Bremsstrahlung and the Photoelectric Effect as Inverse Processes*

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The bremsstrahlung matrix element at the short-wavelength limit of the spectrum is calculated to lowest order in $\alpha \equiv Z/137$, for an unscreened Coulomb field. The result, valid for relativistic incoming electrons, is shown to be exactly $\alpha^{-1}m^{-\frac{1}{2}}$ times the complex conjugate of Sauter's relativistic matrix element for the K-shell photoelectric effect. These matrix elements are the leading terms in an expansion of the exact matrix elements in powers of α, and they are found to be derivable from the first two terms of the expansions in powers of α of the electron wave functions. In this sense their structure is completely analogous to that of the Bethe-Heitler bremsstrahlung matrix element.

This simple relation between the matrix elements derives from an approximate equality (through first order in α) between the Coulomb wave functions for bound and zero-momentum continuum states, which can be understood as due to the neglect of the Coulomb binding energy, a second-order quantity in α.

Finally, the range of validity of Sauter's approximation is examined in detail. The lower bound of this (energy) range is found to be simply related to the radius of convergence of the expansion of the photoeffect matrix element in powers of α.

INTRODUCTION

THE inverse process to the atomic photoelectric effect is the radiative capture of an electron by an ionized atom. Another radiative process which may also, in a sense, be thought of as inverse to the photoeffect is the process of bremsstrahlung at the short-wavelength limit of the spectrum, where the electron is brought "completely to rest," and its initial kinetic energy converted entirely into radiation. The purpose of this paper is to delineate exactly the sense in which this process is inverse to the photoeffect, and to point out some of the practical consequences of such a relationship.

The most striking consequence of the relation is the prediction that to every polarization phenomenon in one of the processes corresponds an analogous polarization phenomenon in the other, a fact which has recently been put to practical use by Motz and Placious.[1] In addition, the relation in question provides a means of investigating the bremsstrahlung process at the spectrum limit, where the usual Born approximation breaks down,[2,3] and conversely it sheds considerable illumination on the meaning and validity of the approximation techniques employed by Sauter[4] in his calculation of the K-shell photoelectric effect. Sauter's calculation is closely related to a Born approximation; we shall show that it is entirely analogous to the Bethe-Heitler[5] calculation of the bremsstrahlung matrix element, and that the results of both calculations can be described as the sum of two "Feynman diagrams."

EXACT STATEMENT OF THE PROBLEM

One of the electron states involved in both the photoeffect and bremsstrahlung at the spectrum tip ("tip-bremsstrahlung," as we shall call it) is a continuum Coulomb state of asymptotic momentum $\mathbf{p}$. In the calculation of the matrix elements for both processes, we shall use for this state the first Born-approximation wave function for a Coulomb field, without screening. With this understanding, we can state the asserted inverse relation, which is an approximate one, in terms of the following recipe: Calculate the K-shell photoeffect martix element using the exact Dirac spinor for the bound state, and calculate the tip-bremsstrahlung matrix element using the exact Dirac spinor for the zero-momentum final state. Expand both matrix elements in powers of $Z/137$, retaining only the lowest-order term in each case. Then these two leading terms are complex conjugates of each other except for a constant factor, which is the ratio of normalization constants for the bound and $p=0$ continuum states. Thus the cross sections for the photoeffect and for tip-bremsstrahlung, in this approximation, are directly related to each other by detailed balancing. The angular distributions, as well as the polarization phenomena, will then be exactly the same for the two processes, in this lowest-order approximation.[6]

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[1] J. W. Motz and R. C. Placious, Phys. Rev. **112**, 1039 (1958).
[2] U. Fano, this issue [Phys. Rev. **116**, 1156 (1959)].
[3] Fano, Koch, and Motz, Phys. Rev. **112**, 1679 (1958).
[4] F. Sauter, Ann. Physik **11**, 454 (1933).
[5] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

[6] It is convenient to refer to this as the "first-order" approximation, or alternatively as the "Sauter approximation." The Born-approximation continuum function by definition contains zeroth- and first-order terms in $Z/137$, and one of the purposes of our discussion is to show that, analogously, if the other wave function entering the matrix element is expanded in powers of $Z/137$, only its first two terms contribute in the present approximation. The lowest-order term of the resulting matrix element vanishes, so only the next term, of first order *relative* to this lowest possible term, contributes. Hence we call it the "first-order" approximation even though, because of normalization factors, it is not actually of first order in $Z/137$.

Our main task in this paper is to construct a mathematical proof of this inverse relationship. This proof is unfortunately rather involved, but the physical ideas behind it are actually very simple. To see this, recall that the matrix elements for tip-bremsstrahlung and for the photoeffect differ only in one of the electron states, which is a zero-momentum continuum state in one case, and a K-shell bound state in the other. The energy eigenvalue of the zero-momentum state is m (if $c=1$); that of the bound state is less, but in a Coulomb field it differs from m only by an amount of order $(Z/137)^2$. Consequently the differential equations satisfied by these two wave functions have eigenvalues which differ only in second order in $(Z/137)$, so that if we expand the functions in powers of $(Z/137)$, we may expect them to agree through *first* order. (Of course this argument applies only to the $s_{\frac{1}{2}}$ part of the continuum state, for the K-shell state is an $s_{\frac{1}{2}}$ function. However, the bremsstrahlung integral comes predominantly from this $s_{\frac{1}{2}}$ partial wave of the $p=0$ continuum state, for it is the one which is largest near the origin.) Since (as we shall show) the radial variable r appears in these functions only as $(Z/137)r$, this implies that the two functions should agree through first order in r, i.e., *near the origin.*

Another way of seeing this is by borrowing from the effective range theory of nuclear forces the argument that two low-energy solutions to the same wave equation are expected to agree closely in any region of space where the difference between their eigenvalues is small compared to the potential energy. In a Coulomb field, this requirement is always satisfied if we approach close enough to the nucleus, so the two wave functions should agree most closely as $r\rightarrow 0$. To be more precise, the difference between the eigenvalues for the above two wave functions is approximately $\frac{1}{2}(Z/137)^2m$ (setting $\hbar=c=1$), so we expect them to agree well in the region where

$$\tfrac{1}{2}(Z/137)^2m\ll(Z/137)/r, \quad \text{or} \quad mr\ll 2(137/Z).$$

Since $a_0m=137$, where a_0 is the Bohr radius, we may write this as

$$r\ll 2a_0/Z,$$

i.e., the functions should differ little if we are well within the radius of the K-shell, $a_K=a_0/Z$.

Of course, this similarity between the wave functions implies a similarity between the matrix elements for tip-bremsstrahlung and the photoeffect only if the major contributions to the integrals come from the region where the wave functions agree. We expect the principal contribution to come from $r\lesssim 1/q$, where q is the momentum transfer to the nucleus; using this estimate, the matrix elements should agree best when

$$1/q\ll 2a_0/Z, \quad \text{or} \quad qa_0\gg Z/2,$$

i.e., at high energy and low Z. The analysis given below shows that the criterion is actually

$$qa_0\gg Z, \quad \text{i.e.,} \quad q/m\gg Z/137.$$

Mathematically, this comes about because the matrix element has a power series expansion in the variable $\alpha=(Z/137)$, with radius of convergence $\alpha=q/m$; the criterion $qa_0\gg\alpha Z$ is merely the requirement that, if we wish to approximate the series by only its first term (the Sauter approximation), then we must be well inside the circle of convergence.

As for the relation to previous calculations, the photoeffect matrix element thus obtained is exactly that of Sauter, given in reference 4. (For a further discussion of this point, see other recent analyses and extensions of Sauter's work.[7,8]) The bremsstrahlung matrix element obtained in this way has a close formal relation to the Bethe-Heitler result. The latter may be thought of as the first two terms of an expansion of the exact matrix element in powers of $Z/137\beta_1$ and $Z/137\beta_2$, where β_1 and β_2 are the asymptotic velocities of the initial and final electron states (in units where $c=1$). This Born-approximation result is of course invalid for tip-bremsstrahlung, where $\beta_2=0$, but is replaced, according to the above recipe, by the first two terms of an *asymmetric* double expansion, in powers of $Z/137\beta_1$ and $Z/137$. The relation to the Born approximation is in fact more than formal, and this "Sauter approximation" bremsstrahlung cross section is shown in reference 2, on the basis of the results of the present paper, to differ from the Bethe-Heitler cross section evaluated at $\beta_2=0$ *only by a normalization factor* (albeit an infinite one). Since the angular distribution and polarization properties of the Bethe-Heitler cross section vary slowly with β_2 near $\beta_2=0$, we expect them to remain similar to those of the photoeffect in the whole region near the tip of the bremsstrahlung spectrum. This expectation is borne out by the experimental results of reference 1.

THE PHOTOELECTRIC EFFECT

We shall first consider in some detail the first-order approximation to the photoeffect matrix element. This approximation is identical with Sauter's,[4] for Sauter explicitly neglected second-order terms in $Z/137$, though only in the last stages of his calculation. The comparison with Sauter's results is discussed in detail in reference 8, so we need not dwell upon the point here. Our purpose in this section will rather be to rederive the Sauter result by an entirely different method, one which is designed to enable us to see exactly what properties of the bound state appear in the matrix element.

As for notational conventions, we shall set $\hbar=c=m=1$, m being the electron mass. The parameter $Ze^2/\hbar c$

[7] Haakon Olson, *Festskrift til Egil Hylleraas* (Bruns Trondheim, 1958).
[8] Fano, McVoy and Albers, Phys. Rev. **116**, 1147 (1959).

$=Z/137$ will occur repeatedly throughout the discussion, so in the interest of economy we shall employ the somewhat unconventional but convenient definition,

$$\alpha \equiv Z/137. \quad (1)$$

In these terms, e.g., the Bohr radius for nuclear charge Z is $a_K=\alpha^{-1}$, so the radial dependence of the lowest bound state of the Coulomb field is $\sim e^{-\alpha r}$.

We shall write the photoeffect matrix element as

$$M_{12}(\alpha)=\int \psi_2^\dagger(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}(\boldsymbol{\alpha}\cdot\mathbf{e})\psi_1(\mathbf{r})d^3r, \quad (2)$$

for the case in which an electron in state ψ_1 absorbs a photon of momentum $\mathbf{k}$ and polarization $\mathbf{e}$, thereby going into the final state ψ_2, a state of the continuum. We have indicated that M_{12} depends on α, for we restrict our considerations to the case in which both ψ_1 and ψ_2 describe electrons in a Coulomb field: $V(r)=-\alpha/r$. M_{12}^*, of course, describes the emission of such a photon, accompanied by an electronic transition from ψ_2 to ψ_1. If ψ_1 is taken to be a continuum rather than a bound state, M_{12}^* is the bremsstrahlung matrix element.

