. H. Pierre Noyes are also gratefully acknowledged. We thank Dr. Sidney Fernbach for his active interest in our work, and Mr. Peter Cziffra for his important contribution at the initial stages of this work. The numerical computations were carried out on the IBM-704 electronic computer at the Lawrence Radiation Laboratory at Livermore.

Note added in proof.—Stimulated by remarks made by B. L. Ioffe, I. Pomeranchuk, V. G. Grishin, and I. Kobzarev at the International Conference on High-Energy Physics in Kiev, July 1959, we carried out the analysis of the data on *p-p* scattering at 310 Mev with the S, P, D, $^3F_3$, and $^3F_4$ states being searched on, and the parameters ϵ_2, $\delta(^3F_2)$, $\delta(^1G_4)$, etc. being fixed by the pole contribution. The result of this analysis is shown in Fig. 11. This 7-parameter analysis is as unreliable, quantitatively, as the 7-parameter analysis described in the text. Qualitatively the new analysis does not exhibit the superiority of Solution 1 over Solution 2, shown in the original 7-parameter analysis (Fig. 8), and the two solutions now seem equally good.