$$d\sigma_{\epsilon\delta}=C(\theta)\{\tfrac{1}{2}[1+b^2(\theta)+(1-a(\theta))^2] +(-1)^{\epsilon-\frac{1}{2}\delta}\tfrac{1}{2}[1+b^2(\theta)-(1-a(\theta))^2]\cos\theta -(-1)^{\epsilon-\frac{1}{2}\delta}b(\theta)[1-a(\theta)]\sin\theta\}d\Omega. \quad (34)$$

The corresponding degree of polarization (helicity) is

$$P_{\mathrm{p}}(\theta) =\frac{\{1+b^2(\theta)-[1-a(\theta)]^2\}\cos\theta-2b(\theta)[1-a(\theta)]\sin\theta}{1+b^2(\theta)+[1-a(\theta)]^2} =\left(\frac{k}{k+2m}\right)^{\frac{1}{2}}\frac{1+\frac{1}{2}(k+m)(k^2-2m^2)(1-\beta\cos\theta)}{1+\frac{1}{2}k(k^2-m^2)(1-\beta\cos\theta)}, \quad (35)$$

and is plotted in Fig. 6. As expected, $P_{\mathrm{p}}(\theta)$ and $P_{\mathrm{k}}(\theta)$ are nearly equal at small θ, where the cross section is largest at relativistic energies, and have opposite sign at $\theta\sim 180°$. It is remarkable that $P_{\mathrm{p}}(\theta)$ approaches 1 for all values of θ in the high-energy limit.

The integral cross sections pertaining to helicity eigenstates are given by

$$\sigma_{\epsilon\delta}=2\pi\frac{Z^5}{137^4}r_0^2\left(\frac{m}{k}\right)^2\left(\frac{k+2m}{k}\right)^{\frac{3}{2}} \times\left\{\frac{4}{3}+\frac{k^2-m^2}{m(k+2m)}\left(1-\frac{m^2}{(k+m)^2}\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}\right) +(-1)^{\epsilon-\frac{1}{2}\delta}\left(\frac{k}{k+2m}\right)^{\frac{1}{2}}\left[\frac{4}{3}+\frac{k+m}{k}\frac{k^2-2m^2}{m(k+2m)} \times\left(1-\frac{m^2}{(k+m)^2}\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}\right)\right]\right\}, \quad (36)$$

and are shown in Fig. 7. The corresponding "average" degree of polarization $\langle P_{\mathrm{p}}\rangle$ is shown in Fig. 5.

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High-Frequency Limit of Bremsstrahlung in the Sauter Approximation

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The expansion in powers of $Z/137$ utilized in the Sauter theory of the photoelectric effect yields a nonzero cross section proportional to Z^3 at the high-frequency limit of the bremsstrahlung spectrum, and also at the low electron energy limit of the pair production spectrum. These cross sections are related to the Sauter photoelectric cross section by detailed balancing. The results apply equally when the effects of spin orientations and x-ray polarization are considered.

THE x-ray emission by electrons is calculated most simply in the Born approximation.[1] However this approximation breaks down completely at the upper end of the x-ray frequency spectrum, because it involves an expansion in powers of $Z/137\beta_0$ and of $Z/137\beta$, where β_0 and β are the electron velocities (in units of c) before and after a radiative collision with a nucleus of atomic number Z. The high-frequency end of the spectrum arises from collisions in which the electron emerges with $\beta=0$, so that $Z/137\beta=\infty$ instead of being $\ll 1$ as required by the approximation method. Procedures have been proposed to compensate this failure of the simple theory through corrections derived by comparing the exact and the Born approximation results in the nonrelativistic limit.[1,2]

This paper presents a modified Born approximation, which applies at $\beta\sim 0$ and involves an expansion in powers of $Z/137\beta_0$ and $Z/137$ instead of $Z/137\beta$. The modified ("Sauter"[3]) approximation yields, like the earlier corrections, a simple renormalization of the Born approximation results. Whereas the earlier corrections were tentative extrapolations and were intended to apply throughout the range $0\leq\beta<1$, the correction introduced here has a more definite foundation and applies only for $\beta\lesssim Z/137$. The result obtained in this paper for the high-energy limit of the bremsstrahlung is directly applicable to the low electron energy limit of pair production and is related by detailed balancing to the cross section for the photoelectric emission of atomic electrons calculated in the same approximation. No attempt is made here to take into account the terms of

[1] See, e.g., W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, Oxford, 1954), third edition, pp. 242–247.