For $\psi_2(\mathbf{r})$, the continuum Coulomb state of asymptotic momentum $\mathbf{p}_2$, we shall use the first Born-approximation wave function. In coordinate space we write this 4-spinor in the "split notation" as

$$\psi_2(\mathbf{r})=(E_2+1)^{-1}[e^{i\mathbf{p}_2\cdot\mathbf{r}}+\alpha F(\mathbf{r};\mathbf{p}_2)]\begin{bmatrix}(E_2+1)w_2(\mathbf{p}_2)\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2(\mathbf{p}_2)\end{bmatrix}, \quad (3)$$

where $w_2(\mathbf{p}_2)$ is a 2-spinor describing the polarization of the state, and $\boldsymbol{\sigma}$ is the Pauli matrix. The function $F(\mathbf{r};\mathbf{p}_2)$ is actually a Dirac matrix operator, described most simply in terms of its Fourier transform. If in general we define the Fourier transform $\varphi(\mathbf{p})$ of a function $\psi(\mathbf{r})$ as

$$\varphi(\mathbf{p})=(2\pi)^{-\frac{3}{2}}\int e^{-i\mathbf{p}\cdot\mathbf{r}}\psi(\mathbf{r})d^3r, \quad (4)$$

the transform of $\psi_2(\mathbf{r})$ is

$$\varphi_2(\mathbf{p})=(2\pi)^{\frac{3}{2}}\left[\delta^3(\mathbf{p}-\mathbf{p}_2)+\frac{\alpha}{2\pi^2}\frac{\boldsymbol{\alpha}\cdot\mathbf{p}+\beta+E_2}{(p^2-p_2^2)(\mathbf{p}-\mathbf{p}_2)^2}\right]\begin{bmatrix}(E_2+1)w_2\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2\end{bmatrix}. \quad (5)$$

It will also be convenient to write ψ_1 in the split notation,

$$\psi_1(\mathbf{r})=\begin{bmatrix}u_1(\mathbf{r})\\v_1(\mathbf{r})\end{bmatrix}. \quad (6)$$

Then M_{12} can be written as the sum of two terms, corresponding to the two terms of ψ_2:

$$(E_2+1)M_{12}\equiv M_A+\alpha M_B, \quad (7)$$

with

$$M_A(\alpha)=\int\begin{bmatrix}(E_2+1)w_2\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2\end{bmatrix}^\dagger(\boldsymbol{\alpha}\cdot\mathbf{e})\begin{bmatrix}u_1(\mathbf{r})\\v_1(\mathbf{r})\end{bmatrix}e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r, \quad (8a)$$

$$M_B(\alpha)=\int\begin{bmatrix}(E_2+1)w_2\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2\end{bmatrix}^\dagger F^\dagger(\mathbf{r};\mathbf{p}_2)(\boldsymbol{\alpha}\cdot\mathbf{e})\begin{bmatrix}u_1(\mathbf{r})\\v_1(\mathbf{r})\end{bmatrix}e^{i\mathbf{k}\cdot\mathbf{r}}d^3r, \quad (8b)$$

with $\mathbf{q}=\mathbf{p}_2-\mathbf{k}$. Note that the α-dependence of M_A and M_B comes entirely from ψ_1.

For the K-shell photoeffect, ψ_1 is the $\kappa=-1$ state, with s and p components, which is conveniently written in the split notation as

$$\psi_1(\mathbf{r})=N\begin{bmatrix}(1+\gamma_1)^{\frac{1}{2}}s\\i(1-\gamma_1)^{\frac{1}{2}}(\boldsymbol{\sigma}\cdot\hat{r})s\end{bmatrix}r^{\gamma_1-1}e^{-\alpha r},$$
$$N=(2\alpha)^{\gamma_1+\frac{1}{2}}[8\pi\gamma_1\Gamma(1+2\gamma_1)]^{-\frac{1}{2}}, \quad (9)$$
$$\gamma_1=(1-\alpha^2)^{\frac{1}{2}},$$

where s is a 2-spinor independent of r and α which describes the polarization of the state; $s^\dagger s=1$; and $\hat{r}$ is a unit vector parallel to $\mathbf{r}$.

Consider first $M_A(\alpha)$, the part of the matrix element which comes from the unperturbed outgoing plane-wave state. Inserting Eq. (9) for ψ_1, the two spatial integrals encountered are

$$U(\alpha)=\int r^{\gamma_1-1}e^{-\alpha r}e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r, \quad (10a)$$

$$V(\alpha)=\int(\boldsymbol{\sigma}\cdot\mathbf{r})r^{\gamma_1-1}e^{-\alpha r}e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r, \quad (10b)$$

which are just the Fourier transforms, with respect to $\mathbf{q}$, of relatively simple functions of $\mathbf{r}$. Since in ψ_2 we have already neglected terms of order α^2 and higher, we wish to expand $U(\alpha)$ and $V(\alpha)$ in powers of α, if possible, and retain only terms of less than second order. A particularly attractive and simple way of doing this would be to expand $e^{-\alpha r}$ in the usual Taylor's series (valid for all r), and integrate term by term, so that the result would automatically be a power series in α if this term-by-term integration could be justified.[9]

[9] This is strictly true if the α which appears in $\gamma_1=(1-\alpha^2)^{\frac{1}{2}}$ is neglected. The most consistent way of discussing the situation is to consider the α which appears in γ_1 as an entirely different parameter, say α', from the α in $e^{-\alpha r}$. Then U, e.g., depends on both α and α', and has a power series expansion in α of the form

$$U(\alpha,\alpha')=\Sigma_n f_n(\alpha')\alpha^n.$$

It is this series that we shall call the "expansion of $U(\alpha)$," and we shall be interested in it at $\alpha'=\alpha$. With this note of explanation as justification, we shall continue to use this slightly inexact but very convenient terminology. Incidentally, we shall meet an entirely analogous situation when we discuss bremsstrahlung.

Since a similar approach to a considerably more complicated integral is very useful in discussing the bremsstrahlung matrix element, we have taken some pains in Appendix A to investigate the existence, form, and range of convergence of the power series expansions of functions of the form of $U(\alpha)$ and $V(\alpha)$. The existence of such a power series is equivalent to the existence of a region of regularity about $\alpha=0$ for the functions in question. Consequently we have considered the basic question to be that of the analytic properties of these functions in the neighborhood of the origin. Since Eq. (10) clearly defines $U(\alpha)$ and $V(\alpha)$ only for $\mathrm{Re}(\alpha)>0$, it is necessary to investigate the possibility of analytically continuing them into the left half of the α-plane. This analytic continuation, into the pocket about the origin shown in Fig. 1, is explicitly constructed in Appendix A, and the singularity nearest the origin is found to lie on the semicircular boundary of the pocket. By this means these functions are shown to possess convergent Taylor's expansions about $\alpha=0$, with known, finite radii of convergence. Finally, by explicitly calculating the coefficients of the expansions, they are found to be exactly what one would obtain by formally inserting the expansion of $e^{-\alpha r}$ and the integrating term by term (including a convergence factor $e^{-\epsilon r}$ to assure the convergence of the resulting integrals).

The results of this Appendix are summarized in Theorem (A-2), which states that the circle of convergence of the Taylor's expansions is $|\alpha|<q$ in both cases, that the integral can be evaluated by a term-by-term integration of the power series representation for $e^{-\alpha r}$, and that the result for any individual term of the form $r^\lambda Y_{lm}(\hat{r})$ is

$$\lim_{\epsilon\to 0}(2\pi)^{-\frac{3}{2}}\int e^{-\epsilon r}r^\lambda Y_{lm}(\hat{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r = (-i)^l S_l(\lambda+2)q^{-(\lambda+3)}Y_{lm}(\hat{q}), \quad (11)$$

for $q\neq 0$, where the numerical coefficient $S_l(\lambda+2)$ is given by Eq. (A-5). One of its important properties is that

$$S_l(\lambda+2)=0, \quad \text{if} \quad \lambda=l(\mathrm{mod}\ 2). \quad (12)$$

Using these results, we can evaluate $U(\alpha)$, the S-wave integral, with sufficient accuracy by making the replacement

$$e^{-\alpha r}\approx 1-\alpha r. \quad (13)$$

From the first term, we get a coefficient $S_0(\gamma_1+1)$. Recalling that $\gamma_1=(1-\alpha^2)^{\frac{1}{2}}$, we can use the approximation $\gamma_1\approx 1$, since we are neglecting α^2. But in this approximation [$S_l(\rho)$ is an analytic function of ρ], the coefficient is $S_0(2)$, which is zero by Eq. (12): the leading term vanishes, and the photoeffect matrix element is thus of first order in α. This result has a direct analog in the calculation of the Born-approximation matrix element for bremsstrahlung, where *both* wave functions have only zeroth-order and first-order terms; the integral coming from the zeroth-order terms of both functions vanishes by momentum conservation, and the lowest-order integrals then come from the two "cross terms."[10]

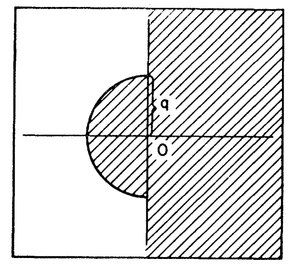

FIG. 1. Region of analyticity, in the α plane, of $U(\alpha)$ and $V(\alpha)$.

The next term of (13) does contribute and gives

$$U(\alpha)\approx -\alpha(2\pi)^{\frac{1}{2}}S_0(\gamma_1+2)q^{-(\gamma_1+3)} \approx -\alpha(2\pi)^{\frac{1}{2}}S_0(3)q^{-4}=8\pi\alpha q^{-4}. \quad (14)$$

In evaluating $V(\alpha)$, the p-wave or "small component" integral, we need to keep only the *first* term of $e^{-\alpha r}$, for in ψ_1 this part of the spinor contains the factor $(1-\gamma_1)^{\frac{1}{2}}\approx\alpha/\sqrt{2}$, which is already first-order in α. Equation (11) then gives for this integral

$$\begin{aligned} V(\alpha)&\approx -i(2\pi)^{\frac{1}{2}}S_1(\gamma_1+1)q^{-(\gamma_1+2)}\boldsymbol{\sigma}\cdot\hat{q} \\ &\approx -i(2\pi)^{\frac{1}{2}}S_1(2)q^{-3}\boldsymbol{\sigma}\cdot\hat{q} \\ &= -8\pi i q^{-4}\boldsymbol{\sigma}\cdot\mathbf{q}. \end{aligned} \quad (15)$$

Inserting these results and Eq. (9) into Eq. (8a), $M_A(\alpha)$ correct through first order in α becomes

$$M_A(\alpha)\approx\alpha^{\frac{3}{2}}4\pi^{\frac{1}{2}}\alpha q^{-4}\begin{bmatrix}(E_2+1)w_2\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2\end{bmatrix}^\dagger(\boldsymbol{\alpha}\cdot\mathbf{e})\begin{bmatrix}2s\\(\boldsymbol{\sigma}\cdot\mathbf{q})s\end{bmatrix}, \quad (16)$$

using $N\approx\alpha^{\frac{3}{2}}(2\pi)^{-\frac{1}{2}}$ to lowest order in α. Note that, because the leading term of the upper component from ψ_1 vanished in this approximation, the contributions from the upper and lower components of ψ_1 are both of order α, as Sauter[4] found in his original calculation.

Next we must evaluate M_B, the term coming from the perturbed part of the outgoing electron function. Inserting Eq. (9) into (8b), the integral involving the upper component of ψ_1 is

$$W(\alpha)=\int F^\dagger(\mathbf{r};\mathbf{p}_2)r^{\gamma_1-1}e^{-\alpha r}Y_{00}(\hat{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r. \quad (17)$$

The investigation of its regularity in a region about $\alpha=0$ is more complicated than the similar investigation of $U(\alpha)$, for this integrand contains the extra factor $F(\mathbf{r},\mathbf{p}_2)$, whose properties play an important role in the argument. Appendix B is devoted to this investigation, and the result is summarized in Theorem (B-2). The conclusion is that, provided F possesses certain detailed

[10] See, e.g., H. A. Bethe and E. E. Salpeter, *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 35, p. 326.

properties, $W(\alpha)$ does have a region of analyticity about the origin in the α-plane. This assures the existence of a Taylor's series for $W(\alpha)$, and within its circle of convergence this series is again found to be just what one would obtain by expanding $e^{-\alpha r}$ and formally integrating term-by-term.

A central role is played in this theorem by a function $G_{lm}(k,p;\mathbf{p}_2)$, defined in terms of $G(\mathbf{p};\mathbf{p}_2)$, the Fourier transform of $F(\mathbf{r};\mathbf{p}_2)$. $G_{lm}(k,p;\mathbf{p}_2)$ is the coefficient of $Y_{lm}(\hat{p})$ in the spherical-harmonic expansion of $G(\mathbf{k}-\mathbf{p};\mathbf{p}_2)$:

$$G_{lm}(k,p;\mathbf{p}_2)=\int G(\mathbf{k}-\mathbf{p};\mathbf{p}_2)Y_{lm}(\hat{p})d\Omega_p. \tag{18}$$

From Eq. (4), the dependence of $G(\mathbf{k}-\mathbf{p};\mathbf{p}_2)$ on the direction of the momentum-space variable $\mathbf{p}$ is given by the function

$$\frac{\boldsymbol{\alpha}\cdot(\mathbf{k}-\mathbf{p})+\beta+E_2}{[(\mathbf{p}-\mathbf{k})^2-p_2^2+i\epsilon][(\mathbf{p}+\mathbf{q})^2+i\epsilon]}, \tag{19}$$

with $\mathbf{q}=\mathbf{p}_2-\mathbf{k}$ as before. We have included the infinitesimal imaginary quantity $(i\epsilon)$ (as, e.g., in reference 10, p. 45) to assure that the singularities of this function do not occur for real p. This is equivalent to employing the Parseval's theorem of Appendix B in the form of Eq. (B-8), with $\epsilon_2\neq0$.

In order that $W(\alpha)$ have the desired Taylor's expansion, $G_{lm}(k,p;\mathbf{p}_2)$ must: (a) possess all its derivatives (with respect to p) on the real p axis; (b) be analytic in some region $|p|<R$ of the complex p plane; and (c) vanish at least as fast as p^{-2} as $p\to\infty$. We shall not prove these statements here, but it is possible to verify that all three are true, that in fact $R=|p_2-k|$, and that G_{lm} actually vanishes at least as fast as p^{-4} as $p\to\infty$.

Theorem (B-2) then states that, for $|\alpha|<|p_2-k|$, $e^{-\alpha r}$ can be expanded and the integral (17) done term by term, the result for any single term r^λ being the function $(-i)^l A_l(\lambda+2;m)$ given by (B-10) or (B-11). Since M_B already carries a factor α from the continuum function, we only need the *first* term of the bound-state function in this case, i.e., the replacement $e^{-\alpha r}\approx1$ is sufficient. Since $l=0$ in (17), the result is $A_0(\gamma_1+1;2)$, and expanding $\gamma_1,\gamma_1\approx1$, we have finally $W(\alpha)\approx A_0(2;2)$, through zeroth order in α, which according to Eq. (B-11) is

$$W(\alpha)\approx A_0(2;2)=(2\pi)^{\frac{1}{2}}G_{00}^\dagger(k,0;\mathbf{p}_2).$$

This is fortunate, for $G_{00}(k,0;\mathbf{p}_2)$ is easy to evaluate, and is in fact just $G(\mathbf{k};\mathbf{p}_2)/Y_{00}=(4\pi)^{\frac{1}{2}}G(\mathbf{k};\mathbf{p}_2)$, as can readily be inferred from the form of the expression (19): since $\mathbf{p}$ occurs only in dot-products with other vectors, if we think of $G(\mathbf{k}-\mathbf{p};\mathbf{p}_2)$ expanded in spherical harmonics of the direction $\hat{p}$, it is clear that at $p=0$, only the s-wave part will remain, which is just $Y_{00}G_{00}(k,0,\mathbf{p}_2)$. Consequently, using Eq. (5) for G, we have

$$\begin{aligned}W(\alpha)&\approx2\sqrt{2}\pi G^\dagger(\mathbf{k};\mathbf{p}_2)\\&=-4\pi^{\frac{1}{2}}\alpha\frac{\boldsymbol{\alpha}\cdot\mathbf{k}+\beta+E_2}{(p_2^2-k^2)(\mathbf{p}_2-\mathbf{k})^2}\begin{bmatrix}(E_2+1)w_2\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2\end{bmatrix}.\end{aligned} \tag{20}$$

Furthermore, the lower component of ψ_1 with its factor $(1-\gamma_1)^{\frac{1}{2}}\sim\alpha$ does not contribute at all in this first order approximation, because of the α from ψ_2, so $W(\alpha)$ is the only contribution to M_B. Incorporating it into Eq. (8b) and recalling that the $Y_{00}=(4\pi)^{-\frac{1}{2}}$ in $W(\alpha)$ does not actually occur in ψ_1, we have

$$M_B(\alpha)\approx-\alpha^{\frac{3}{2}}\frac{4\pi^{\frac{1}{2}}\alpha}{(p_2^2-k^2)q^2}\begin{bmatrix}(E_2+1)w_2\\(\boldsymbol{\sigma}\cdot\mathbf{p}_2)w_2\end{bmatrix}^\dagger\times(\boldsymbol{\alpha}\cdot\mathbf{k}+\beta+E_2)(\boldsymbol{\alpha}\cdot\mathbf{e})\begin{bmatrix}2s\\0\end{bmatrix}. \tag{21}$$

This completes our derivation of the Sauter-approximation matrix element for the K-shell photoelectric effect. The essence of what we have accomplished by this alternative method of derivation may be summarized rather loosely in the statement that the Sauter matrix element depends on only three very simple properties of the bound-state wave function, $\psi_1(\mathbf{r})$: on the magnitude, $g(r=0)$, and the first radial derivative, $g'(r=0)$, of the "large" component, and on the magnitude, $f(r=0)$, of the "small" component, all evaluated at $r=0$. This statement is "loose" in the sense that the factor r^{γ_1-1}, with $\gamma_1=(1-\alpha^2)<1$, which appears both in $f(r)$ and $g(r)$, makes all three of these quantities infinite. However, if α^2, and hence this entire factor, is neglected throughout, the statement is exact.

These quantities are actually the first two coefficients in the expansion of ψ_1 in powers of α, as is readily verified. The approximate wave function obtained by retaining only these two terms is, from Eq. (9),

$$\tilde{\psi}_1(\mathbf{r})\equiv\alpha^{\frac{3}{2}}(2\pi)^{-\frac{1}{2}}\begin{bmatrix}2(1-\alpha r)s\\\alpha i(\boldsymbol{\sigma}\cdot\hat{r})s\end{bmatrix}. \tag{22}$$

This function, obtained by making the approximations $e^{-\alpha r}\approx1-\alpha r$ (upper component) and $e^{-\alpha r}\approx1$ (lower component), is not intended to have any significance in itself; its behavior for large r, e.g., is entirely meaningless. Its significance in the present problem is seen only through the work of Appendixes A and B, which demonstrate that the desired expansion of $M_{12}(\alpha)$ can formally be regarded as the result of integrating the corresponding expansion of ψ_1 (of which $\tilde{\psi}_1$ is the first two terms) term by term.[11]

[11] In this sense $\tilde{\psi}_1(r)$ is essentially what Lighthill calls a "generalized function." See M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge University Press, Cambridge, 1958).

The Sauter matrix element for the photoeffect, which employs ψ_1 in this way, is thus seen to be entirely analogous to the Bethe-Heitler matrix element for bremsstrahlung. For, (a) it is the sum of two terms, M_A and M_B, corresponding to the two terms of the Born approximation wave function for the outgoing electron; (b) the leading ("zeroth-order") term of the matrix element vanishes because of energy-momentum conservation; and (c) the remaining contribution is the sum of the two "cross terms," each containing the zeroth-order term of one wave function and the first-order term of the other.

Finally, it should be realized that our use of the Born-approximation wave function for the continuum state ψ_2 makes this entire calculation invalid as $p_2 \to 0$, as was emphasized by Sauter.[4] The above method of derivation shows that, in fact, there is an additional restriction upon the range of validity of the Sauter approximation. As we have seen, the matrix element calculated from the Born-approximation ψ_2 and the exact ψ_1 has a power series expansion in α, whose radius of convergence is finite and equal to $q_{\min} = |p_2 - k|$. Neglecting the binding energy of the K-shell, this means that the series converges only if $2k(E_2 - p_2) > \alpha^2$. Thus no matter how many terms of this series are retained, it cannot be used at energies lower than this limit. Since the Sauter approximation keeps only the first term of the series, it is valid only if we are *well* within the circle of convergence, i.e., only if $2k(E_2 - p_2) \gg \alpha^2$. From the kinematics of the problem, this restriction can be seen to be essentially $\alpha/\beta_2 \ll 1$, i.e., the same as the restriction imposed by the use of the Born-approximation wave function itself.

BREMSSTRAHLUNG

We can now make immediate use of this discussion of the photoeffect to give a very simple derivation of the Sauter-approximation matrix element for bremsstrahlung at the spectrum tip, and to prove that it differs from the Sauter photoeffect matrix element only by a constant factor.

As was mentioned earlier, the fact that the energy of a bound state in the Coulomb field differs from m only by terms of order α^2 or higher suggests that a bound-state wave function of given angular momentum and parity will agree with the corresponding zero momentum continuum function, through first order in α. This can be demonstrated directly, and the comparison is made especially simple if we employ the notation of Akhiezer and Berestetsky,[12] who give both the bound and continuum Coulomb functions in exactly the same form. Their notation is quite standard, and in particular $\gamma = (\kappa^2 - \alpha^2)^{\frac{1}{2}}$, where κ is the usual Dirac quantum number; for the principal quantum number in the bound-state case they use n, with $n = |\kappa|$, $|\kappa|+1$, etc. Their bound-state functions are normalized to unity, and the continuum functions "per unit energy."

TABLE I. Radial functions for the bound (B) and threshold (T) states.

(κ)	$(\alpha m/n)^{-\frac{3}{2}} g_B(r)$	$(\alpha m^2)^{-\frac{1}{2}} g_T(r)$
(-1)	$2(1-\alpha mr)$	$2(1-\alpha mr)$
$(+1)$	$[(n^2-1)/n^2]^{\frac{1}{2}} \frac{2}{3}\alpha mr$	$\frac{2}{3}\alpha mr$
(-2)	$[(n+1)/n^3(n-1)]^{\frac{1}{2}} \frac{2}{3}\alpha mr$	$\frac{2}{3}\alpha mr$
	$(\alpha m/n)^{-\frac{3}{2}} f_B(r)$	$(\alpha m^2)^{-\frac{1}{2}} f_T(r)$
(-1)	$-\alpha$	$-\alpha$
$(+1)$	$-[(n^2-1)/n^2]^{\frac{1}{2}}\alpha$	$-\alpha$

We need the limit of the continuum function as $p \to 0$, which we shall call the "threshold function." The limit is facilitated by a form of Stirling's approximation,[13]

$$\lim_{|p|\to 0} |\Gamma(\gamma + i\alpha E/p)| e^{\pi|\alpha E/2p|} |p|^{\gamma - \frac{1}{2}} = (2\pi)^{\frac{1}{2}} (\alpha m)^{\gamma - \frac{1}{2}}. \quad (23)$$

Also, one of the factors which appears in the continuum function for general asymptotic momentum p is $e^{i\eta}(\gamma + i\alpha E/p)$; this can be written more conveniently as

$$p^{-1}\{-(\kappa/|\kappa|)[\tfrac{1}{2}(E-m)(\kappa-\gamma)(E\kappa - m\gamma)]^{\frac{1}{2}} + i[\tfrac{1}{2}(E+m)(\kappa+\gamma)(E\kappa-m\gamma)]^{\frac{1}{2}}\} \approx -\tfrac{1}{2}(\kappa-\gamma) + i\alpha m/p, \quad \text{if} \quad p \ll m, \quad (24)$$

a form well suited for considering the limit $p \to 0$.

The radial functions $g(r)$ and $f(r)$ of the "large" and "small" components, respectively, of the continuum function have power series expansions in r of the form

$$g(r) = r^{\gamma-1} \sum_{n=0}^{\infty} a_n(p,\alpha) r^n,$$
$$f(r) = r^{\gamma-1} \sum_{n=0}^{\infty} b_n(p,\alpha) r^n. \quad (25)$$

It can readily be verified that $a_n(0,\alpha)$ and $b_n(0,\alpha)$ have power series expansions in α, with leading term α^m, where m is linearly related to n. In other words, if we take the limit $p \to 0$ term-by-term, the series in r becomes a series in (αr), of which we need keep only the first few terms for our purposes. In this way we obtain the results shown in Table I, correct only through first order in α, for the bound (B) and threshold (T) states. All higher partial waves are proportional to at least the second power of α [because of the factor $\alpha^{\gamma-\frac{1}{2}}$ of Eq. (23)] and so are of no interest to us. The proportionality between the corresponding first-order bound and threshold functions is clear, the constant

[12] A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics*, a translation of *Kvantovaya Elektrodinamika* (Gosudarstvennoe Izdatelstvo Tekhniko-Teoreticheskoi Literatury, Moskva, 1953); available from Technical Services, Department of Commerce, Washington 25, D. C.

[13] *Bateman Manuscript Project*, edited by H. Erdelyi (McGraw-Hill Book Company, New York, 1953), Vol. I, p. 47.

of proportionality being just the ratio of normalization constants.