[2] G. Elwert, Ann. Physik **34**, 178 (1939); E. Guth, Phys. Rev. **59**, 329 (1941); P. Kirkpatrick and L. Wiedemann, Phys. Rev. **67**, 321 (1945). In addition, F. Nagasaka, thesis, University of Notre Dame, 1955 (unpublished), and M. V. Mihailovic, Nuovo cimento **9**, 331 (1958), have carried out calculations at the high-frequency limit using Sommerfeld-Maue wave functions; however, they do not discuss the significance of these wave functions at $\beta=0$.

[3] F. Sauter, Ann. Physik **9**, 217 (1931) and **11**, 454 (1931).

order $(Z/137\beta_0)^2$ and $(Z/137)^2$ which are usually far from negligible.

The expansion in powers of $Z/137$, carried out to first order only and for $\beta_0 \sim 1$, is the same as in the Sauter theory[3] of the photoelectric effect. It was suggested by an analysis of the Sauter theory[4] and it is based on properties of the relativistic Coulomb "zero-energy" wave functions which are studied in detail in a separate paper.[5] The theory of the photoelectric effect to lowest order in $Z/137$ disregards implicitly the binding of an atomic electron, inasmuch as its total energy departs from mc^2 by terms of order $(Z/137)^2$. Within this approximation the photoelectric process is indeed inverse to the process of x-ray emission at the high-frequency limit, which leaves the electron with zero residual energy. We shall apply the term "zero energy" to any free or bound electron state whose energy, inclusive of the rest mass, departs from mc^2 by an amount of order $\leq (Z/137)^2$.

Among the zero-energy states, those with the spectroscopic classification $s_{\frac{1}{2}}$ are most likely to partake in high-energy radiative transitions because electrons in these states are not subjected to centrifugal force and therefore can stay in the region close to the nucleus where the high field strength permits high-frequency oscillations. More specifically, all stationary bound states are eigenstates of the angular momentum and are classified by the symbols $s_{\frac{1}{2}}$, $p_{\frac{1}{2}}$, $p_{\frac{3}{2}}$, $\cdots$, which correspond, respectively, to the values $-1, 1, -2, \cdots$ of the quantum number κ.[6] Stationary free states may be eigenstates of angular momentum or may be superpositions of such eigenstates. It is shown in reference 5 that radiative transitions between zero energy states $s_{\frac{1}{2}}$ and high-energy states with $\beta_0 \sim 1$ have probability amplitudes of lower order in $Z/137$ than transitions that involve only states other than $s_{\frac{1}{2}}$. Therefore the total probability of bremsstrahlung emission in which the electron is left with a given very low energy and travels out in any direction is equal to the probability of radiative transition to the $s_{\frac{1}{2}}$ state of the given energy, to lowest order in $Z/137$.

The value of this probability is further shown, in reference 5, to depend only on a few characteristics of the Dirac $s_{\frac{1}{2}}$ wave function, namely, on the magnitude of the "large" and "small" radial functions, and on the first derivative of the "large" one, evaluated at the position of the nucleus.[7] These characteristics are represented, for magnetic quantum number $\frac{1}{2}$, by the formula

$$\psi_{s_{\frac{1}{2}}}(r,\theta,\varphi) \sim N \begin{vmatrix} 1-(Z/137)(mc/\hbar)r \\ 0 \\ \frac{1}{2}i(Z/137)\cos\theta \\ \frac{1}{2}i(Z/137)\sin\theta e^{i\varphi} \end{vmatrix}, \qquad (1)$$

in which only the normalization factor depends on whether the state is free or bound. Equation (1) may be regarded as the expansion of $\psi_{s_{\frac{1}{2}}}$ into powers of $Z/137$, limited to the first order term. This expansion should not be considered by itself, but only in the context of its intended application; its mathematical discussion and justification proved laborious and is given separately in reference 5.