These "wave functions" have meaning in exactly the sense which the "approximate function" $\tilde{\psi}_1$ of Eq. (22) has meaning—i.e., not as point functions, but only in terms of integrals over them.[11] For the bound states, we have already shown that the power series in r, of which these functions are truncations, can formally be substituted for ψ_1 in the integral of Eq. (2) and the integral done term by term to give an expansion of the photoeffect matrix element in powers of α. If we can establish the same for the threshold function series, which enters the tip-bremsstrahlung matrix element, we can compare the two matrix elements directly from the entries in Table I.

It is inconvenient to establish this directly, by the method used for the photoeffect matrix element. The investigation of the region of regularity of $M_A(\alpha)$ and $M_B(\alpha)$ for the photoeffect employed integral representations extensively; the threshold functions are less amenable to this treatment, for they are Bessel functions of argument $(r)^{\frac{1}{2}}$—and this leads to integral representations which are unmanageable by the techniques employed above. An alternative approach is to calculate the matrix element $M_{12}(\alpha)$ [Eq. (2)] for "inverse bremsstrahlung," using for $\psi_1(\mathbf{r})$ the exact continuum function of momentum $p_1\neq 0$. In this case the radial functions have the form $g(r)\sim r^{\gamma-1}e^{ip_1r}F(\gamma+1+i\alpha E/p_1,\, 2\gamma+1,\, 2ip_1r)$ [where $F(a,b,z)$ is the confluent hypergeometric function], and $f(r)$ is similar. These functions do have simple integral representations, and by using them we have shown in Theorems (A-2) and (B-2) of the Appendixes that $M_A(\alpha,p_1)$ and $M_B(\alpha,p_1)$ are analytic in p_1 for $|p_1|<|p_2-k|$ and can formally be calculated by a term-by-term integration of the series representation, Eq. (25) of the continuum function ψ_1. This result may be of considerable interest in itself,[14] but we will use it only as a means of obtaining $M_A(\alpha,0)$ and $M_B(\alpha,0)$. For $p_1\neq 0$, M_A and M_B contain p_1 only in the functions $a_n(p_1,\alpha)$ and $b_n(p_1,\alpha)$, which appear in the series resulting from the term-by-term integration of (25). But we obtained the $p_1\rightarrow 0$ limits of Table I by taking the limit of the series (25) term by term; clearly if we do the same for the present expansion of $M_{12}(\alpha)$, again only the first two terms at most will be of less than second order in α, and the result, through first order in α, will be exactly what one would obtain by the use of the "functions" of Table I. In other words, we can calculate tip-bremsstrahlung from the threshold states of Table I exactly as we calculated the photoeffect from the $\kappa=-1$ bound state of Table I, and the proportionality between the bound and threshold states implies exactly the same proportionality between the Sauter-approximation matrix elements, in the same angular momentum states, for the photoeffect and tip-bremsstrahlung.

For $p_1\neq 0$, the continuum function ψ_1 of bremsstrahlung must asymptotically be a momentum eigenstate, and thus contain all angular momentum substates. These partial waves are all present even in the $p_1\rightarrow 0$ limit, and our sole remaining task is to show that only the $\kappa=-1$ state contributes terms to $M_{12}(\alpha)$ which are less than second order in α. This, however, is trivial, if we use the results of Appendix A to do the integrals arising from the $\kappa=+1,\,-2$ functions of Table I. (They all contain a factor α, and so at worst can contribute to M_A, whose integrals are considered in Appendix A.) The $\kappa=+1$ $(p_{\frac{1}{2}})$ state integral has a factor $S_1(3)$ from the large component and a factor $S_0(2)$ from the small component, where $S_l(\rho)$ is defined in Eq. (A-5). But as is discussed there, $S_l(\rho)=0$ if $\rho=l(\text{mod } 2)$, so both these integrals are zero. Similarly, the integral from the upper component of the $\kappa=-2$ $(p_{\frac{3}{2}})$ state has the factor $S_1(3)$, which again is zero.

In summary, only the $\kappa=-1$ $(s_{\frac{1}{2}})$ state contributes to the tip-bremsstrahlung matrix element in the Sauter approximation. The matrix element can be calculated from the threshold functions of Table I exactly as the photoeffect matrix element was calculated from the bound state of Table I, and from the proportionality factor between the two wave functions we obtain our final result:

$$M_{T\text{-brems.}}(\alpha)=(1/\alpha m^{\frac{1}{2}})M_{\text{photo.}}{}^*(\alpha), \qquad (26)$$

in the Sauter approximation. It is in this exact sense that the two processes are inverse to each other, and, by detailed balancing, have identical angular distributions and polarization properties in this approximation.

ACKNOWLEDGMENTS

We wish to express our appreciation to Dr. J. Schwartz of the Institute of Mathematical Sciences, New York University, for many helpful suggestions regarding the mathematical treatment given in the Appendixes.

APPENDIX A. FOURIER-BESSEL TRANSFORMS[15]

We shall define the Fourier-Bessel transform of order l,[FBT(l)], of the function $f(x)$ as

$$g_l(p)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)j_l(px)x^2dx, \qquad \text{(A-1)}$$

provided the integral exists. If it does not, we use the definition

$$g_l(p)=\lim_{\epsilon\rightarrow 0+}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\epsilon x}f(x)j_l(px)x^2dx, \quad \text{(A-2)}$$

[14] For example in the investigation of bremsstrahlung near as well as at the spectrum limit [G. W. Ford and C. J. Mullin (private communication)].

[15] The calculations in these Appendixes were carried out by only one of the authors (K.W.M.)

provided *it* exists. In either case, $j_l(x)$ is the spherical Bessel function of (integral) order l.

This transform is of interest here because it is the "radial part" of the 3-dimensional Fourier transform of a function $F(\mathbf{r})$ which has the special form $F(\mathbf{r}) = f_l(r)Y_{lm}(\hat{r})$. This is easily seen by using the standard partial-wave expansion of a plane wave,

$$e^{i\mathbf{p}\cdot\mathbf{r}} = 4\pi \sum_{l,m} i^l j_l(pr) Y_{lm}(\hat{r}) Y_{lm}^*(\hat{p}),$$

from which the Fourier transform of $f_l(r)Y_{lm}(\hat{r})$ is

$$(2\pi)^{-\frac{3}{2}} \int f_l(r) Y_{lm}(r) e^{-i\mathbf{p}\cdot\mathbf{r}} d^3r = i^{-l} g_l(p) Y_{lm}(\hat{p}),$$

where $g_l(p)$ is just the FBT(l) of $f_l(r)$.

We are concerned in this Appendix with functions f of the special form $f(cx)$, c being a constant parameter, and wish to know under what conditions $f(cx)$ can be expanded as a power series in (cx) and integrated term-by-term, to get $g_l(c,p)$ in the form of a power series in c.

Problems of a similar nature, but without the complications of infinite series, have been considered recently in a very lucid fashion by Lighthill,[11] following the distribution-theory approach of L. Schwartz. Although Lighthill's work is exceedingly attractive, it unfortunately cannot be used directly here because it does not take advantage of the complex-variable techniques which are essential to the solution of our problem. However, the method of defining the transform of x^n, as well as several other devices we use, will be recognized as inspired by Lighthill's book.

Since the direct justification of term-by-term integration of such a series, as well as a determination of the radius of convergence of the resulting expansion in c, is very difficult, we shall use a different strategy. We first define the FBT(l) of $f(cx)$ as

$$g_l(c,p) = \lim_{\epsilon\to 0+} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\epsilon x} f(cx) j_l(px) x^2 dx. \quad \text{(A-3)}$$

For the function f in which we are interested, it will be possible by direct inspection to establish the analyticity (in c) of $g_l(c,p)$ in a region of the complex c-plane about $c=0$. Thus we know that a Taylor's series in c exists, and can find it by the usual methods. It is then possible to verify *a posteriori* that this series is exactly the one we would have obtained by a term-by-term transformation of the series for $f(cx)$.

In order to be able to recognize this fact, we must know the transform of a power of x, which we shall formulate as a lemma:

Lemma A-1.—The FBT(l) of x^λ $(\lambda > -1)$ is

$$\lim_{\epsilon\to 0} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{-\epsilon x} x^\lambda j_l(px) x^2 dx = \frac{S_l(\lambda+2)}{p^{\lambda+3}}, \quad p \neq 0, \quad \text{(A-4)}$$

with

$$S_l(\lambda+2) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\lambda+l+3)\Gamma[-(\lambda+l+2)/2]}{2^{l+1}\Gamma[(l-\lambda)/2]} \times \sin[\tfrac{1}{2}\pi(\lambda+l)]. \quad \text{(A-5)}$$

Note that if λ is integral, and $\lambda > l-2$, then $S_{\lambda+2}{}^l = 0$ if $(\lambda+l)$ is even; in this case there is a gamma-function pole in both numerator and denominator, which cancel, and the zero of the sine makes the whole expression zero.

For this proof, as well as all that follow, we shall employ an integral representation of the spherical Bessel function[16]:

$$j_l(z) = \frac{1}{2i^l} \int_{-1}^{1} e^{isz} P_l(s) ds, \quad \text{(A-6)}$$

and valid for all z; $P_l(s)$ is the Legendre polynomial of order l. This representation, in which z appears only as e^{isz}, is particularly convenient, for in an integral like (A-2) the convergence factor $e^{-\epsilon x}$ enables us to perform the x (or z) integration *first*, thus finding the Fourier transform (with respect to s) of $x^2 e^{-\epsilon x} f(x)$, which is then to be multiplied by $P_l(s)$ and integrated over s. By this device, many of the tricks commonly used for Fourier transforms can be carried over to Fourier-Bessel transforms, provided the subsequent s-integration can be carried out.

Since the integrand of (A-6) is analytic in s for all z, we can deform the contour of integration from the real axis in any way we like. In particular, we shall use

$$j_l(z) = \frac{-1}{2i^l} \int_\Gamma e^{isz} P_l(s) ds, \quad \text{(A-7)}$$

where Γ is the upper half of the unit circle in the positive sense, from $+1$ to -1.

Using this representation and interchanging the orders of integration, we have for the integral of (A-4),

$$-\frac{1}{2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_\Gamma P_l(s) ds \int_0^\infty dx\, e^{-(\epsilon - ips)x} x^{\lambda+2}.$$

Note that for s on Γ and $p>0$, $\mathrm{Re}(-ips) \geqslant 0$, so $\mathrm{Re}(\epsilon - ips) > 0$. But from the definition of the Γ-function we know that

$$\int_0^\infty x^\mu e^{-ax} dx = \frac{\Gamma(\mu+1)}{a^{\mu+1}}, \quad \text{provided } \mathrm{Re}(a) > 0,$$

and $\mu > -1$, so the above integral is

$$-\frac{\Gamma(\lambda+3)}{2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_\Gamma \frac{P_l(s)ds}{(\epsilon - ips)^{\lambda+3}} \to -\frac{\Gamma(\lambda+3)}{2i^{(l-\lambda-3)} p^{\lambda+3}} \times \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_\Gamma P_l(s) s^{-\lambda-3} ds,$$

[16] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, New York, 1953), p. 1575.

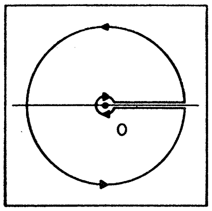

FIG. 2. Contour employed in the evaluation of the integral in Eq. (A-8).

as $\epsilon \to 0$. Thus we need only evaluate the indicated s integral. A convenient way of doing this is as follows. By Rodrigues' formula,

$$P_l(z)=\frac{(-1)^l}{2^l l!}\left(\frac{d}{dz}\right)^l(1-z^2)^l.$$

We can insert this in our integral and integrate by parts l times, noting that the integrand vanishes at the end points each time, and obtaining

$$\int_\Gamma P_l(z)z^{-\lambda-3}dz=\frac{(-1)^l}{2^l l!}\frac{\Gamma(\lambda+l+3)}{\Gamma(\lambda+3)}\times\int_\Gamma(1-z^2)^l z^{-\lambda-l-3}dz.$$

$$=\frac{(-1)^l}{2^{l+1}l!}\frac{\Gamma(\lambda+l+3)}{\Gamma(\lambda+3)}\times\oint(1-w)^l w^{-\frac{1}{2}(\lambda+l+2)-1}dw,$$

the latter integral, completely around the unit circle, coming from the change of variable $z^2=w$.

Now consider the integral

$$\int(1-z)^l z^{\alpha-1}dz=0,$$

around the closed contour C of Fig. 2, under the assumption $\mathrm{Re}(\alpha)>0$, $l\geqslant 0$ and integral. The contribution of the small circle vanishes as its radius $\to 0$; z^α is multiple-valued, and if we take it as real for z real and increasing from zero to 1, and recognize the integral over this interval as a beta function, we get

$$\oint_{|z|=1}(1-z)^l z^{\alpha-1}dz=\frac{\Gamma(l+1)\Gamma(\alpha)}{\Gamma(l+1+\alpha)}(e^{2\pi i\alpha}-1).$$

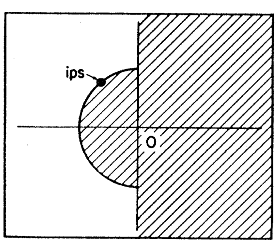

FIG. 3. Region of analyticity of $T_{\lambda+2}{}^l(c,p)$ in the c plane.

This was established for $\mathrm{Re}(\alpha)>0$, but by analytic continuation clearly holds for any α not a negative integer. Then using $i^{(\lambda+2-l)}=(-1)^l e^{\frac{1}{2}i\pi(\lambda+l)}$, we finally obtain

$$\frac{\Gamma(\lambda+3)}{2i^{(l-\lambda-3)}}\int_\Gamma P_l(z)z^{-\lambda-3}dz=\frac{\Gamma(\lambda+l+1)\Gamma[-(\lambda+l+2)/2]}{2^{l+1}\Gamma[(l-\lambda)/2]}\times\sin[\tfrac{1}{2}\pi(\lambda+l+2)]. \quad \text{(A-8)}$$

This is valid for λ nonintegral; however, the numerator contains a pole only if $(\lambda+l+2)$ is an even integer, and in this case the zero from the sine cancels the pole, so by analytic continuation, it holds for all λ, and thus establishes Lemma A-1. As we noted earlier, if $p>0$ and $n>l-2$, the FBT(l) of x^n is zero if $n=l$ (mod 2). As can readily be checked, if $l=0$ (A-4) agrees with Lighthill's result[17] for the sine transform of $x^{\lambda+1}$, as it should.

The simplest function of the form $f(cx)$ which we wish to consider is $x^\lambda e^{-cx}$. Its FBT(l) can be written as follows:

Lemma A-2.—Let $\mathrm{Re}(c)\geqslant 0$. Then FBT($l$) of $x^\lambda e^{-cx}$ is analytic in c for $0\leq|c|<p$, and in this circle has the Taylor's expansion

$$T_{\lambda+2}{}^l(c,p)\equiv\lim_{\epsilon\to 0}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-(\epsilon+c)x}x^{\lambda+2}j_l(px)dx$$

$$=\sum_{n=0}^\infty\frac{S_l(\lambda+n+2)}{p^{\lambda+n+3}}\frac{(-c)^n}{n!}, \quad \text{(A-9)}$$

which is clearly what would be obtained by transforming the series for $x^{\lambda+2}e^{-cx}$ term by term.

Using the same integral representation as before, the x integration gives again

$$T_{\lambda+2}{}^l(c,p)=-\lim_{\epsilon\to 0}\frac{\Gamma(\lambda+3)}{2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\Gamma P_l(s)\times(\epsilon+c-ips)^{-\lambda-3}ds.$$

Now for s on Γ, the upper half of the unit circle, and $p>0$, (ips) is a point on the *left* half of the circle about the origin with radius p. Then for $\mathrm{Re}(c)\geqslant 0$, $\mathrm{Re}(\epsilon+c)>0$, and so the integrand can have no singularities for c in the right half-plane, so $T_{\lambda+2}{}^l(c,p)$ is analytic in c in this region. In fact since (ips) is on the left half of the circle of radius p, the region of analyticity in c also includes the region $|c|<|ps|=|p|$, i.e., a pocket of radius p about the origin in the c-plane, as shown in Fig. 3. Thus $T(c,p)$ has a Taylor's series in c about $c=0$, with radius of convergence $|c|=p$.

[17] See reference 13, p. 43.

We can find its coefficients by differentiating the above integral under the integral sign (which is justified since the integral is uniformly convergent in c inside the circle of radius p), giving

$$\left(\frac{\partial}{\partial c}\right)^n T_{\lambda+2}{}^l(c,p) = -\lim_{\epsilon\to 0}\frac{(-1)^n}{2i^l}\Gamma(\lambda+n+3)\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \times\int_\Gamma P_l(s)(\epsilon+c-ips)^{-(\lambda+n+3)}ds.$$

Letting c and $\epsilon\to 0$, we find as before,

$$\left(\frac{\partial}{\partial c}\right)^n T_{\lambda+2}{}^l(c,p)\,|_{c=0} = (-1)^n\frac{S_l(\lambda+n+2)}{p^{\lambda+n+3}},$$

which establishes the Taylor's expansion given by (A-9). The radius of convergence $R=p$ of this series is determined, of course, by the end points $s=\pm 1$ of the contour Γ.

With the help of these two lemmas, we can readily obtain our desired result, the transform of the radial Coulomb wave function.

Theorem A-1.—Let $\mathrm{Re}(b)>\mathrm{Re}(a)>0$, $\mathrm{Re}(c)\geqslant 0$, $\mathrm{Re}(d)\leqslant\mathrm{Re}(c)$, $p>0$. Then

$$G_{\lambda+2}{}^l(c,d;\,p,a,b)\equiv \lim_{\epsilon\to 0}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-(\epsilon+c)x}x^\lambda F(a,b,dx)j_l(px)x^2dx$$

$$=\sum_{n=0}^\infty\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}T_{\lambda+n+2}{}^l(c,p)d^n \qquad \text{(A-11)}$$

$$=\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}\frac{S_l(\lambda+m+n+2)}{p^{\lambda+m+n+3}}\frac{(-1)^m}{m!}c^md^n, \qquad \text{(A-12)}$$

where $T_\rho{}^l$ is defined in Lemma (A-2) and $S_l(\rho)$ in Lemma (A-1). The region of convergence of the series is at least $|c|<p/2$, $|d|<p/2$ for c and d unrelated, but in the special case $c=d$, it is $|c|<p$. It converges to the given integral if $\mathrm{Re}(c)\geqslant 0$ and $\mathrm{Re}(c)\geqslant\mathrm{Re}(d)$.

It is clear from the expansion

$$F(a,b,dx)=\sum_{n=0}^\infty\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}(dx)^n,$$

and the definition of $T_\rho{}^l$, that the form (A-11) is exactly what one would obtain by expanding the confluent hypergeometric function (but not e^{-cx}) and doing the integral term by term. The final form (A-12) is thus exactly the double series which would result from expanding *both* e^{-cx} and $F(a,b,dx)$ in powers of x and integrating the resulting double series term by term.

This result may be obtained by the use of the standard integral representation for the confluent hypergeometric function,

$$F(a,b,dx)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{dxt}t^{a-1}(1-t)^{b-a-1}dt,$$

valid for all x and for $\mathrm{Re}(b)>\mathrm{Re}(a)>0$. This representation is particularly convenient, for the x dependence of the integrand is exponential. Consequently, by doing the t integration last, the other manipulations become very similar to those used in the proof of Lemma (A-2). Using the same integral representation as before for the spherical Bessel function, and recalling that the assumptions made guarantee that $\mathrm{Re}(\epsilon+c-dt-ips)>0$, we again do the x integration first:

$$G_{\lambda+2}{}^l(c,d;\,p,a,b) = -\lim_{\epsilon\to 0}\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\frac{1}{2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \times\int_\Gamma P_l(s)ds\int_0^1 t^{a-1}(1-t)^{b-a-1}dt \times\int_0^\infty x^{\lambda+2}e^{-(\epsilon+c-td-ips)x}dx$$

$$= -\lim_{\epsilon\to 0}\frac{\Gamma(b)\Gamma(\lambda+3)}{\Gamma(a)\Gamma(b-a)}\frac{1}{2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \times\int_\Gamma P_l(s)ds\int_0^1 t^{a-1}(1-t)^{b-a-1} \times(\epsilon+c-td-ips)^{-\lambda-3}dt. \qquad \text{(A-13)}$$

Since $\mathrm{Re}(\epsilon+c-td)>0$ and $\mathrm{Re}(ips)<0$, the integrand, and by uniform convergence the resulting integral, are analytic functions of d for $\mathrm{Re}(d)\leq\mathrm{Re}(c)$, provided $\mathrm{Re}(c)\geqslant 0$. Recalling that (ips) is a point on the left half of the circle about the origin of radius p, it is clear that, as in Lemma (A-2), we can use (A-13) as a means of analytically continuing this function of d into a pocket about the origin in the d-plane. All that is necessary is to be sure that no zeros of $(c-td-ips)$ are encountered, which will be the case if we require, e.g., that $|c-td|<p$. The simplest and most symmetric way of assuring this is to require that

$$|c|<p/2 \quad\text{and}\quad |d|<p/2,$$

although this is clearly not the only set of regions which will do. If on the other hand we consider the special case $c=d$, then if we wish to guarantee that $|c-tc|=|c|(1-t)<p$, we must clearly have $|c|<p$; it is this special case with which we shall be most concerned in our applications.

In either case, the function at hand is thus seen to have a Taylor's expansion in d about $d=0$, so again we wish its derivatives at $d=0$. Differentiating n times under the integral sign, we obtain as before.

$$\left(\frac{\partial}{\partial d}\right)^n G_{\lambda+2}{}^l(c,d;\,p,a,b)=$$

$$-\lim_{\epsilon\to 0}\frac{\Gamma(b)\Gamma(\lambda+n+3)}{\Gamma(a)\Gamma(b-a)2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\Gamma P_l(s)ds$$

$$\times\int_0^1 t^{a+n-1}(1-t)^{b-a-1}(\epsilon+c-td-ips)^{-(\lambda+n+3)}dt.$$

Letting $d\to 0$, the t integration conveniently becomes independent of the s integration. Recognizing it as a beta function,

$$\int_0^1 t^{a+n-1}(1-t)^{b-a-1}dt=\frac{\Gamma(a+n)\Gamma(b-a)}{\Gamma(b+n)},\quad\text{(A-14)}$$

we have

$$\left(\frac{\partial}{\partial d}\right)^n G_{\lambda+2}{}^l(c,0;\,p,a,b)$$

$$=\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)}(-1)\lim_{\epsilon\to 0}\frac{1}{2i^l}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_\Gamma P_l(s)$$

$$\times(\epsilon+c-ips)^{-(\lambda+n+3)}ds,$$

which by (A-10) gives just the stated Taylor's series, (A-11). Either setting $a=0$, or $a=b$ and $c=0$, reproduces Lemma (A-2), and $a=0$, $c=0$ reproduces Lemma (A-1), as it must.

Finally, the connection between Fourier-Bessel transforms and three-dimensional Fourier transforms enables us to state this theorem in terms of a three-dimensional Fourier transform, for the case $c=d$ in which we are most interested:

Theorem (A-2).—Let $\mathrm{Re}(b)>\mathrm{Re}(a)\geqslant 0$, $\mathrm{Re}(c)\geqslant 0$, and $p>0$. Then the following expansion of the Fourier transform of $r^\lambda e^{-cr}F(a,b,cr)Y_{lm}(\hat{r})$ holds:

$$\lim_{\epsilon\to 0}(2\pi)^{-\frac{3}{2}}\int e^{-\epsilon r}r^\lambda e^{-cr}F(a,b,cr)Y_{lm}(\hat{r})e^{-i\mathbf{p}\cdot\mathbf{r}}d^3r$$

$$=i^{-l}Y_{lm}(\hat{p})\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}$$

$$\times\frac{S_l(\lambda+m+n+2)}{p^{\lambda+m+n+3}}\frac{(-1)^m}{m!}c^{m+n}.\quad\text{(A-15)}$$

The series converges for $|c|<p$, and converges to the integral if $\mathrm{Re}(c)\geqslant 0$. This is exactly the expansion which would be obtained by expanding both e^{-cr} and $F(a,b,cr)$ in powers of (cr), and integrating the resulting double series term by term.