The approximate wave function (1) can be reduced, by Fourier transformation, to the form of the wave functions utilized in the Born approximation treatment of bremsstrahlung. Indicating a Dirac free-particle wave function by $\psi_{\mathbf{p}}(\mathbf{r}) = u(\mathbf{p})\exp(i\mathbf{p}\cdot\mathbf{r}/\hbar)$, one obtains from (1)[8]:

$$\psi_{s_{\frac{1}{2}}}(\mathbf{r}) \sim N(2\pi)^{-3}\int d\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \begin{vmatrix} (2\pi)^3\delta(\mathbf{p})+8\pi(Z/137)mc/p^4 \\ 0 \\ 4\pi(Z/137)p_z/p^4 \\ 4\pi(Z/137)(p_x+ip_y)/p^4 \end{vmatrix}$$

$$= N\left\{\psi_0(\mathbf{r}) + \sum \psi_{\mathbf{p}}(\mathbf{r})\frac{1}{mc^2-E(\mathbf{p})} \times \int d\mathbf{r}\,\psi_{\mathbf{p}}^*(\mathbf{r})\frac{Ze^2}{r}\psi_0(\mathbf{r})\right\}, \qquad (2)$$

where the sum extends over free-particle states with all momenta $\mathbf{p}$, with either spin orientation, and with $E(\mathbf{p}) = \pm(m^2c^4+c^2p^2)^{\frac{1}{2}}$. The expression in the braces is recognized as the wave function of a zero-energy free particle perturbed to first order by the interaction with a Coulomb field. It is the wave function which represents the zero-energy final state of the electron in the usual theory of bremsstrahlung.[1]

Once the wave function (1) is cast in the form (2), it becomes unnecessary to calculate anew the radiative transition probability from an initial high-energy state with momentum p_0 to a final zero-energy $s_{\frac{1}{2}}$ state. The calculation of reference 1 utilizes wave functions normalized, like the expression $\{\cdots\}$ of (2), to one particle per unit volume, but then it introduces in the final resuls a factor representing the density of final electron statet per unit energy range. This factor is

$$\frac{p^2}{(2\pi\hbar)^3}\frac{dp}{dE}\sin\theta\,d\theta\,d\varphi = \frac{pE}{(2\pi\hbar)^3c^2}\sin\theta\,d\theta\,d\varphi, \qquad (3)$$

[4] Fano, McVoy, and Albers, preceding paper [Phys. Rev. **116**, 1147 (1959)].

[5] K. W. McVoy and U. Fano, this issue [Phys. Rev. **116**, 1168 (1959)].

[6] See, e.g., H. A. Bethe and E. Salpeter, *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1957), Vol. 35, p. 151.

[7] The Dirac $s_{\frac{1}{2}}$ radial wave functions are singular at the nucleus through a factor r^b, where $b = (1-Z^2/137^2)^{\frac{1}{2}}-1$, but this factor has no effect on calculations to lowest order in $Z/137$, as shown in reference 5, and will therefore be disregarded.

[8] The Fourier transformation of r, $\cos\theta$, and $\sin\theta\exp(i\varphi)$ must be suitably defined; this is done in Appendix A of reference 5.

where $\mathbf{p}$ represents the final electron momentum which we want to set equal to zero and $E=(m^2c^4+p^2c^2)^{\frac{1}{2}}$. The probability of radiative transition to a zero energy state $s_{\frac{1}{2}}$ is then obtained by replacing, in the formulas of reference 1, the factor (3) by the square of the normalization factor N indicated in (1) and (2). This normalization factor is[5]

$$N^2=\frac{Z}{137}\frac{m^2c}{\pi\hbar^3}=\frac{Z}{137}\left(\frac{mc}{\hbar}\right)^3\frac{1}{\pi mc^2}. \quad (4)$$