APPENDIX B. INTEGRALS OVER FOURIER-BESSEL TRANSFORMS

The purpose of this Appendix is to consider the Taylor's expansion, in the parameter c, of a radial integral which commonly occurs in matrix-elements involving Coulomb wave functions,

$$f_{\lambda+2}(c;\,a,b)\equiv\lim_{\epsilon\to 0+}\int_0^\infty e^{-\epsilon x}x^\lambda e^{-cx}F(a,b,cx)F(x)x^2dx,\quad\text{(B-1)}$$

where $F(a,b,z)$ is the confluent hypergeometric function, $\lambda>-2$, $\mathrm{Re}(c)\geqslant 0$, and $F(x)$ is such as to assure the convergence of the integral. In order that the Taylor's expansion exist and can be found explicitly, we shall need to impose further restrictions on $F(x)$ and its Fourier-Bessel transforms. Provided these conditions are satisfied, the Taylor's expansion will be found to have exactly the form one would get by inserting the expansions

$$F(a,b,cx)=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}(cx)^n,$$

$$e^{-cx}=\sum_{m=0}^{\infty}\frac{(-cx)^m}{m!},\quad\text{(B-2)}$$

and integrating the resulting double series term by term.

The functions $F(x)$ of interest here are bounded as $x\to\infty$, but do not necessarily vanish there, so that in general we can take the limit $\epsilon\to 0$ only after performing the integration. Although we cannot actually do the integration without knowing the exact form of $F(x)$, it is possible to transform it into an integral over the FBT(l)'s of the functions involved, and we shall then find, as in Appendix A, that we can let $\epsilon\to 0$ before performing this integration. The net result is that the coefficients in the desired Taylor's expansion in c are obtained in terms of integrals over the FBT(l) of $F(x)$—a form which is particularly convenient in the problem at hand.

We first need certain properties of these transforms. Let $xf(x)$ be absolutely integrable on the interval $(0,\infty)$; then we take as the FBT(l) of $f(x)$ the function

$$g_l(p)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(x)j_l(px)x^2dx,\quad\text{(B-3)}$$

and under this assumption the inversion theorem holds,[18]

$$f(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty g_l(p)j_l(px)p^2dp.\quad\text{(B-4)}$$

[18] This is a special case of Hankel's inversion theorem. See, e.g., *Bateman Manuscript Project*, edited by H. Erdelyi (McGraw-Hill Book Company, New York, 1953), Vol. II, p. 73.

Then by evaluating the double integral

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty\int_0^\infty x^2dxp^2dpf^1(x)g^2(p)$$

in two different ways, we immediately have the very useful Parseval's theorem,

$$\int_0^\infty f^1(x)f^2(x)x^2dx=\int_0^\infty g_l^1(p)g_l^2(p)p^2dp, \quad \text{(B-5)}$$

valid for all l, where $g_l^1(p)$ and $g_l^2(p)$ are the FBT(l), respectively, of $f^1(x)$ and $f^2(x)$.

It will actually be more convenient for us to use a slightly different form of this theorem. The integral (B-3) is ordinarily thought of as defining $g_l(p)$ only for $p>0$. However, we may take the same integral as defining its analytic continuation for $p<0$. and from the fact that $j_l(-x)=(-1)^lj_l(x)$, $g_l(p)$ then has a definite parity,

$$g_l(-p)=(-1)^lg_l(p).$$

Consequently the integrand of the second integral of (B-5) is always even [independently of the properties of $f^1(x)$ and $f^2(x)$], so we may rewrite Parseval's theorem as

$$\int_0^\infty f^1(x)f^2(x)x^2dx=\tfrac{1}{2}\int_{-\infty}^\infty g^1(p)g^2(p)p^2dp, \quad \text{(B-6)}$$

a form which will be convenient later, when we shall wish to deform the contour of integration to something other than the real p axis.

Finally, if $f^1(x)$ and $f^2(x)$ are too divergent as $x\to\infty$ to have transforms, but $e^{-\epsilon x}f^1(x)$ and $e^{-\epsilon x}f^2(x)$ are not, we define

$$g_l(p;\epsilon)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\epsilon x}f(x)j_l(px)x^2dx, \quad \text{(B-7)}$$

and use Parseval's theorem in the form

$$\lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\int_0^\infty e^{-(\epsilon_1+\epsilon_2)x}f^1(x)f^2(x)x^2dx$$
$$=\lim_{\epsilon_1\to 0}\lim_{\epsilon_2\to 0}\tfrac{1}{2}\int_{-\infty}^\infty g_l^1(p;\epsilon_1)g_l^2(p;\epsilon_2)p^2dp. \quad \text{(B-8)}$$

With these preliminaries established, we shall proceed, as in Appendix A, by considering the simplest examples of our desired theorem first. As in Appendix A, the essential difficulties occur even in the case $c=0$, i.e., when the confluent hypergeometric function of (B-1) is replaced by unity. We state the result for this case as a somewhat lengthy lemma:

Lemma B-1.—Consider the integral

$$A(\lambda+2)\equiv\lim_{\epsilon\to 0+}\int_0^\infty e^{-\epsilon x}x^{\lambda+2}F(x)dx, \quad \text{(B-9)}$$

where (a) $\lambda>-2$, (b) $\int_0^\infty x|F(x)|dx$ exists, and (c) $G_l(p)$, the FBT(l) of $F(x)$, possesses all its derivatives everywhere on the real p-axis, and $G_l^{(n)}(p)=O(p^{-2})$ as $|p|\to\infty$, for all n and l. If in addition m is the next integer larger than $\lambda+2$, so that $m-1<\lambda+2\leqslant m$, the integral (B-9) can be transformed into the integral

$$A(\lambda+2)\equiv A_l(\lambda+2;m)=S_l(\lambda+2)\frac{\Gamma(\lambda+3-m)}{\Gamma(\lambda+3)}$$
$$\times\tfrac{1}{2}\int_{-\infty}^\infty \tilde{G}_l^{(m)}(p)[\varepsilon(p)]^{l+m}p^{m-\lambda-3}dp, \quad \text{(B-10)}$$

valid for all l, where $S_l(\lambda+2)$ is the function given by Eq. (A-5), $\tilde{G}_l(p)\equiv p^2G_l(p)$, and $\varepsilon(p)$ is the usual step-function, $+1$ for $p>0$ and -1 for $p<0$.

The interpretation of the integral in (B-10) is straightforward for λ not integral, for the singularity is then integrable; if we impose the additional restriction that $\lambda>l-2$, we may take the limit $\lambda\to m-2$ $(m>l)$ and find

$$A_l(m;m)=\frac{i^{(l+m)}\Gamma(l+m+1)\Gamma[-\frac{1}{2}(l+m)]}{(2\pi)^{\frac{1}{2}}2^{l+1}\Gamma(m+1)\Gamma[\frac{1}{2}(l-m+2)]}$$
$$\times\begin{cases}\pi\tilde{G}_l^{(m)}(0), & (l+m)\text{ even}\\ -i\int_{-\infty}^\infty[\tilde{G}_l^{(m)}(p)/p]dp, & (l+m)\text{ odd},\end{cases} \quad \text{(B-11)}$$

where the last integral is to be interpreted as a principal-value integral. Finally, if $F(x)$ does not possess the requisite properties but $e^{-\delta x}F(x)$ does, the lemma holds with $G_l(p)$ replaced by $G_l(p;\delta)$, as in Eq. (B-7).

This lemma by itself would appear to be rather uninteresting, especially as the integrals (B-10) and (B-11) are considerably more complicated than the original one, (B-9). Its purpose, however, just as in Appendix A, is to establish a theorem for the special case in which the integrand is simply $x^\lambda F(x)$; then when we prove a similar theorem for a more complicated integrand, say $W(x)F(x)$, where $W(x)$ has a Taylor's expansion valid for all x, we will be able to decide easily whether or not the same result would have been obtained by inserting this Taylor's expansion and integrating term by term.

The lemma is proved by the use of Parseval's theorem in the extended form (B-8); we shall assume for simplicity that $F(x)$ is convergent enough not to require a convergence factor, but $x^{\lambda+2}$ clearly does. To find the FBT(l) of x^λ, we shall, as in Appendix A, use an integral representation for $j_l(x)$, but in this case it will be more convenient to choose the straight-line contour of (A-6) rather than the semicircle of (A-7). Then just as in the proof of Lemma (A-1), the FBT(l) of $x^\lambda e^{-\epsilon x}$ is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{-\epsilon x}x^\lambda j_l(px)x^2dx$$
$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\lambda+3)}{2i^l}\int_{-1}^1 P_l(x)(\epsilon-ips)^{-\lambda-3}ds.$$

If we now note that, if m is any positive integer,

$$\frac{\Gamma(\lambda+3)}{(\epsilon-ips)^{\lambda+3}}=\frac{\Gamma(\lambda+3-m)}{(is)^m}\left(\frac{d}{dp}\right)^m(\epsilon-ips)^{m-\lambda-3}, \quad \text{(B-12)}$$

we can write the FBT(l) of $e^{-\epsilon x}x^\lambda$ as

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\lambda+3-m)}{2i^{(l+m)}}\left(\frac{d}{dp}\right)^m\int_{-1}^1 P_l(s)s^{-m}$$
$$\times(\epsilon-ips)^{m-\lambda-3}ds, \quad \text{(B-13)}$$

a form which will be useful later. In applying Parseval's theorem, (B-8), the assumption that $G_l(p)$ has all its derivatives allows us to do the p-integration by parts, thus transferring the derivatives to $G_l(p)$. The choice of m is of course important; we take it to be the next integer larger than $\lambda+2$, so that $-1\leqslant m-\lambda-3<0$. Then the assumed behavior of $G_l^{(n)}(p)$ as $p\rightarrow\infty$ assures that we get no integrated terms from the integration by parts. Because of the factor p^2 which appears in the Parseval theorem integral, it is convenient in doing the integration by parts to define

$$\tilde{G}_l(p)\equiv p^2G_l(p). \quad \text{(B-14)}$$

Then we find, using (B-13),

$$A(\lambda+2)\equiv\lim_{\epsilon\to0}\int_0^\infty e^{-\epsilon x}x^\lambda F(x)x^2dx\equiv A_l(\lambda+2;m)$$
$$=\lim_{\epsilon\to0}\frac{\Gamma(\lambda+3-m)}{2(2\pi)^{\frac{1}{2}}i^{(l+m)}}(-1)^m\int_{-\infty}^\infty dp\int_{-1}^1 ds$$
$$\times P_l(s)s^{-m}(\epsilon-ips)^{m-\lambda-3}\tilde{G}_l^{(m)}(p). \quad \text{(B-15)}$$

Taking the limit under the integral sign, the singularity in the p integral is integrable, and the s integral becomes

$$I_{lm}{}^\lambda\equiv\int_{-1}^1 P_l(s)s^{-m}(-ips)^{m-\lambda-3}ds, \quad \text{(B-16)}$$

the evaluation of which is the core of our proof. It appears at first sight to be divergent, but we shall find by explicit evaluation that it is not.

Note first that for x real,

$$(ix)^\alpha=|x|^\alpha e^{i(\pi/2)\alpha\varepsilon(x)}, \quad \text{(B-17)}$$

where $\varepsilon(x)=+1$ for $x>0$ and -1 for $x<0$, so that, since p and s are real,

$$(-ips)^\alpha=|ps|^\alpha[\cos\tfrac{1}{2}\pi\alpha-i\varepsilon(p)\varepsilon(s)\sin\tfrac{1}{2}\pi\alpha].$$

Consequently the integral is

$$I_{lm}{}^\lambda(p)=|p|^{m-\lambda-3}\int_{-1}^1 P_l(s)|s|^{-\lambda-3}$$
$$\times\{[\varepsilon(s)]^m\cos\tfrac{1}{2}\pi(m-\lambda-3)-i\varepsilon(p)[\varepsilon(s)]^{m+1}$$
$$\times\sin\tfrac{1}{2}\pi(m-\lambda-3)\}ds.$$

Using the fact that $P_l(-s)=(-1)^lP_l(s)$, we have

$$I_{lm}{}^\lambda(p)=2|p|^{m-\lambda-3}\cos\tfrac{1}{2}\pi(m-\lambda-3)\times\int_0^1 P_l(s)s^{-\lambda-3}ds, \quad (m+l)\text{ even}$$
$$=-2i|p|^{m-\lambda-3}\varepsilon(p)\sin\tfrac{1}{2}\pi(m-\lambda-3)\times\int_0^1 P_l(s)s^{-\lambda-3}ds, \quad (m+l)\text{ odd}. \quad \text{(B-18)}$$

An argument entirely analogous to that used in establishing Eq. (A-8) gives the result,

$$\int_0^1 P_l(s)s^{-\lambda-3}ds$$
$$=\frac{(-1)^l}{2^{l+1}}\frac{\Gamma(\lambda+l+3)\Gamma[-\frac{1}{2}(\lambda+l+2)]}{\Gamma(\lambda+3)\Gamma[\frac{1}{2}(l-\lambda)]}, \quad \text{(B-19)}$$

valid for all nonintegral λ.

If we now note that

$$\cos\tfrac{1}{2}\pi(m-\lambda-3)=i^{m+l}\sin\tfrac{1}{2}\pi(\lambda+l), \quad (m+l)\text{ even}$$

and

$$\sin\tfrac{1}{2}\pi(m-\lambda-3)=-i\times i^{m+l}\sin\tfrac{1}{2}\pi(\lambda+l), \quad (m+l)\text{ odd},$$

we find, comparing with Eq. (A-5),

$$I_{lm}{}^\lambda(p)=\frac{(2\pi)^{\frac{1}{2}}i^{(m+l)}(-1)^l}{\Gamma(\lambda+3)}\times S_l(\lambda+2)|p|^{m-\lambda-3}\begin{cases}1, & (l+m)\text{ even}\\ -\varepsilon(p), & (l+m)\text{ odd}.\end{cases} \quad \text{(B-20)}$$

Inserting this in (B-15) gives the stated result (B-10).

Finally, in order to take the limit $\lambda\rightarrow m-2$, an integer, we must impose the further restriction that $m>l$; for as λ approaches a positive integer, one of the Γ functions in the numerator of $S_l(\lambda+2)$ can have a pole, and only if $\lambda>l-2$ does another pole occur in the denominator to cancel it. Fortunately, in Coulomb functions it *is* true that $\lambda>l-2$.

If $(l+m)$ is odd, the limiting process is simple, for

$$\lim_{\delta\to0}\int_{-\infty}^\infty G(p)\varepsilon(p)|p|^{\delta-1}dp=\int_{-\infty}^\infty G(p)p^{-1}dp,$$

where the result is a principal-value integral.

If $(l+m)$ is even, the factor $\sin\frac{1}{2}\pi(l+\lambda)$ in $S_l(\lambda+2)$ tends to zero as $\lambda\to m-2$; in this case it can be rewritten as

$$\sin\tfrac{1}{2}\pi(\lambda+l)=i^{l+m}\sin\tfrac{1}{2}\pi(m-\lambda-2),$$

and we note that

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty}G(p)\frac{\sin\frac{1}{2}\pi\delta}{|p|^{1-\delta}}dp=\lim_{\delta\to 0}\tfrac{1}{2}\pi\int_{-\infty}^{\infty}G(p)\frac{\delta}{|p|^{1-\delta}}dp.$$

Provided $G(p)$ is differentiable for all real p, the evaluation of the second limit by integrating by parts gives just $\pi G(0)$. Inserting these limits and using the definition (A-5) of $S_l(\lambda+2)$, as well as

$$\sin\tfrac{1}{2}\pi(m+l)=-i^{(m+l+1)},$$

for $(m+l)$ odd, we obtain the limiting form (B-11) for $A_l(m;m)$, provided $m>l$.

With these results established, the rest is easy. The next most complicated function to consider is $x_\lambda e^{-cx}$:

Lemma B-2.—Let the conditions of Lemma B-1 hold. Then if in addition $G_l(p)$ is analytic in a region $|p|<R_l$, for some $R_l>0$, and $\mathrm{Re}(c)\geqslant 0$, the expansion

$$B_{\lambda+2}(c)\equiv\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon x}x^\lambda e^{-cx}F(x)x^2dx$$
$$=\sum_{n=0}^{\infty}A(\lambda+n+2)\frac{(-c)^n}{n!}\quad\text{(B-21)}$$

holds, for any l, in that part of the region $|c|<R_l$ for which $\mathrm{Re}(c)\geqslant 0$. By comparison with Lemma (B-1), this is just the expansion one would obtain by expanding e^{-cx} in powers of $(-cx)$ and integrating term by term.

Proof.—Since the FBT(l) of $e^{-(\epsilon+c)x}x^\lambda$, for $\mathrm{Re}(c)\geqslant 0$, is, as in (B-12) and (B-13),

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\lambda+3)}{2i^l}\int_{-1}^{1}P_l(s)(\epsilon+c-ips)^{-\lambda-3}ds$$
$$=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\Gamma(\lambda+3-m)}{2i^{l+m}}\left(\frac{d}{dp}\right)^m\int_{-1}^{1}P_l(s)s^{-m}$$
$$\times(\epsilon+c-ips)^{m-\lambda-3}ds,$$

Parseval's theorem allows us to write the integral of (B-21) in either of the two forms

$$B_{\lambda+2}(c)=\lim_{\epsilon\to 0}\frac{\Gamma(\lambda+3)}{2(2\pi)^{\frac{1}{2}}i^l}\int_{-\infty}^{\infty}dp\int_{-1}^{1}dsP_l(s)$$
$$\times(\epsilon+c-ips)^{-\lambda-3}p^2G_l(p)\quad\text{(B-22a)}$$
$$=\lim_{\epsilon\to 0}\frac{\Gamma(\lambda+3-m)}{2(2\pi)^{\frac{1}{2}}i^{(l+m)}}(-1)^m\int_{-\infty}^{\infty}dp\int_{-1}^{1}ds$$
$$\times P_l(s)s^{-m}(\epsilon+c-ips)^{m-\lambda-3}\tilde{G}_l^{(m)}(p).\quad\text{(B-22b)}$$

We wish to establish the analyticity (in c) of $B_{\lambda+2}(c)$ in a region about $c=0$, in order to prove that it has a Taylor's expansion about the origin. Under the assumed conditions of $F(x)$, the integral of (B-21) is not necessarily convergent for $\mathrm{Re}(c)<0$, so it cannot serve as a means of analytically continuing $B(c)$ into the left half-plane. The definition (B-22a), on the other hand, can provide the continuation.

To see this, we must first know something about the singularities of the integrand,

$$f(p,c)\equiv\int_{-1}^{1}P_l(s)(\epsilon+c-ips)^{-\lambda-3}ds,$$

which results from doing the s integration. Since $P_l(s)$ is a polynomial, the singularities of $f(p,c)$ are simplest to locate by considering the terms of the Legendre polynomial separately. We can do the integral

$$\int_{-1}^{1}\frac{s^n}{(a+ps)^{\lambda+3}}ds$$

by integrating by parts n times, the final integral having the form

$$\frac{1}{p^n}\int_{-1}^{1}(a+ps)^{n-\lambda-3}ds$$
$$=\frac{n-\lambda-2}{p^{n+1}}[(a+p)^{n-\lambda-2}-(a-p)^{n-\lambda-2}].$$

Since $n\leqslant l$, this has a pole at $p=0$ of order at most $(l+1)$, and possibly also poles at $p=\pm a$. It is clear that the integrated terms resulting from the integrations by parts are of a similar form, so that $f(p,c)$ has singularities at only these three points. Further, it is not difficult to see that any FBT(l) has a *zero* of order l at $p=0$; consequently $p^2G_l(p)$ has a zero of order $(l+2)$ there, which cancels the poles of all the terms of $f(p,c)$ at $p=0$.

After the s integration is performed, then, the integrand of the p integral, considered as a function of c, has singularities only at $\epsilon+c=\pm ip$, and these occur in separate (additive) terms. Consider first that part whose singularity occurs at $\epsilon+c=+ip$. Since $\mathrm{Re}(\epsilon+c)>0$ if $\mathrm{Re}(c)\geqslant 0$, this part of the integrand is analytic in c for any p such that $\mathrm{Re}(ip)=-\mathrm{Im}(p)\leqslant 0$, i.e., for any p not in the lower half of the p plane—in particular, for p real. But it is also analytic in p in this region, provided $G_l(p)$ is analytic there. If in particular $G_l(p)$ is analytic for $|p|<R_l$, then this part of the integrand is analytic in p in the shaded portion of the p plane shown in Fig. 4, of radius $\rho_l<R_l$, and we are then free to deform the contour of integration from the real p axis to the contour of Fig. 4. But $|p|\geqslant\rho_l$ everywhere on this contour, so this part of the integrand can have no singularity whenever $|c|<\rho_l<R_l$ (for $\epsilon\to 0$), and

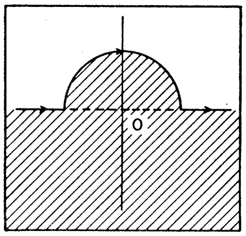

FIG. 4. Contour employed, in the p plane, in the evaluation of $B_{\lambda+2}(c)$.

we thus have the analytic continuation of this part of the integral into the left half of the c plane, in the pocket $|c|<\rho_l<R_l$, i.e., the region similar to the shaded part of Fig. 3, of Appendix A. Similarly, the part of the integral whose integrand is singular at $\epsilon+c=-ip$ can be analytically continued into the same region by deforming the p contour into the lower half of the p plane, and in this way we have established the fact that $B_{\lambda+2}(c)$ has a continuation which is analytic in a region $|c|<R_l$ about the origin. Thus it must have a Taylor's series, which we can find most conveniently by differentiating (B-22b) under the integral sign.

Using the fact that

$$\frac{d}{dc}(\epsilon+c-ips)=\frac{-1}{is}\frac{d}{dp}(\epsilon+c-ips),$$

we may transform c derivatives into p derivatives, and then transfer them to $\tilde{G}_l(p)$ by integrating by parts, so the result of differentiating (B-22b) n times is

$$B_{\lambda+2}{}^{(n)}(c)=\lim_{\epsilon\to 0}\frac{\Gamma(\lambda+3-m)}{2(2\pi)^{\frac{1}{2}}i^{(l+m+n)}}(-1)^m \times\int_{-\infty}^{\infty}dp\int_{-1}^{1}ds P_l(s)s^{-m-n} \times(\epsilon+c-ips)^{m-\lambda-3}\tilde{G}_l{}^{(m+n)}(p). \quad \text{(B-23)}$$

But, setting $c=0$, we see by comparison with Eq. (B-15) that

$$B_{\lambda+2}{}^{(n)}(0)=(-1)^n A_l(\lambda+n+2;m+n) \equiv(-1)^n A(\lambda+n+2),$$

so that the Taylor's expansion of $B_{\lambda+2}(c)$ is

$$B_{\lambda+2}(c)=\sum_{n=0}^{\infty}A(\lambda+n+2)\frac{(-c)^n}{n!},$$

the radius of convergence, R_l, being determined by the position of the singularity of $G_l(p)$ nearest $p=0$. (Since the Taylor's expansion is actually independent of l, we may conclude that R_l is the same for all l.)