Therefore one can simply multiply the cross section (13) of reference 1 by the ratio

$$\left(\frac{Z}{137}\frac{m^2c}{\pi\hbar^3}\right)\Big/\left(\frac{pE}{(2\pi\hbar)^3c^2}\sin\theta\, d\theta\, d\varphi\right) = 2\pi\frac{Z}{137}\frac{mc}{p}\frac{mc^2}{E}\frac{4\pi}{\sin\theta d\theta d\varphi}, \quad (5)$$

and set $p=0$, $E=mc^2$. The result is

$$d^2\Phi=\frac{Z^3}{137^2}\left(\frac{e^2}{mc^2}\right)^2\frac{(mc^2)^5}{k^2E_0^3}\beta_0\frac{dk}{k}\frac{4\pi\sin^3\theta_0 d\theta_0}{(1-\beta_0\cos\theta_0)^4} \times\left\{1+\frac{1}{2}\frac{E_0(E_0-mc^2)(E_0-2mc^2)}{(mc^2)^3}(1-\beta_0\cos\theta_0)\right\}, \quad (6)$$

where E_0 is the energy and β_0 the velocity of the incident electron, $k\sim E_0-mc^2$ the (maximum) energy of the emitted photon, and θ_0 the angle between the directions of emission and incidence. Integration of (6) over θ_0 yields

$$d\Phi=4\pi\frac{Z^3}{137^2}\left(\frac{e^2}{mc^2}\right)^2\frac{dk}{k}\frac{E_0mc^2}{(E_0-mc^2)^2}\beta_0 \times\left\{\frac{4}{3}+\frac{E_0(E_0-2mc^2)}{mc^2(E_0+mc^2)}\left[1-\left(\frac{mc^2}{E_0}\right)^2\frac{1}{2\beta_0}\ln\frac{1+\beta_0}{1-\beta_0}\right]\right\}. \quad (7)$$

The ratio (5) is equivalent to the Elwert corrective factor[2] applied in the limit $Z/137\beta_0\ll1$, $Z/137\beta\gg1$, where $\beta=p/mc$. In this limit the factor $f(Z/137\beta,Z/137\beta_0)$ given by Eq. (19), reference 1 reduces to $2\pi(Z/137)(mc/p)$, which is the essential constituent of (4). (The remaining factors are $mc^2/E=1$ and $4\pi/\sin\theta d\theta d\varphi$ which represents an integration over the direction of the outgoing electron.)

The spectrum of pair production particles at the limit of high positron energy and zero-energy electron presents the same problem as the high photon energy limit in the bremsstrahlung. Equation (6) on p. 257 of reference 1 can be modified by the same factor (4) as the bremsstrahlung formula. (The problem is different at the opposite limit of the spectrum, for low positron energy, owing to the repulsion exerted by the nucleus on the positron; the spectrum vanishes in this limit exponentially rather than in proportion to p_+ as predicted by the Born approximation.)

Because the wave function (1) coincides, except for the normalization factor, with the relevant part of the K electron wave function in the Sauter theory of the photoelectric effect, the cross sections (6) and (7) are the same as those obtained by Sauter for the photoeffect by unpolarized x-rays. The normalization factor (4) of zero-energy states per unit energy is replaced for K electrons, by

$$N_K^2=\left(\frac{Z}{137}\frac{mc}{\hbar}\right)^3\frac{1}{\pi}. \quad (8)$$

A factor k^2 in the bremsstrahlung formulas, proportional to the spectral density of states available for bremsstrahlung emission, is replaced with a factor $c^3p_0^2dp_0/dE_0=cp_0E_0$ for the photoelectric process, and a factor $1/\beta_0$ in the cross section for bremsstrahlung by incident electrons is replaced with 1 for the photoelectric absorption of incident photons. Thereby the photoelectric and bremsstrahlung cross sections are related to *lowest order* in $Z/137$, by

$$\Phi_{\text{photo}}=\left(\frac{Z}{137}\right)^2 mc^2\frac{E_0+mc^2}{E_0-mc^2}\frac{d\Phi}{dk}. \quad (9)$$

Entering in (9) the expressions (6) or (7) of $d\Phi/dk$ yields, respectively, the Sauter differential and integral cross sections,[3] averaged over the polarization of the incident x-rays.

The relationships among cross sections obtained in this paper apply equally when the polarization of the incident or emitted x-rays or electrons is taken into account rather than averaged out, because they involve only normalization constants independent of spin and polarization.

Applications and comparisons with experimental results have been presented separately.[9]

[9] J. W. Motz and R. C. Placious, Phys. Rev. **112**, 1039 (1958); Fano, Koch, and Motz, Phys. Rev. **112**, 1679 (1958).