In terms of these Lemmas, the last generalization is straightforward:

Theorem (B-1).—Let the condition of Lemma (B-2) hold, and in addition suppose $\mathrm{Re}(b)>\mathrm{Re}(a)>0$, $\mathrm{Re}(c)\geqslant 0$, $\mathrm{Re}(d)\leqslant\mathrm{Re}(c)$. Then we have the following double series expansion:

$$\begin{aligned}D_{\lambda+2}(c,d;a,b)&\equiv\lim_{\epsilon\to 0}\int_0^{\infty}e^{-\epsilon x}x^{\lambda}e^{-cx}F(a,b,dx)F(x)x^2dx\\&=\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}B_{\lambda+n+2}(c)d^n\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}\\&\quad\times A(\lambda+m+n+2)\frac{(-1)^m}{m!}c^m d^n,\end{aligned} \quad \text{(B-25)}$$

where $F(a,b,z)$ is the confluent hypergeometric function. The region of convergence is at least $|c|<R/2$, $|d|<R/2$, for c and d unrelated, and is $|c|<R$ if $c=d$; R is defined as the smallest distance from the origin of a singularity of $G_l(p)$. The series converges to the integral wherever the integral exists, i.e., at least for $\mathrm{Re}(c)\geqslant 0$, $\mathrm{Re}(d)\leqslant\mathrm{Re}(c)$.

The single series (B-24) is seen, by the definition (B-21) of $B_\lambda(c)$ and the expansion (B-2) of the confluent hypergeometric function, to be the result one would obtain by inserting this expansion and integrating term by term. The form (B-25) then follows from Lemma (B-2), and as in Appendix A is the result which would be obtained by expanding *both* $F(a,b,dx)$ and e^{-cx} in powers of x and integrating this double series term by term.

By employing the integral representation

$$F(a,b,dx)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{dtx}t^{a-1}(1-t)^{b-a-1}dt,$$

in which the x dependence is exponential, the proof of this theorem becomes very similar to the proof of Lemma (B-2). Since by assumption $\mathrm{Re}(\epsilon+c-dt-ips)>0$, we can use a slight generalization of (B-12) and (B-13) to find the FBT(l) of the first factor in the integrand. Using Parseval's theorem and differentiating under the integral sign n times with respect to d, we find, as in (B-23),

$$\left(\frac{\partial}{\partial d}\right)^n D_{\lambda+2}(c,d;a,b) =\lim_{\epsilon\to 0}\frac{(-1)^{m+n}\Gamma(b)\Gamma(\lambda+3-m)}{2(2\pi)^{\frac{1}{2}}i^{(l+m+n)}\Gamma(a)\Gamma(b-a)} \times\int_{-\infty}^{\infty}dp\int_{-1}^{1}ds\int_0^1 dt\, P(s)s^{-m-n} \times(\epsilon+c-dt-ips)^{m-\lambda-3}\tilde{G}_l{}^{(m+n)}(p) \times t^{a+n-1}(1-t)^{b-a-1}. \quad \text{(B-26)}$$

For $n=0$, this can be written in a form analogous to Eq. (B-22a). By again distorting the contour of the p integration to that of Fig. 4, with a semicircle of radius R, we can use this form to analytically continue (B-26) as a function of d from the region $\mathrm{Re}(d)\leqslant\mathrm{Re}(c)$, for $\mathrm{Re}(c)\geqslant 0$, to include a pocket about $d=0$. As in the proof of theorem (B-1), singularities in d are avoided if we require $|c-dt|<R$, since $|p|\geqslant R$ on the new contour, and this can be accomplished (for $0\leqslant t\leqslant 1$) by requiring, e.g., that $|c|<R/2$, $|d|<R/2$ if c and d are unrelated, or $|c|<R$ if $c=d$.

By this means we see that $D_{\lambda+2}(c,d;a,b)$ is analytic in d in a region about the origin and thus has a Taylor's expansion in d. The coefficients may be found from (B-26), which is valid for $\mathrm{Re}(c)\geqslant 0$, $\mathrm{Re}(d)\leqslant\mathrm{Re}(c)$, by setting $d=0$. The t-integration then separates from the other two, and by comparison with (B-22b) we see that

$$\left(\frac{\partial}{\partial d}\right)^n D_{\lambda+2}(c,0;a,b)$$
$$=B_{\lambda+n+2}(c)\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a+n-1}(1-t)^{b-a-1}dt$$
$$=B_{\lambda+n+2}(c)\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)},$$

using Eq. (A-15), and this establishes the expansion (B-24).

Finally, the form of the matrix element for the emission or absorption of a photon, to which we wish to apply this theorem, requires that we state it in a slightly more general form. We need first the definition of $G(\mathbf{p})$, the three-dimensional Fourier transform of $F(\mathbf{r})$:

$$G(\mathbf{p})=(2\pi)^{-\frac{3}{2}}\int F(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}}d^3r. \qquad \text{(B-27)}$$

The inverse theorem is then

$$F(\mathbf{r})=(2\pi)^{-\frac{3}{2}}\int G(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}}d^3p, \qquad \text{(B-28)}$$

and the form of Parseval's theorem which we need is

$$\int F_1(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}F_2(\mathbf{r})d^3r=\int G_1(\mathbf{p})G_2(\mathbf{k}-\mathbf{p})d^3p. \qquad \text{(B-29)}$$

Furthermore, if $F(\mathbf{r})$ has the "partial wave" form,

$$F(\mathbf{r})=f_l(r)Y_{lm}(\hat{r}),$$

its Fourier transform is

$$G(\mathbf{p})=i^{-l}g_l(p)Y_{lm}(\hat{p}), \qquad \text{(B-30)}$$

where $g_l(p)$ is exactly the FBT(l) of $f_l(r)$, as defined above. Consequently, in this special case, Parseval's theorem takes the form

$$\int f_l(r)Y_{lm}(\hat{r})e^{i\mathbf{k}\cdot\mathbf{r}}F(\mathbf{r})d^3r$$
$$=i^{-l}\int_0^\infty g_l(p)G_{lm}(k,p)p^2dp, \qquad \text{(B-31)}$$

where $g_l(p)$ is the FBT(l) of $f_l(r)$, $G(\mathbf{p})$ is the three-dimensional Fourier transform of $F(\mathbf{r})$, and

$$G_{lm}(k,p)=\int d\Omega_{\mathrm{p}}\, Y_{lm}(\hat{p})G(\mathbf{k}-\mathbf{p}). \qquad \text{(B-32)}$$

Thus we have reduced the right-hand side of (B-31) to a one-dimensional integral; using this as a basis, we can readily prove the following generalization of Theorem (B-1), for the case $c=d$ in which we are most interested.

Theorem (B-2).—Let the following conditions hold: (a) $\lambda>l-2$; (b) $F(\mathbf{r})$ is absolutely integrable over all space and has the Fourier transform $G(\mathbf{p})$; (c) $G_{lm}(k,p)\equiv\int G(\mathbf{k}-\mathbf{p})Y_{lm}(\hat{p})d\Omega_{\mathrm{p}}$; (d) $G_{lm}(k,p)$ possesses all its derivatives with respect to p everywhere on the real p axis, for $k>0$, and $d^nG_{lm}/dp^n\equiv G_{lm}^{(n)}(k,p)=O(p^{-2})$ as $|p|\rightarrow\infty$, for all n, l, and m, and for $k>0$; (e) $G_{lm}(k,p)$ is analytic in p, for $0\leqslant|p|<R(k)$ for all l, and for $k>0$; (f) $\mathrm{Re}(b)>\mathrm{Re}(a)\geqslant 0$; $\mathrm{Re}(c)\geqslant 0$. Then

$$\lim_{\epsilon\to 0}\int e^{-\epsilon r}r^\lambda e^{-cr}F(a,b,cr)Y_{lm}(\hat{r})e^{i\mathbf{k}\cdot\mathbf{r}}F(\mathbf{r})d^3r$$
$$=(-1)^l\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)n!}$$
$$\times A(\lambda+m+n+2)\frac{(-1)^m}{m!}c^{m+n}, \qquad \text{(B-33)}$$

where $F(a,b,z)$ is the confluent hypergeometric function, and $A(\lambda+m+n+2)$ is defined just as in (B-10), but with $G_{lm}(k,p)$ substituted for $G_l(p)$. The series converges for $|c|<R(k)$, and converges to the stated integral wherever the integral exists, i.e., at least for $\mathrm{Re}(c)\geqslant 0$. If we set $c=0$, the series reduces to the single term $m=n=0$, and from the result for this case we see that (B-33) is just the series which would be obtained by expanding both e^{-cr} and $F(a,b,cr)$ in powers of (cr) and integrating the resulting double series term by term.

To prove this, we note that by (B-31) the integral of (B-33) can be transformed to

$$(-i)^l \lim_{\epsilon\to 0} \frac{(-1)^m\Gamma(b)\Gamma(\lambda+3-m)}{2(2\pi)^{\frac{1}{2}} i^{(l+m)}\Gamma(a)\Gamma(b-a)} \int_{-\infty}^{\infty} dp \int_{-1}^{1} ds \int_{0}^{1} dt \times P_l(s) s^{-m}(\epsilon+c-ct-ips)^{m-\lambda-3}\tilde{G}_{lm}{}^{(m)}(k,p) \times t^{a-1}(1-t)^{b-a-1}, \quad \text{(B-34)}$$

by the same manipulations as were used to obtain Eq. (B-26). But except for the factor $(-i)^l$, this is identical with the integral of (B-26), with $n=0$, $c=d$, and $G_l(p)$ replaced by $G_{lm}(k,p)$. Consequently the remainder of the proof follows identically the proof of Theorem (B-1).

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Scattering of 18-Mev Alpha Particles by C^{12}, O^{16}, and S^{32}†

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The scattering of 18-Mev alpha particles by gaseous C_3H_8, O_2, and H_2S targets was studied with a multiplate scattering chamber. The elastic angular distributions exhibit the diffraction-like pattern typical of light elements. Carbon and oxygen show a sharp rise above the Rutherford cross sections at the backward angles, with values σ/σ_R of 660 for carbon and 350 for oxygen near 173°. A good fit to the angular distribution for inelastic scattering leading to the first excited state of C^{12} (4.43 Mev, 2^+) is obtained using a $[j_2(qR)]^2$ dependence with $R=5.5\times10^{-13}$ cm. No direct-interaction analysis is possible for the alpha-particle groups corresponding to the 7.65-Mev and 9.61-Mev levels in C^{12} and to the excited states of O^{16} up to the 8.87-Mev level. All these distributions show strong forward peaking. In the case of inelastic scattering by S^{32} ($Q=-2.44$ Mev), an interaction radius of 6.5×10^{-13} cm can be deduced from the angular distribution, though the agreement with $[j_2(qR)]^2$ is rather poor. A summary of elastic scattering results for elements in the range from $Z=6$ to $Z=47$ is presented. Expressions for the second-order geometry and the multiple-scattering corrections are given.

I. INTRODUCTION

THIS investigation is part of a program to study the scattering of 18- to 19-Mev alpha particles by light and medium-heavy nuclei. The scattering cross sections of Ne, Al, A, Cu, and Ag have been discussed in earlier reports.[1–3] In the present study, C, O, and S were investigated. The carbon and oxygen targets were chosen because the elastic cross section for neon had shown a significant rise at large angles and it seemed desirable to check this trend at lower Z. Sulfur was included as one of the heavier $n\alpha$-type nuclei and because it was hoped that eventually an accurate theory might allow the determination of the nuclear deformation from the angular distribution of the inelastically scattered alpha particles leaving S^{32} in its first excited state.[4]

With the present measurements, a fairly complete survey of the elastic alpha-particle scattering at 18 to 19 Mev is now available in the range of elements from $Z=6$ to $Z=47$.

II. EXPERIMENTAL PROCEDURE

The experimental methods used were essentially those described by Seidlitz *et al.*[2] The external alpha-particle beam of the 37-inch cyclotron was focused by means of a magnetic quadrupole lens into a 19-inch diameter scattering chamber and collimated within a cone of 0.56° half-angle before passing through the target. The beam was collected by a Faraday cup and measured with an integrator of the type designed by Higinbotham and Rankowitz.[5] The maximum error in the number of incident alpha particles is 1.5%. The average incident alpha-partcle energy was obtained, with an estimated maximum error of 1%, by measuring the mean range in aluminum.[6]

The target materials used were reagent grade propane (C_3H_8), oxygen (O_2) and hydrogen sulfide (H_2S), and, for one auxiliary run, a polyethylene foil. The gases were contained in a brass target cell with a ½-mil thick Mylar window, described in detail by Corelli *et al.*[7] The metal parts of the target cell did not obstruct the paths of particles scattered in the range of angles from 10°

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